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STABLE PROCESSES ON THE BOUNDARY OF A REGULAR TREE

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We define a class of processes on the boundary of a regular tree that can be viewed as "stable" Lévy processes on $(\mathbb{Z}/n_0\mathbb{Z})^{\mathbb{N}}$. We show that the range of these processes can be compared with a Bernoulli percolation as in Peres which easily leads to various results on the intersection properties. We develop an alternative approach based on the comparison with a branching random walk. By this method we establish the existence of points of infinite multiplicity when the index of the process equals the dimension of the state space, as for planar Brownian motion.

1. Introduction. The goal of this article is to study the intersection properties of a class of self-similar processes, namely "stable" Lévy processes on the group $G = (\mathbb{Z}/n_0\mathbb{Z})^{\mathbb{N}}$, $n_0 \geq 2$. These processes are a special instance of Lévy processes on a totally disconnected group, which were already studied by Evans [6].

Let us recall briefly their construction. As for real-valued Lévy processes with bounded variation, they are defined by a sum of jumps derived from a Poisson point process. For every n, let G_n be the subset of G where the first n coordinates equal 0 and μ_n the uniform measure on G_n with total mass 1. Fix a real $m > 1$ and let Π be the Lévy measure of X,

$$\Pi = \sum_{n=0}^{\infty} m^n \mu_n. \tag{1}$$

We can construct a Poisson point process N on $\mathbb{R}_+ \times G$ with intensity measure $dt \otimes \Pi$, which yields a random measure on $\mathbb{R}_+ \times G$ with Dirac point masses at, say, $\{(s, \Delta_s), s \geq 0\}$. Then X is given by the formula

$$X_t - X_0 = \sum_{s \leq t} \Delta_s. \tag{2}$$

One easily checks the self-similarity properties of X, which accounts for the denomination "stable process." More precisely, choosing a parameter $\beta > 1$, the Gromov metric with parameter β (the relevant definition is given in Section 2) gives to G a Hausdorff dimension $d = \log n_0 / \log \beta$ and to X a stability index $\nu = \log m / \log \beta$. An important difference with stable processes on $\mathbb{R}^n$ is that the dimension d and the index ν can take any positive real value.

Evans [6] used the special structure of N to establish precise results on the modulus of continuity, slow points, the exact Hausdorff measure, etc.

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We want to focus on the problem of intersections. To this aim, our main tool will consist in identifying G with the boundary (i.e., the set of lines of descent) of an n_0-regular tree Γ, and in using Peres' method of comparison with a percolation [20].

THEOREM 1. *Let K be a closed subset of G and let X start uniformly on G. Let $\mathbf{B}$ be a Bernoulli percolation on Γ where the survival probability p_n of an edge between levels $n-1$ and n is given by*

$$\begin{array}{ll} p_n = m/n_0 & \text{if } \nu < d, \\ p_n = n/(n+1) & \text{if } \nu = d. \end{array}$$

Then if $\nu < d$,

$$\mathbb{P}([X] \cap K \neq \varnothing) \asymp \mathbb{P}(K \text{ survives } \mathbf{B}) \asymp \mathrm{Cap}_{\nu-d}(K)$$

and if $\nu = d$,

$$\mathbb{P}([X] \cap K \neq \varnothing) \asymp \mathbb{P}(K \text{ survives } \mathbf{B}) \asymp \mathrm{Cap}_{\log}(K).$$

This leads easily to various results on multiple points.

THEOREM 2. (i) *Single points are polar if and only if $\nu \leq d$.*

(ii) *Let $n \geq 1$ be an integer and $X_1, \ldots, X_n$ independent stable processes with respective indices $\nu_1, \ldots, \nu_n$, with $\nu_k < d$ for every k. Then the event*

$$I = \{[X_1] \cap \cdots \cap [X_n] \neq \varnothing\}$$

has positive probability if and only if

$$d' := nd - (\nu_1 + \cdots + \nu_n) < d$$

and in that case, conditionally on I we have almost surely

$$\dim([X_1] \cap \cdots \cap [X_n]) = d - d'.$$

(iii) *The process X has multiple points of order n if and only if $\nu/d > (n-1)/n$. In particular, the process is self-avoiding for $d \geq 2\nu$.*

As for stable processes on $\mathbb{R}^d$ [11, 21, 20], one could summarize the results by saying that the Hausdorff dimension of the range of X is ν if $\nu \leq d$ and that the intersection properties satisfy the algebraic principle "the codimension of the intersection is the sum of the codimensions of the paths."

Theorem 1 can be established by comparing the potential theory for X with the potential theory of Bernoulli percolation derived from Lyons' theorem [17]. Alternatively, using ideas more related to trees, we shall show that the range of X can be compared with a branching random walk on Γ. This enables us to study the problem of points of infinite multiplicity when $\nu = d$, which is the analogue of planar Brownian motion [5, 14, 15].

THEOREM 3. *Assume that $\nu = d$. Let K be a compact, totally discontinuous subset of $\mathbb{R}$. Then almost surely, there exists a point $x \in G$ such that the set of visit times of x $\{t \in \mathbb{R}_+, X_t = x\}$ is in increasing bijection with K.*

We give the construction of the process and prove the first results on intersections in the next section. Then we develop the approach in terms of a branching random walk in Section 3, and apply it in Section 4 to the study of the critical case, that is, when $\nu = d$.

2. Construction and first properties.

2.1. *The state space.* As mentioned in the Introduction, we shall use the identification of G with the boundary of a tree. Recall that a ray of a rooted tree is a maximal self-avoiding sequence of adjacent vertices starting at the root, and that the set of rays is called the boundary of the tree.

Let Γ be an n_0-regular tree, that is, a tree where every vertex has n_0 children. Every vertex at level n can be represented by a finite sequence of integers $(a_0, a_1, \ldots, a_n)$ where $a_i \in \{0, 1, \ldots, n_0 - 1\}$ for each i, and every ray can be represented likewise by an infinite sequence $(a_0, a_1, \ldots)$. Thus the boundary $\partial\Gamma$ can be identified with G. Denote by p^n the projection of the first n coordinates,

$$p^n(a_0, a_1, \ldots, a_{n-1}, \ldots) = (a_0, a_1, \ldots, a_{n-1}) \in (\mathbb{Z}/n_0\mathbb{Z})^n,$$

and by val the valuation on G,

$$\mathrm{val}(a_0, a_1, \ldots, a_n, \ldots) = \sup\{n \in \mathbb{N}, a_0 = \cdots = a_n = 0\}$$

with the convention $\sup \varnothing = -1$. If $\beta > 1$, we can equip G with the Gromov distance with parameter β: $\delta(x, y) = \beta^{-\mathrm{val}(x-y)}$. This gives to G a Hausdorff dimension $\log n_0 / \log \beta$. For every integer n we consider the subgroup $G_n = \{x \in G; \mathrm{val}(x) \geq n\}$ and μ_n the uniform measure on G_n with total mass 1. We define the shift s on G by

$$s(a_0, a_1, \ldots, a_n, \ldots) = (a_1, a_2, \ldots, a_n, \ldots).$$

If we think of G as the additive group of formal series with coefficients in $(\mathbb{Z}/n_0\mathbb{Z})$, we can enlarge it to the group $\overline{G}$ of Laurent series, that is, of elements of the form $x = (a_n)_{n \in \mathbb{Z}}$ where for some $k \in \mathbb{Z}$, $a_n = 0$ if $n \leq k$. One can define in the same manner G_n, μ_n, $n \in \mathbb{Z}$ and the shift operator.

2.2. *Construction of the process.* Let Π be the measure given by (1) and N the Poisson point process on $\mathbb{R}_+ \times G$ with intensity measure $dt \otimes \Pi$. It is easy to check that the series in (2) converges almost surely for every t and defines a process X with stationary, independent increments, that is, a Lévy process [6]. In particular, X has the strong Markov property.

In the sequel, we will often kill X at an independent exponential time ζ with parameter $1/(m-1)$. We also introduce a family $(N^n, n \geq 0)$ of independent

Poisson point processes on $\mathbb{R}_+ \times G$ with respective intensity measures $m^n dt \otimes \mu_n$ and redefine N as $\sum_{n=0}^{\infty} N^n$. We denote by $[X]$ the range of X,

$$[X] = \overline{\{X_t, t \le \zeta\}}.$$

Finally, we can extend X to the process $\overline{X}$ on $\overline{G}$ with Lévy measure,

$$\overline{\Pi} = \sum_{n \in \mathbb{Z}} m^n \mu_n.$$

From the obvious scaling properties of $\overline{\Pi}$ we have

$$(s(\overline{X}_t))_{t \ge 0} \overset{(d)}{=} (\overline{X}_{mt})_{t \ge 0}.$$

As the shift operator multiplies the distances by β, we shall say that $\overline{X}$ is a stable process on $\overline{G}$ with index $\nu = \log m / \log \beta$. Notice that the ratio ν/d does not depend on β. Loosely speaking, the paths of X and $\overline{X}$ are the same for small times. More precisely, we can construct $\overline{X}$ from a family $(N^n, n \in \mathbb{Z})$ of independent Poisson point processes with respective intensity measures $m^n dt \otimes \mu_n$. Then X has the same law as $\overline{X}$ killed at the first jump due to the process $\sum_{n<0} N^n$.

REMARK. In the case when p is prime, random walks and processes on p-adics have been studied by Albeverio; see [1] and the references therein.

2.3. *Potential theory and the link with percolation.* Using easy bare-hand calculations or rephrasing Evans' results [7] in our context, we can easily compute the densities p_t (resp. $\bar{p}_t$) of the semigroup of X (resp. $\overline{X}$): if $\delta(x, y) = \beta^{-n}$,

$$p_t(x, y) = \sum_{k=0}^{n} n_0^k (1 - \exp(-t m^k)) \exp\left(-\frac{t m^k}{m-1}\right),$$

$$\bar{p}_t(x, y) = \sum_{k=-\infty}^{n} n_0^k (1 - \exp(-t m^k)) \exp\left(-t \frac{m^k}{m-1}\right).$$

We deduce the expression of the Green function: if $\delta(x, y) = \beta^{-n}$,

$$\begin{aligned} \int_{\mathbb{R}_+} p_t(x, y)\, dt &= (m-1) \sum_{k=0}^{n} n_0^k \left(\frac{1}{m^k} - \frac{1}{m^{k+1}}\right), \\ \int_{\mathbb{R}_+} \bar{p}_t(x, y)\, dt &= (m-1) \sum_{k=-\infty}^{n} n_0^k \left(\frac{1}{m^k} - \frac{1}{m^{k+1}}\right). \end{aligned} \tag{3}$$

In particular,

$$\int_{\mathbb{R}_+} \bar{p}_t(x, y)\, dt < \infty$$

if and only if $m < n_0$. The following is a classical consequence [10].

PROPOSITION 1. *The process $\overline{X}$ is transient if and only if $m < n_0$.*

For random walks on p-adics, finer results on recurrence and transience can be found in [1].

We now proceed to the proof of Theorem 1. We deal here with the killed version of X. Recall that if h is a positive, decreasing function $\mathbb{R}_+ \to \mathbb{R}_+$ and if θ is a finite measure on G, the h-energy of θ ($x > 0$) is defined by

$$\mathscr{E}_h(\theta) = \int_{G^2} h(\delta(y, z))\theta(dy)\theta(dz).$$

The h-capacity of a closed subset K is given by

$$\mathrm{Cap}_h(K) = \sup(\mathscr{E}_h(\theta)^{-1}),$$

where the supremum is taken over all the probability measures θ supported by K with total mass 1. In particular, for $x > 0$, the x-energy $\mathscr{E}_x$ of θ and the corresponding x-capacity are associated to the function $t \to t^{-x}$.

As the process is symmetric and the semigroup absolutely continuous with respect to the uniform measure μ on G, which is the invariant measure, and as we kill the process at an exponential time, it is well known [10] that the hitting probability of a Borel set K for the process started according to μ can be expressed as $C\mathrm{Cap}_G(K)$, where C is a constant and G is the Green function,

$$G(a) = \int_{\mathbb{R}_+} p_t(x, y)\, dt$$

if x and y are two points at distance a. Hence if $\delta(x, y) = \beta^{-n}$ and $m < n_0$, (3) entails

$$\int_{\mathbb{R}_+} p_t(x, y)\, dt \asymp (n_0/m)^n = \delta(x, y)^{\nu - d},$$

where $\asymp$ means that the ratio is bounded above and below by two positive constants that are independent of n. If $m = n_0$,

$$\int_{\mathbb{R}_+} p_t(x, y)\, dt \asymp n \asymp \log \delta(x, y).$$

This proves the result of capacity in Theorem 1. The result of comparison with a percolation follows readily using the capacitary version of Lyons' theorem [17].

As an application of Theorem 1, single points are polar if $\nu \le d$. Conversely, if $\nu > d$, we check that the transition probabilities $p_t(\cdot)$ are continuous, which entails that single points are not polar [10]. This proves part (i) of Theorem 2. In the sequel we shall distinguish three cases: the subcritical case ($\nu < d$), the critical case ($\nu = d$) and the supercritical case ($\nu > d$).

2.4. *Intersections in the subcritical case.* We prove here parts (ii) and (iii) of Theorem 2.

PROOF OF (ii). Using Peres' arguments (Section 2 in [20]) together with Theorem 1, we see that the problem of intersection of $[X_1], \dots, [X_n]$ reduces to the problem of intersection of independent Bernoulli percolations with respective parameters $\beta^{\nu_1}/n_0, \dots, \beta^{\nu_n}/n_0$, which is the same as a single Bernoulli percolation with parameter

$$p = \frac{\beta^{\nu_1+\cdots+\nu_n}}{n_0^n} = \beta^{-d'}.$$

This percolation yields a Galton–Watson process with mean $pn_0 = \beta^{d-d'}$, which is supercritical if and only if $pn_0 > 1$, that is, $d' < d$.

To compute the Hausdorff dimension of the intersection, it suffices to remark that almost surely, I intersects the range of an independent process X if and only if the index ν of X satisfies $\exp(\nu + d') > n_0$, that is, $\nu + d' > d$. According to Theorem 1, this means that the x-capacity of I is positive if and only if $x < n_0 - d'$. The connection between capacity and Hausdorff dimension induced by Frostman's lemma (see [12]) yields the result. Notice that for $n = 1$, this provides a proof that the index of the process is the Hausdorff dimension of its range.

REMARK. As an alternative to Peres' method, the most precise results relating intersections of paths to capacity can be found in [9].

PROOF OF (iii). Let us prove, for instance, that if $\nu/d > 1/2$, X has double points almost surely. Consider t_1 and t_2 the first two jump times due to N^0,

$$t_1 = \inf\{t > 0, N^0(t \times G) > 0\},$$
$$t_2 = \inf\{t > t_1, N^0(t \times G) > 0\}.$$

The event $\{t_2 < \zeta\}$ occurs with positive probability and on that event we can define the ranges

$$[X^1] = \overline{\bigcup_{0 \le t < t_1} X_t},$$
$$[X^2] = \overline{\bigcup_{t_1 \le t < t_2} X_t}.$$

It is clear from the self-similarity of X and the strong Markov property that $[X^1]$ and $[X^2]$ are independent and that

$$s([X^1]) \stackrel{d}{=} s([X^2]) \stackrel{d}{=} [X].$$

Consequently, since $\nu/d > 1/2$, the event

$$\{s([X^1]) \cap s([X^2]) \neq \varnothing\}$$

has positive probability, and the same holds for the event

$$\{[X^1]\cap[X^2]\neq\varnothing\}=\{p^1(X_0)=p^1(X_{t_1})\}\cap\{s([X^1])\cap s([X^2])\neq\varnothing\}.$$

Recall that every jump time is the kth jump of the process N^i for some integers k and i and is therefore a stopping time. Using the strong Markov property, the polarity of single points and the fact that jump times are denumerable, we deduce that with probability 1, for every jump time τ,

$$\lim_{t\to\tau-} X_t$$

is a simple point of $[X]$. This entails that, conditionally on the event,

$$\{[X^1]\cap[X^2]\neq\varnothing\}$$

there exists almost surely two times s, s', $0<s<t_1$ and $t_1<s'<t_2$, such that $X_s=X_{s'}$. Hence X has double points with positive probability.

Finally, using the scaling property, we deduce that X has a double point almost surely. The case of multiple points of order n can be proved by the same arguments.

2.5. *The supercritical case.* We shall rather consider the (nonkilled) process $\overline{X}$ here. The results stated are the same for X.

PROPOSITION 2. *Suppose that the process $\overline{X}$ is supercritical. Then single points are not polar and regular. The set of return times R_x to a fixed single point x, conditionally on nonemptiness, is a regenerative set with Hausdorff dimension $1-d/\nu$ almost surely.*

PROOF. Fix a point $x\in G$. It is a classical fact that R_x, as the set of return times to a point for a strong Markov process, is the range of a (killed) subordinator σ, that is, a regenerative set. One can determine the Laplace transform of σ from the resolvents $\overline{u^q}$ of $\overline{X}$ (see, e.g., [2], Chapter IV),

$$\mathbb{E}(e^{-q\sigma_t})=\exp(-t\Phi(q)),\qquad q>0,$$
$$\frac{1}{\Phi(q)}=\bar{u}^q(0).$$

The resolvents can be easily computed from the transition probabilities: if $\mathrm{val}(x)=n$,

$$\bar{u}^q(x)=\int_0^\infty e^{-qt}p_t(0,x)\,dt=(m-1)\sum_{k=-\infty}^{n}\frac{n_0^k}{q(m-1)+m^k}-\frac{n_0^k}{q(m-1)+m^{k+1}}.$$

As a consequence,

$$\Phi(mq)=\frac{n_0}{m}\Phi(q). \tag{4}$$

Furthermore, since Φ is continuous, increasing on $\mathbb{R}_+$, we deduce that there exist two positive constants c_1 and c_2 such that for every positive q,

$$c_1 q^{1-d/\nu} \leq \Phi(q) \leq c_2 q^{1-d/\nu}$$

and classical results on subordinators [2] entail that $\dim(R_x) = 1 - d/\nu$. $\square$

REMARK. Subordinators satisfying (4) are called strictly semistable in the literature; see, for instance, [3].

3. The associated branching random walk.

3.1. *The main result.* Let us first introduce some more definitions and notations on trees. We denote by $|v|$ the level of a vertex v. If u and v are two vertices, $u \leq v$ means that u is an ancestor of v. As an additional structure, one can put a labelling on a tree, that is, for every vertex, a total order on the set of its children. A labelled tree can be thought as a tree embedded in the plane. In particular, the labelling enables us to consider the left-most vertex at a given level, or the left-most ray going through a given vertex.

Notice that G_n can be viewed as the set of rays issued from the vertex $(0, 0, \ldots, 0)$ at level n. Similarly, the group $(\mathbb{Z}/n_0\mathbb{Z})^n$ can be viewed as the boundary (or, equivalently in this case, the set of leaves) of the tree Γ_n obtained by cutting Γ at level n. Hence, for a ray r, $p^n(r)$ can be identified with the vertex of r at height n. We shall use the following result of elementary topology.

LEMMA 1. *Let K be a closed subset of $\partial\Gamma$. Then K is the boundary of the subtree of Γ defined by the set of vertices*

$$\bigcup_{n\geq 0} p^n(K).$$

We now give our tree construction of the process. Define the pseudogeometric distribution with mean m as the probability distribution on $\{1, 2, \ldots\}$ where the probability of an integer n is $(m-1)^{n-1}m^{-n}$ for $n \geq 1$. Consider a branching random walk W on Γ defined as follows:

1. There is one particle at time 0 on the root.
2. Every particle on a vertex $u \in \Gamma$ at level n has a random offspring according to a pseudogeometric distribution with mean m.
3. Each child goes with equal probability to one of the vertices at level $n+1$ connected to u.

All these events are independent.

Say that a ray is visited by W if all the vertices of this ray are occupied by particles of W. We want to prove that the set of rays visited by W has the same law as $[X]$.

To state the result in terms of trees, construct a random labelled Galton–Watson tree $\mathbb{T}$ with pseudogeometric offspring distribution with mean m ($\mathbb{T}$ can

be thought as the family tree of W). Associate to each vertex $u \in \mathbb{T}$ an independent random variable $g(u)$, uniformly distributed on $\{0, 1, \ldots, n_0 - 1\}$. To each ray r of $\mathbb{T}$, viewed as an infinite sequence of adjacent vertices $(v_0, v_1, \ldots)$, associate the sequence

$$g(r) := (g(v_0), g(v_1), \ldots) \in G.$$

THEOREM 4. *Denote by $\partial\mathbb{T}$ the boundary of $\mathbb{T}$ and set*

$$F = \bigcup_{r \in \partial\mathbb{T}} g(r).$$

Let X start according to the uniform distribution. Then

$$F \overset{(d)}{=} [X].$$

REMARK. The process X can be started according to any probability measure. It suffices to replace the uniform law on $\{0, 1, \ldots, n_0 - 1\}$ by the appropriate distribution for every vertex that is the left-most at its level on $\mathbb{T}$.

3.2. *The tree associated to the process.* To establish Theorem 4, we prove that the construction of X with the N^n's yields a pseudogeometric Galton–Watson tree T. Recall that we kill X at an independent, exponential time ζ with parameter $1/(m-1)$.

We construct T as follows. For every n, the time arrivals of the jumps of X due to the Poisson point process $\sum_{k=0}^{n} N^k$ determine a subdivision of $[0, \zeta]$ into subintervals. Say that these subintervals form the set of vertices at level n of T. Define fatherhood as follows: a vertex v at level i is an ancestor of w at level $j > i$ if, as intervals of $[0, \zeta]$, they satisfy the inclusion relation $w \subset v$. The labelling on T is induced by the natural order on the intervals of a subdivision.

For each vertex $v = [t, t']$ at level n, where t and t' are the times of two consecutive jumps due to the Poisson point process $\sum_{k=0}^{n} N^k$, we put $l(v) = t' - t$. Let $f(v) \in \{0, 1, \ldots, n_0 - 1\}$ be nth component of X_t: if $X_t = (0, a_1, \ldots, a_n, \ldots)$, then $f(v) = a_n$. To each ray r of T, viewed as an infinite sequence of adjacent vertices $(v_0, v_1, \ldots)$, associate the sequence

$$f(r) := (f(v_0), f(v_1), \ldots) \in G.$$

The link between T and the path of X is the following.

PROPOSITION 3. *Denote by ∂T the boundary of T and set*

$$A = \overline{\bigcup_{r \in \partial T} f(r)}.$$

Then $A = [X]$.

PROOF. For every integer n, if $u_1 = [t_0, t_1], \dots, u_k = [t_{k-1}, t_k]$ are the vertices of T at level n, then $p^n(A) = \{p^n(X_{t_0}), \dots, p^n(X_{t_{k-1}})\}$. As $X^n := p^n(X)$ can only jump at times t_i, $1 \le i \le k-1$, it follows that $p^n(A) = [X^n]$. On the other hand, it is clear that $p^n([X]) = [X^n]$. Hence

$$\bigcup_{n=0}^{\infty} p^n(A) = \bigcup_{n=0}^{\infty} p^n([X]).$$

As A and $[X]$ are closed, Lemma 1 entails that $A = [X]$. □

To be more precise, for any time $t < \zeta$, let $[q_1, q_1'], [q_2, q_2'], \dots$ be the only sequence of vertices of T such that for every n, $[q_i, q_i']$ is at level n and $t \in [q_i, q_i')$. This sequence of vertices forms a ray r and one easily checks that

$$X_t = f(r).$$

Furthermore, as X is cadlag, it appears that

$$[X] = \{x = X_t \text{ or } x = X_{t-} \text{ for some } t\}$$

and that $X_t \neq X_{t-}$ if and only if t is a jump time. Let t be a jump time, corresponding to a jump due to the process N^n. Let $v(t)$ be the only vertex of T at level n associated to an interval of the form $[s, t]$. Then it should be clear that

$$X_{t-} = f(r),$$

where r is the right-most ray going through the vertex $v(t)$. We shall denote by ρ the function that associates to every ray r a time t.

The joint law of $(T, f(v), v \in T)$ is characterized by the following.

PROPOSITION 4. *Let X start according to the uniform probability on G. Then*:

(i) *The tree T has the law of a Galton–Watson tree with pseudogeometric offspring distribution with mean m.*

(ii) *Conditionally on T, the variables $(f(v), v \in T)$ are independent and uniformly distributed on $\{0, 1, \dots, n_0 - 1\}$.*

PROOF. (i) To establish this assertion, we prove by induction that for every n:

(a) The tree T_n obtained by cutting T at level n has the law of a Galton–Watson tree with pseudogeometric offspring distribution with mean m, cut at level n.

(b) Conditionally on T_n, denoting by k the number of leaves of T_n and by

$$0 = t_0 < t_1 < \cdots < t_k = \zeta,$$

the jump times of the process $\sum_{i=0}^{n} N^i$, the random variables $t_{i+1} - t_i$ are i.i.d. with exponential distribution with parameter $m^{n+1}/(m-1)$.

This is true for $n = 0$, because of the choice of the killing time ζ. Assume it is true for an integer n. The number of children of a vertex $v = [t, t']$ at level n is $1 + b$, where b denotes the number of jumps in $[t, t']$ due to the process N^{n+1}. In particular, it is independent of T_n according to the second part of the induction hypothesis and independent of the offspring of the other vertices at level n because of the independence properties of Poisson point processes.

Moreover, recall that if e and e' are independent exponential random variables with respective parameters a and a', then conditionally on $e' < e$, e' has the exponential distribution with parameter $a' + a$ and $e - e'$ is independent of e' and has the exponential distribution with parameter a. Applying this for the jumps of the process N^{n+1} between two jumps of the process $\sum_{i=0}^{n} N^i$, we see that b has the geometric distribution with parameter m and that the second part of the induction hypothesis holds.

(ii) If $t = 0$, as X starts according to the the uniform probability on G, $f(v)$ is uniformly distributed on $\{0, 1, \ldots, n_0 - 1\}$. Otherwise, t is the time of a jump due to the Poisson point process $\sum_{k=0}^{n} N^k$. As the projection of its intensity measure on the nth coordinate is the uniform measure on $\{0, 1, \ldots, n_0 - 1\}$, it is clear that $f(v)$ is again uniformly distributed on $\{0, 1, \ldots, n_0 - 1\}$, and that it is independent of $(T, f(u), u \neq v)$. □

REMARK. Here again, an analogous result could be stated for the process started according to another probability measure. It suffices to replace the uniform law on $\{0, 1, \ldots, n_0 - 1\}$ by the appropriate distribution for every vertex that is the left-most at its level in the planar embedding of T.

The fact that $(T, f(v), v \in T)$ has the same law as $(\mathbb{T}, g(u), u \in \mathbb{T})$ entails Theorem 4.

3.3. *Some comments.* The representation of the time interval $[0, \zeta]$ as the boundary of T can be thought of as a generalization of the representation of $[0, 1]$ by the boundary of an n-regular tree, using the n-ary developement. An important difference is that the tree structure of T is naturally connected to the process X, since the vertices of T represent time intervals between two jumps of X. This is not the case when one studies a process using the n-ary developement of the set of times. However, the latter method can be used for the study of exceptional times; see [4, 13] for some recent developements.

Remark that if $\nu < \alpha$, Theorem 1 compares X with a Bernoulli percolation that yields a random subtree of Γ. This subtree is a Galton–Watson tree with binomial distribution with mean m. In particular, it has the same mean as $\mathbb{T}$, which makes the result of intersection-equivalence perhaps more natural in that framework than in the case of stable processes on $\mathbb{R}^d$ [20].

Moreover, the Hausdorff dimension of $\partial\mathbb{T}$, equipped with the Gromov metric with parameter β, is ν. Hence the construction by a branching random walk can be viewed as a random mapping (induced by the function g) of a random object of dimension ν (the boundary of $\mathbb{T}$) to a deterministic object of dimension

d (the boundary of Γ). This provides an intuitive interpretation of the results of Theorem 2. The case when the tree $\mathbb{T}$ is deterministic was studied by Evans [7].

The construction by a branching random walk only constructs the range of X. Nevertheless, it is possible to recover the time parametrization from $(\mathbb{T}, g(u), u \in \mathbb{T})$. Let us describe the results, whose proofs can be easily derived from the properties of Poisson point processes (see [18]).

For every ray $r \in \partial\mathbb{T}$ and every integer n, denote by $l(r, n)$ the number of vertices of $\mathbb{T}$ at level n on the left of r. Then almost surely, for every ray r, the limit

$$\rho'(r) := \lim_{n\to\infty} l(r, n)/m^n$$

exists. The function ρ' is the analogue for $\mathbb{T}$ of the function ρ defined in Section 3.2 for the tree T. In particular, the equality $\rho'(r) = \rho'(r'), r \neq r'$, occurs only if r is the right-most ray going through some vertex v and r' is the left-most ray going through the vertex v', v' being the brother following v. Furthermore, the function $\rho'\colon \partial\mathbb{T} \to \mathbb{R}_+$, induces almost surely a one-to-one correspondance between the set of rays that are not the left-most ray going through some vertex v of $\mathbb{T}$ and some interval $[0, \zeta) \subset \mathbb{R}_+$. Then we have the proposition.

PROPOSITION 5. *Put* $Y_t = g(\rho^{-1}(t))$. *Then*

$$Y \overset{(d)}{=} X.$$

Finally, let us mention that Theorem 1 could be proved by remarking that the *snake* of W (see [16] for this notion of snake) is a random walk on Γ and by using the comparison between random walks and percolation on a tree induced by Lyons' theorem; see [18]. Moreover, the results on intersections of the previous section can be reinterpreted using the tree representation. This is the topic of the end of this section.

3.4. *Polarity.* The problem of polarity has a simple interpretation in terms of the branching random walk W. Consider a fixed point x of G and let the process start uniformly on G. According to Theorem 4, the problem of polarity of x for the process X amounts to the question of whether x is hit by W or not.

Recall that x can be viewed as a ray of $\partial\Gamma$, or, equivalently, as a sequence of vertices of Γ. Define by W_x the set of particles of W located on a vertex of x. As the particles of W are the vertices of $\mathbb{T}$, W_x can be viewed as a subtree $\mathbb{T}_x$ of $\mathbb{T}$. The process visits x if and only if $\mathbb{T}_x$ survives forever.

It is clear that $\mathbb{T}_x$ has the same law as a subtree of $\mathbb{T}$ obtained by performing on $\mathbb{T}$ a Bernoulli percolation with parameter $1/n_0$. In particular, it is a Galton–Watson tree with mean m/n_0. This tree is infinite a.s. if and only if $m > n_0$, which is equivalent to $\nu > d$. Hence points are not polar if and only if $\nu > d$.

3.5. *Hausdorff dimension on T.* We want to relate the Hausdorff dimension on T to the Hausdorff dimension of the set of times.

Recall that the natural mapping from the boundary of a (deterministic) regular tree with degree n^d to the unit cube of $\mathbb{R}^d$ obtained by taking the n-ary development of each coordinate preserves capacity [19]. Likewise, the function ρ defined in Section 3.4 yields a mapping from the boundary of the random tree T to an interval of $\mathbb{R}$, and we can state a similar result.

THEOREM 5. *Let $\mathbf{T}$ be a subtree of T where each vertex has at least one child. Put on $\mathbf{T}$ the Gromov metric δ' with parameter m. Then $\partial\mathbf{T}$ and $\rho(\partial\mathbf{T}) \subset \mathbb{R}_+$ have the same Hausdorff dimension.*

In particular, using the notations of the previous subsection, we see that when $\nu > d$, the Hausdorff dimension of $\mathbb{T}_x$, equipped with the Gromov metric with parameter m, is $1 - \nu/d$. Hence Proposition 2 can be deduced from Theorem 5. Note that Theorem 1 is less precise than the result of capacity-equivalence stated in [19] for regular trees. The choice of the Gromov metric with parameter m is natural: as T is a Galton–Watson tree with mean m, the parameter that gives to the whole boundary a Hausdorff dimension 1 is m.

We shall follow the proof of Theorem 3.1 in [19]. As we are dealing with random trees, we need some estimates that will enable us to compare them with the deterministic case. Recall that every vertex $v \in T$ is an interval of the form $[t, t']$, and that $l(v) = t' - t$.

LEMMA 2. (i) *For every real $m' \in (0, m)$, there exists almost surely a constant C such that for every vertex v,*

$$l(v) < C m'^{-|v|}. \tag{5}$$

(ii) *For every real $m' > m$, there exists almost surely an integer n such that for every $k > n$, any interval of length m'^{-k} contains at most A jumps due to the process $\sum_{i=0}^{k} N^i$, with*

$$A = 1 + 2(\ln m' - \ln m)/(\ln m').$$

PROOF. (i) Fix an integer n. Recall that the vertices at level n are the intervals of the form $[t, t']$ where t, t' are two consecutive jumps of the process $\sum_{i=0}^{n} N^i$. Let Z_n be the number of vertices of T at level n. Then conditionally on Z_n, for each vertex $v = [t, t']$ at level n, $l(v) = t' - t$ has the exponential distribution with mean $m^{n+1}/(m-1)$.

Choose a real $m' \in (0, m)$. For every vertex v at level n,

$$\mathbb{P}(l(v) > m'^{-(+1)n}) = \exp\left(-\frac{1}{m-1}\left(\frac{m}{m'}\right)^{n+1}\right).$$

Taking an arbitrary real $m'' > m$, as Z_n/m^n is a martingale, we have by the Markov inequality,

$$\mathbb{P}(Z_n > m''^n) \le \left(\frac{m}{m''}\right)^n,$$

which leads to

$$\mathbb{P}\left(\sup_{|v|=n} l(v) > m'^{-n}\right) \le \left(\frac{m}{m''}\right)^n + m''^n \exp\left(-\frac{1}{m-1}\left(\frac{m}{m'}\right)^{n+1}\right).$$

Summing on n yields

$$\sum_{n=1}^{\infty}\left(\sup_{|v|=n} l(v) > m'^{-n}\right) \le \sum_{n=0}^{\infty}\left(\frac{m}{m''}\right)^n + m''^n \exp\left(-\frac{1}{m-1}\left(\frac{m}{m'}\right)^{n+1}\right) < \infty.$$

Thus by the Borel-Cantelli lemma, there exists almost surely an integer k such that for $n \ge k$,

$$\sup_{|v|=n} l(v) \le m'^{-(n+1)}.$$

This proves (i).

(ii) Let $m' > m$. Conditioning on the lifetime ζ of the process, we divide the interval $[0, \zeta]$ into subintervals of the form

$$I_i^n = [im'^{-n}, (i+1)m'^{-n}]$$

for every integer n. For fixed n, the number of such intervals is the integer part of $\zeta m'^n$+1. In each of these intervals, the number of jumps of the process $\sum_{i=0}^n N^i$ is an independent random variable with Poisson distribution with mean $c_n = m'^{-n}(1 + m + \cdots + m^n)$. Thus the probability that at least one of these intervals contains at least a jumps of the process $\sum_{i=0}^n N^i$ is less than

$$\zeta m'^{n+1} \sum_{k=a}^{\infty} \frac{c_n^k}{k!} \exp(-c_n) \le \zeta m'^{n+1} c_n^a \asymp \frac{m^{an}}{m'^{(a-1)n}}.$$

Choosing a such that $m'^{a-1} > m^a$ and summing on n yields

$$\sum_{n=1}^{\infty} \zeta m'^n c_n^a < \infty.$$

Using again the Borel–Cantelli lemma, this entails that almost surely, for n sufficiently large, all the intervals I_i^n contain at most a jumps of the process $\sum_{i=0}^n N^i$. Hence almost surely, for n sufficiently large, every interval of length m'^{-n} contains at most $2a$ jumps of the process $\sum_{i=0}^n N^i$. □

PROOF OF THEOREM 5. First, we show that $\dim(\partial\mathbf{T}) \geq \dim(\rho(\partial\mathbf{T}))$. Heuristically, it suffices to prove that if the distance between two rays $r, r' \in \partial\mathbf{T}$ is small, so is the distance between $\rho(r)$ and $\rho(r')$.

Consider two rays r, r' and suppose that $\delta'(r, r') \leq m^{-n}$. This means that r and r' share the same common vertex $v = [t, t']$ at level n. Hence we have $\rho(r) \in [t, t']$, $\rho(r') \in [t, t']$, and $|\rho(r) - \rho(r')| \leq t' - t < Cm'^{-n}$ where the constant C is the same as in (5) and, therefore, does not depend on v, v'. This being true for every $m' < m$, we deduce by standard arguments that

$$\dim(\partial\mathbf{T}) \geq \dim(\rho(\partial\mathbf{T})).$$

Let us show the converse inequality. Consider a probability measure θ on $\partial\mathbf{T}$. If v is a vertex, let $\partial T(v)$ denote the set of rays going through v and set $\theta(v) := \theta(\partial T(v))$. For $x > 0$, the same calculation as in [19] yields

$$\mathscr{E}_x(\theta) = \sum_{k=0}^{\infty} (m^{-kx} - m^{(1-k)x}) \sum_{|v|=k} \theta(v)^2. \tag{6}$$

Consider an $\varepsilon > 0$ and define the real m' such that $m'^{x-\varepsilon} = m$. The $(x-\varepsilon)$-energy of the measure $\theta\rho^{-1}$ on $\mathbb{R}$ satisfies

$$\overline{\mathscr{E}_{x-\varepsilon}}(\theta\rho^{-1}) \leq \sum_{k=0}^{\infty} (m^{-kx} - m^{(1-k)x})\ (\theta\rho^{-1} \times \theta\rho^{-1})\{(t, t'), |t - t'| \leq m'^{1-k}\} \tag{7}$$

by the same argument as for (10) in [19]. The problem is to compute

$$(\theta\rho^{-1} \times \theta\rho^{-1})\{(t, t'), |t - t'| \leq m'^{1-k}\}.$$

Suppose that $|t-t'| \leq m'^{1-k}$. Recall that t is in a vertex v (which is an interval of $\mathbb{R}_+$) of T at level $k-1$ and t' is in a vertex v' of T at level $k-1$. Then (ii) of Lemma 2 implies that for k sufficiently large, this can only happen if the number of vertices of T at level k that are on the right of v and on the left of v' is less than $2(\ln m' - \ln m)/\ln m')$. Hence for a fixed vertex v at level $k-1$,

$$\begin{aligned}\text{Card}\ \{v', |v'| = k-1, &\text{there exist } t \in v, t' \in v' \text{ with } |t - t'| \leq m'^{1-k}\} \\ &\leq 4(\ln m' - \ln m)/\ln m') + 1.\end{aligned}$$

Using the inequality $\theta(u)\theta(v) \leq (\theta(u)^2 + \theta(v)^2)/2$, we see that for k sufficiently large,

$$(\theta\rho^{-1} \times \theta\rho^{-1})\{(t, t'), |t-t'| \leq m'^{1-k}\} \leq (4(\ln m' - \ln m)/\ln m') + 1) \sum_{|v|=k-1} \theta(v)^2.$$

Comparing the latter inequality with (6) and (7), we deduce that if $\mathscr{E}_x(\theta) < \infty$, then $\overline{\mathscr{E}_{x-\varepsilon}}(\theta\rho^{-1}) < \infty$. This being true for every probability measure θ on G and any two positive reals ε and x, we deduce

$$\dim(\partial\mathbf{T}) \leq \dim(\rho(\partial\mathbf{T}))$$

and the theorem is proved.

4. Multiple points in the critical case. Before proving Theorem 3 we establish a weaker result.

PROPOSITION 6. *Suppose that the process X is critical; that is, $\alpha = d$. Then there exists almost surely a multiple point of infinite, nondenumerable order.*

4.1. *A growing sequence of trees.* We want to show that if the index of X' is greater than the index of X, one can construct the range of X' by enlarging the range of X.

Let Z be a pseudogeometric random variable with mean m. Then

$$\sum_{n=0}^{\infty} \mathbb{P}(Z = n)x^n = \frac{(1-p)x}{1-px},$$

with $p = (m-1)/m$. Let $m' > m$, $p' = (m'-1)/m' > p$ and consider a random variable **Imm**, independent of Z with generating function

$$\sum_{n=0}^{\infty} \mathbb{P}(\mathbf{Imm} = n)x^n = \frac{(1-p')(1-px)}{(1-p)(1-p'x)}.$$

Then $Z + \mathbf{Imm}$ is a pseudogeometric random variable with mean m'.

More generally, let $\mathbb{T}$ be a pseudogeometric, labelled Galton–Watson tree with mean m and let $m' > m$. Conditioning on $\mathbb{T}$ we define the m'-enlargement of $\mathbb{T}$ as a random tree $\mathbb{T}'$ constructed as follows.

Step 1. To every vertex $v \in \mathbb{T}$ associate an independent random variable $\mathbf{Imm}_v$ with the same distribution as **Imm**.

Step 2. To the offspring of every $v \in \mathbb{T}$, add $\mathbf{Imm}_v$ children and put them on the left with respect to the labelling.

Step 3. For every vertex v' added at Step 2, construct an independent, pseudogeometric Galton–Watson tree with mean m'.

Of course, the variables $\mathbf{Imm}_v$, $v \in \mathbb{T}$ are assumed independent. The tree $\mathbb{T}'$ consists of the vertices of $\mathbb{T}$ and of the vertices added at Steps 2 and 3, with the induced genealogical relations. It is clear that $\mathbb{T}'$ has the law of a pseudogeometric Galton–Watson tree with mean m'. Indeed, a vertex $v' \in \mathbb{T}'$ is either a vertex of $\mathbb{T}$, in which case the initial offspring in $\mathbb{T}$ plus the children added at Step 2 yields a pseudogeometric offspring with mean m', or a vertex added at Step 2 or 3, in which case its offspring is by definition a pseudogeometric offspring with mean m'. The independence properties follow immediately from the construction.

Finally, if $(\mathbb{T}, g)$ is a pseudogeometric branching random walk with mean m, we can construct an m'-enlargement in the same way. We first construct $\mathbb{T}'$ the m'-enlargement of $\mathbb{T}$ as above and then associate to every vertex v' added at Step 2 or 3 an independent random variable $g'(v')$, uniformly distributed on $\{0, 1, \ldots, n_0 - 1\}$. If $v \in \mathbb{T}$, we put $g'(v) := g(v)$. Then $(\mathbb{T}', g')$ is a pseudogeometric branching random walk with mean m'.

Consider now (m_n) an increasing sequence of reals with

$$m := \lim_{n\to\infty} m_n < \infty.$$

Define by induction a growing sequence of pseudogeometric branching random walks $(\mathbb{T}_n, g_n)$, $n \geq 0$ with respective means m_n. Let $(\mathbb{T}_0, g_0)$ be a pseudogeometric branching random walk with mean m_0 and for $n \geq 0$, let $(\mathbb{T}_{n+1}, g_{n+1})$ be the m_{n+1}-enlargement of $(\mathbb{T}_n, g_n)$. We define the limit branching random walk $(\mathbb{T}, g)$, where $\mathbb{T}$ is the set of vertices

$$\mathbb{T} = \bigcup_{n=0}^{\infty} \mathbb{T}_n,$$

the genealogical relations and the function g being given by the natural compatibility relations. Then it is clear by the same arguments as above that $(\mathbb{T}, g)$ is a pseudogeometric branching random walk with mean m.

4.2. *Proof of Proposition* 6. As above, we construct $(\mathbb{T}, g)$ as the limit of a growing sequence of branching random walks $(\mathbb{T}_n, g_n)$ with respective means m_n. We choose m_n such that

$$\nu_n = \frac{\ln m_n}{\ln n_0} > \frac{2^n - 1}{2^n} \qquad \text{for every } n, \lim_{n\to\infty} m_n = n_0.$$

Note that this choice of m_n entails the existence of multiple points of order 2^n for the branching random walk $(\mathbb{T}_n, g_n)$. So the idea is to construct a sequence $x_1, x_2, \ldots$ such that x_n is a multiple point of order 2^n for $(\mathbb{T}_n, g_n)$. If suitably constructed, this sequence yields a limit point x that will be a multiple point of infinite, nondenumerable order.

To define $x_1, x_2, \ldots$, we need a result that enables us to choose in a deterministic way the multiple points.

LEMMA 3. *Let* $(\mathbb{T}, g), (\mathbb{T}', g')$ *be two deterministic branching walks on* Γ. *Suppose that*

$$\mathbb{I} := \bigcup_{r\in\partial\mathbb{T}} g(r) \bigcap \bigcup_{r\in\partial\mathbb{T}'} g(r) \neq \varnothing.$$

Then there exists a left-most ray $r \in \partial\mathbb{T}$ *such that* $g(r) \in \mathbb{I}$, *and a left-most ray* $r' \in \partial\mathbb{T}'$ *such that* $g'(r') = g(r)$.

PROOF. As sequences of vertices, the rays r and r' are defined by induction as follows. First, u_0 is the root of $\mathbb{T}$ and v_0 is the root of $\mathbb{T}'$. Then we choose for u_n the left-most vertex u of $\mathbb{T}$ at level n such that

$$\bigcup_{r\in\partial\mathbb{T}(u_n)} g(r) \bigcap \bigcup_{r\in\partial\mathbb{T}'(v_{n-1})} g'(r) \neq \varnothing$$

and for v_n the left-most vertex v of $\mathbb{T}'$ at level n such that

$$\bigcup_{r\in\partial\mathbb{T}(u_n)} g(r) \bigcap \bigcup_{r\in\partial\mathbb{T}'(v_n)} g(r) \neq \varnothing.$$

The rays r and r' constructed by this method clearly satisfy the conditions of the lemma. □

Of course, this lemma has a generalization for an arbitrary number of branching walks whose intersection is nonempty. Notice that the possibility of choosing multiple points in a deterministic way is intimately connected with the tree structure. In the case of planar Brownian motion, such a deterministic choice is not possible and intersection local times are needed to establish the existence of points of any kind of multiplicity [14].

We now proceed to the proof of the proposition. Recall that if v is a vertex of $\mathbb{T}_n$, $\mathbb{T}_n(v)$ denotes the set of ancestors and descendants of v in $\mathbb{T}_n$.

Consider a deterministic growing sequence of branching random walks $(\mathbb{T}_n, g_n)$ that has a limit $(\mathbb{T}, g)$. In order to find a multiple point of infinite, nondenumerable order for $(\mathbb{T}, g)$, we construct the following sequence.

Let n_1 be the minimal level such that there exist two distinct vertices $v, v' \in \mathbb{T}_1$ at level n_1 satisfying

$$g(\partial\mathbb{T}_1(v)) \cap g(\partial\mathbb{T}_1(v')) \neq \varnothing.$$

If $n_1 < \infty$, we can consider the left-most vertex v_1^1 and the left-most vertex $v_2^1 \neq v_1^1$ for which this property holds. Then we can choose as in Lemma 3 the left-most ray $r_1^1 \in \mathbb{T}_1(v_1^1)$ and the left-most ray $r_2^1 \in \mathbb{T}_1(v_2^1)$ such that $g(r_1^1) = g(r_2^1)$.

By induction, we define a sequence of rays $(r_i^k,\ k \geq 1,\ 1 \leq i \leq 2^k)$, as follows. Let n_{k+1} be the minimal integer $> n_k$ such that:

1. For every $i \leq 2^k$, the vertex of r_i^k at level n_{k+1} has exactly two children in $\mathbb{T}_{k+1} - \mathbb{T}_k$. Call them v_{2i}^{k+1}, v_{2i+1}^{k+1}.
2. $\bigcap_{i=1}^{2^{k+1}} g(\partial\mathbb{T}_{k+1}(v_i^{k+1})) \neq \varnothing$.

If $n_{k+1} < \infty$, we have 2^{k+1} branching random walks $\mathbb{T}_{k+1}(v_i^{k+1})$, $1 \leq i \leq 2^{k+1}$, whose intersection is nonempty, and we can construct in a deterministic way the rays $r_i^{k+1} \in \partial\mathbb{T}_{k+1}(v_i^{k+1})$ as in Lemma 3, with $g(r_1^{k+1}) = g(r_2^{k+1}) = \cdots$.

Define the tree $T^k \subset \mathbb{T}_k \subset \mathbb{T}$ as the union of the vertices of the rays r_i^k, $1 \leq i \leq 2^k$. Then for $k' > k$, T^k has the same vertices as $T^{k'}$ up to level $n_k - 1$.

Moreover, if the whole sequence is well defined, that is, if for every k, $n_k < \infty$, we can build a tree T^∞ whose vertices are the same as the vertices of T^k up to level $n_k - 1$. Then the boundary of T^∞ is nondenumerable, since there is an obvious bijection between ∂T^∞ and the boundary of the infinite binary tree. Finally, for any two rays $r, r' \in \partial T^\infty$, $g(r) = g(r')$.

As a consequence, to establish Proposition 6 it suffices to prove that for almost every growing sequence of branching random walks constructed as in Section 4.1, the sequence $(r_i^k,\ k \geq 1,\ 1 \leq i \leq 2^k)$ is well defined. So we want to

show by induction that almost surely, for every k, $n_k < \infty$, which is sufficient to construct the sequence of rays.

According to Theorem 2, $(\mathbb{T}_1, g_1)$ contains double points almost surely, which entails $n_1 < \infty$. Suppose that $n_k < \infty$, so that the sequence (r_i^j) is defined for $j \le k$. Condition on $(\mathbb{T}_k, g_k, r_i^j,\ j \le k, 1 \le i \le 2^j)$. Then for every integer $n' > n_k$, the events $E_{n'}^k$:

1. For every $i \le 2^k$, the vertex of r_i^k at level n' has exactly two children, say, v_{2i}^{k+1} and v_{2i+1}^{k+1}, in $\mathbb{T}_{k+1} - \mathbb{T}_k$.
2. $\bigcap_{i=1}^{2^{k+1}} g(\partial \mathbb{T}_{k+1}(v_i^{k+1})) \neq \varnothing$

are independent and have the same positive probability, thanks to the choice of m_{k+1}. Hence there exists almost surely a level n' such that the event $E_{n'}^k$ occurs. Therefore $n_{k+1} < \infty$ almost surely, which completes the proof. □

4.3. *Proof of Theorem* 3. We use the same growing sequence of trees as in the proof above, with the same notations. To establish Proposition 6, we have constructed an infinite binary tree T^∞ associated to a multiple point of infinite, nondenumerable order. The basic idea is that we can impose any shape for the tree T^∞, which corresponds to giving to the corresponding subset of $\mathbb{R}_+$ any structure of compact, totally discontinuous set.

Let K be a compact, totally discontinuous subset of $\mathbb{R}$. Then it is a classical result that K is in increasing bijection with a closed subset of $\{0, 1\}^{\mathbb{N}}$. In other words, there is an increasing bijection between K and the boundary of a labelled tree $\mathscr{T}$, where every vertex has one or two children. Here "increasing" refers to the natural order on $\partial\mathscr{T}$ induced by the labelling. Therefore we want to construct a subtree T' of $\mathbb{T}$ such that there exists an increasing bijection between $\partial\mathscr{T}$ and $\partial T'$, and such that $g(\partial T')$ is reduced to one point $x \in G$.

We first consider a deterministic growing sequence of branching random walks $(\mathbb{T}_n, g_n)$ that has a limit $(\mathbb{T}, g)$. As in the proof of Theorem 4, we want to define a deterministic sequence of rays $(r_i^k,\ k \ge 1,\ 1 \le i \le c_k)$, where c_k is the number if vertices of $\mathscr{T}$ at level k.

Define by convention r_1^0 as the left-most ray of $\mathbb{T}_1$, and put $n_0 = 0$. Let n_{k+1} be the minimal integer n_k such that:

1. For every $i \le c_k$, the vertex of r_i^k at level n_{k+1} has exactly the same number of children in $\mathbb{T}_{k+1} - \mathbb{T}_k$ as the ith vertex of $\mathscr{T}$ at level k. Call these children u_i^k, u'^k_i, with $u'^k_i = u_i^k$ if the ith vertex of $\mathscr{T}$ at level k has one child.
2. $\bigcap_{i=1}^{c_k} \big(g(\partial \mathbb{T}_{k+1}(u_i^k)) \cap g(\partial \mathbb{T}_{k+1}(u'^k_i))\big) \neq \varnothing$.

Then if $n_{k+1} < \infty$ we can choose the rays r_i^{k+1}, $i \le c_{k+1}$ as in the proof of Proposition 6. We can construct likewise the trees T^k and $T^\infty = T'$ if the whole sequence $(r_i^k,\ k \ge 1,\ 1 \le i \le c_k)$ is well defined. Then there exists an increasing bijection between $\partial\mathscr{T}$ and $\partial T'$, and $g(\partial T')$ is reduced to one point $x \in G$.

The proof that the sequence $(r_i^k,\ k \geq 1,\ 1 \leq i \leq c_k)$ is almost surely well defined, in the case of the random growing sequence of trees constructed in Section 4.1, is the same as in the proof of Proposition 6.

We have to prove that there is no other ray r such that $g(r) = x$. Equivalently, we have to prove that for every n and every vertex $v \notin T'$ at level n, $x \notin g(\partial\mathbb{T}(v))$.

So we condition on $(\mathbb{T}_n, T'_n, g(v), v \in \mathbb{T}_n)$ (recall that the $\mathbb{T}_n$, T'_n are the trees $\mathbb{T}, T'$ cut at level n). Let $\mathbf{a}$ be a configuration of $(\mathbb{T}_n, T'_n, g(v), v \in \mathbb{T}_n)$ and write

$$E_{\mathbf{a}} = \{(\mathbb{T}_n, T'_n, g(v), v \in \mathbb{T}_n) = \mathbf{a}\}.$$

Notice that the set of all possible configurations is denumerable and that each one has positive probability. In particular, if an event has zero probability on the whole probability space, it also has zero probability conditionally on $E_{\mathbf{a}}$.

Denote by u the left-most vertex of T' and by σ the corresponding vertex on Γ. Then conditionally on $E_{\mathbf{a}}$, the ray x defined above is uniformly distributed on $\partial\Gamma(\sigma)$. Thus, for every vertex v at level n on $\mathbb{T}_n - T'_n$, conditionally on $E_{\mathbf{a}}$, $x \notin g(\partial\mathbb{T}(v))$, almost surely, since single points are polar. So we have proved that for every n, almost surely, for every vertex $v \in \mathbb{T}_n - T'_n$ at level n, $x \notin g(\partial\mathbb{T}(v))$. It suffices now to replace "for every n, almost surely" by "almost surely, for every n," to complete the proof.

Finally, every ray of $\partial T'$ contains infinitely many vertices added at Step 2 in the construction of the growing sequence of trees. As these vertices are added on the left, none of these rays is the right-most going through some vertex of $\mathbb{T}$. Hence ρ' induces a bijection between $\partial T'$ and some interval $[0, \zeta)$ of $\mathbb{R}_+$ and the image of $\partial T'$ by this bijection has the required topological structure.

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