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BOOTSTRAPPING THE STUDENT t-STATISTIC

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Let $X_1, \dots, X_n$, $n \geq 1$, be independent, identically distributed random variables and consider the Student t-statistic T_n based upon these random variables. Giné, Götze and Mason (1997) proved that T_n converges in distribution to a standard normal random variable if and only if X is in the domain of attraction of a normal random variable and $EX = 0$. We shall show that roughly the same holds true for the bootstrapped Student t-statistic T_n^* . In the process we shall disclose all the possible subsequential limiting laws of T_n^*. The proofs introduce a number of amusing tricks that may be of independent interest.

1. Introduction and statement of main result.

Let X, X_1, $X_2, \dots,$ be independent, nondegenerate random variables with common distribution function F. For each integer $n \geq 2$, let

$$F_n(x) = n^{-1} \sum_{i=1}^{n} 1\{X_i \leq x\}, \qquad x \in \mathbf{R},$$

$$\overline{X}_n = n^{-1} \sum_{i=1}^{n} X_i$$

and

$$s_n^2 = \frac{\sum_{i=1}^n \{X_i - \overline{X}_n\}^2}{n-1},$$

denote the empirical distribution function, sample mean and sample variance, respectively, based upon $X_1, \dots, X_n$. Consider the Student t-statistic

$$T_n = \frac{\sqrt{n}\ \overline{X}_n}{s_n}. \tag{1.1}$$

Recently Giné, Götze and Mason [9] solved the question concerning when the Student t-statistic is asymptotically distributed as a standard normal random variable Z. They showed that

$$\mathscr{L}(T_n) \to \mathscr{L}(Z) \qquad \text{as } n \to \infty, \quad \text{if and only if } EX = 0 \text{ and } X \in DN, \tag{1.2}$$

where for any random variable Y, $\mathscr{L}(Y)$ denotes its law and $X \in DN$ signifies that X is in the domain of attraction of a nondegenerate normal random

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variable. The latter means that there exist sequences of norming and centering constants $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ such that

$$\mathscr{L}\left(a_n \sum_{i=1}^{n} X_i - b_n\right) \to \mathscr{L}(Z) \qquad \text{as } n \to \infty. \tag{1.3}$$

We shall prove that roughly the same holds for the bootstrapped version of T_n. For any integer $m \geq 2$, conditioned on F_n, sample $X_1^*, \ldots, X_m^*$ i.i.d. F_n and form the bootstrapped mean and sample variance

$$\overline{X}^*_{n,m} = m^{-1} \sum_{i=1}^{m} X_i^* \qquad \text{and} \qquad s^{*2}_{n,m} = \sum_{i=1}^{m} \{X_i^* - \overline{X}^*_{n,m}\}^2/(m-1). \tag{1.4}$$

Let $\mathscr{L}_n^*$ denote the conditional law given $X_1, \ldots, X_n$. For a sequence of random variables $Y, Y_1, Y_2, \ldots,$

$$\mathscr{L}_n^*(Y_n) \to_P \mathscr{L}(Y), \tag{1.5}$$

shall mean that, with probability 1, for every subsequence $\{n_i'\}_{i\geq 1}$ of $\{n\}_{n\geq 1}$ there is a subsequence $\{n_l\}_{l\geq 1}$ of $\{n_i'\}_{i\geq 1}$ such that almost surely

$$\mathscr{L}_{n_l}^*(Y_{n_l}) \to \mathscr{L}(Y) \qquad \text{as } l \to \infty.$$

Introduce the condition on a sequence of positive integers $\{m_n\}_{n\geq 1}$ that for all n large enough

$$\lambda_1 \leq m_n/n \leq \lambda_2, \tag{1.6}$$

for some constants $0 < \lambda_1 < \lambda_2 < \infty$. Assuming (1.6), Giné and Zinn [10] proved that there exists a sequence of positive norming constants $\{\gamma_n\}_{n\geq 1}$ for which

$$\mathscr{L}_n^*\left(\frac{\sqrt{m_n}\left\{\overline{X}^*_{n,m_n} - \overline{X}_n\right\}}{\gamma_n}\right) \to_P \mathscr{L}(Z) \qquad \text{if and only if } X \in DN. \tag{1.7}$$

Shortly afterward, by borrowing an idea of theirs, S. Csörgő and Mason [6] showed similarly that whenever (1.6) is satisfied,

$$\mathscr{L}_n^*\left(\frac{\sqrt{m_n}\left\{\overline{X}^*_{n,m_n} - \overline{X}_n\right\}}{s_n}\right) \to_P \mathscr{L}(Z) \qquad \text{if and only if } X \in DN. \tag{1.8}$$

Following close after, Hall [11] obtained a result that contains both (1.7) and (1.8). We shall discuss his result later on in Section 3. Given the Giné and Zinn [10] and S. Csörgő and Mason [6] bootstrap results (1.7) and (1.8), and in light of the Giné, Götze and Mason [9] result (1.2) for the Student t-statistic, it is natural to consider the question as to when the bootstrapped Student t-statistic

$$T^*_{n,m} = \frac{\sqrt{m}\left\{\overline{X}^*_{n,m} - \overline{X}_n\right\}}{s^*_{n,m}}, \tag{1.9}$$

is asymptotically standard normal. This is our main result.

THEOREM 1.1. *Whenever* (1.6) *holds,*

$$\mathscr{L}_n^*\left(T^*_{n,m_n}\right) \to_P \mathscr{L}(Z) \qquad \text{if and only if } X \in DN. \tag{1.10}$$

Moreover,

$$\mathscr{L}_n^*\left(T^*_{n,m_n}\right) \to \mathscr{L}(Z) \qquad \text{as } n \to \infty \text{ a.s., if and only if } EX^2 < \infty. \tag{1.11}$$

In the process of proving Theorem 1.1 we shall, in fact, describe all the possible subsequential laws of

$$\mathscr{L}_n^*\left(\frac{\sqrt{m_n}\left\{\overline{X}^*_{n,m_n} - \overline{X}_n\right\}}{s_n}\right) \qquad \text{and} \qquad \mathscr{L}_n^*\left(T^*_{n,m}\right). \tag{1.12}$$

Like the proofs of (1.2) and (1.8), that of Theorem 1.1 will hinge upon the fact due to O'Brien [13] that for a nondegenerate random variable X, $X \in DN$ if and only if

$$\max_{1\le i\le n} \frac{X_i^2}{\sum_{i=1}^n X_i^2} \to_P 0 \qquad \text{as } n \to \infty. \tag{1.13}$$

The proof of Theorem 1.1 is given in Section 2 and supplementary results and remarks are detailed in Section 3.

2. Proofs. Set

$$K^*_{n,m_n} = \frac{m_n\left\{\overline{X}^*_{n,m_n} - \overline{X}_n\right\}}{\sqrt{\sum_{i=1}^{m_n}\left\{X_i^* - \overline{X}_n\right\}^2}}. \tag{2.1}$$

By writing

$$\begin{aligned} T^*_{n,m_n} &= \frac{m_n\left\{\overline{X}^*_{n,m_n} - \overline{X}_n\right\}}{\sqrt{\frac{m_n}{m_n-1}\left[\sum_{i=1}^{m_n}\left\{X_i^* - \overline{X}_n\right\}^2 - m_n\left\{\overline{X}^*_{n,m_n} - \overline{X}_n\right\}^2\right]}} \\ &= \frac{K^*_{n,m_n}}{\sqrt{\left(m_n - \left(K^*_{n,m_n}\right)^2\right)/(m_n-1)}}, \end{aligned} \tag{2.2}$$

it is easy to verify that for some random variable Y, with probability 1,

$$\mathscr{L}_{n'}^*\left(T^*_{n',m_{n'}}\right) \to \mathscr{L}(Y) \tag{2.3}$$

along some subsequence $\{n'\}$ of $\{n\}$ if and only if

$$\mathscr{L}_{n'}^*\left(K^*_{n',m_{n'}}\right) \to \mathscr{L}(Y). \tag{2.4}$$

Thus to prove Theorem 1.1, it is equivalent to establish:

THEOREM 2.1. *Whenever* (1.6) *holds,*

$$\mathscr{L}_n^*\left(K_{n,m_n}^*\right) \to_P \mathscr{L}(Z) \qquad \textit{if and only if } X \in DN. \tag{2.5}$$

Moreover,

$$\mathscr{L}_n^*\left(K_{n,m_n}^*\right) \to \mathscr{L}(Z) \qquad \textit{as } n \to \infty,\ \textit{a.s. if and only if } EX^2 < \infty. \tag{2.6}$$

Let

$$v_n^2 = \sum_{i=1}^n (X_i - \overline{X}_n)^2. \tag{2.7}$$

Now $X \in DN$ is equivalent to (1.13), which is equivalent to

$$\max_{1 \le i \le n} \frac{\left(X_i - \overline{X}_n\right)^2}{v_n^2} \to_P 0 \qquad \text{as } n \to \infty, \tag{2.8}$$

which, in turn, is equivalent to the existence, with probability 1, for every subsequence $\{n_i'\}_{i\ge1}$ of $\{n\}_{n\ge1}$, a subsequence $\{n_l\}_{l\ge1}$ of $\{n_i'\}_{i\ge1}$ such that

$$\max_{1 \le i \le n_l} \frac{\left(X_i - \overline{X}_{n_l}\right)^2}{v_{n_l}^2} \to 0 \qquad \text{as } l \to \infty. \tag{2.9}$$

The equivalence of (1.13) and (2.8) is trivial to see when X is nondegenerate and $EX^2 < \infty$. When $EX^2 = \infty$, the equivalence follows readily from the fact that in this case, almost surely,

$$n^{-1}\left(\sum_{i=1}^n X_i\right)^2 \Bigg/ \sum_{i=1}^n X_i{}^2 \to 0 \qquad \text{as } n \to \infty.$$

(See Corollary 4 of Chen and Rubin [5].)

Our aim is to show that along some subsequence $\{n'\}$ of $\{n\}$,

$$\mathscr{L}_{n'}^*\left(K_{n',m_{n'}}^*\right) \to \mathscr{L}(Z), \tag{2.10}$$

if and only if with probability 1

$$\max_{1 \le i \le n'} \frac{\left(X_i - \overline{X}_{n'}\right)^2}{v_{n'}^2} \to 0 \tag{2.11}$$

along $\{n'\}$, which by the above equivalences would complete the proof of Theorem 2.1. This will be shown to be a consequence of two propositions, which may be of separate interest.

Consider the statistics

$$W_n := \sum_{i=1}^{m_n}\{X_i^* - \overline{X}_n\}/v_n \quad \text{and} \quad S_n := \sum_{i=1}^{m_n}\left\{X_i^* - \overline{X}_n\right\}^2/v_n^2. \tag{2.12}$$

Clearly, for each $n \geq 2$,

$$K^*_{n,m_n} = \frac{W_n}{\sqrt{S_n}}. \tag{2.13}$$

(We define $0/0 := 0$.)

PROPOSITION 2.1. *Whenever* (1.6) *holds, then, with probability* 1, *the sequence of conditional laws*

$$\mathscr{L}^*_n\{(W_n, S_n)\} \tag{2.14}$$

is stochastically compact, and each subsequential limit law is of the form

$$\mathscr{L}\left\{\left(W + \sigma\sqrt{\lambda}Z, S + \lambda\sigma^2\right)\right\}, \tag{2.15}$$

with

$$W = \sum_{i=1}^{\infty} a_i(N_i - \lambda) - \sum_{i=1}^{\infty} b_i(N'_i - \lambda),$$

$$S = \sum_{i=1}^{\infty} a_i^2 N_i + \sum_{i=1}^{\infty} b_i^2 N'_i,$$

where Z is a standard normal random variable, independent of $N_1, N'_1, N_2, N'_2, \ldots$, a sequence of i.i.d. Poisson mean λ random variables with $\lambda_1 \leq \lambda \leq \lambda_2$, and $1 \geq a_1 \geq a_2 \geq \cdots$ and $1 \geq b_1 \geq b_2 \geq \cdots$, are sequences of non-negative constants satisfying

$$\sum_{i=1}^{\infty} a_i^2 + \sum_{i=1}^{\infty} b_i^2 = 1 - \sigma^2 \tag{2.16}$$

with $0 \leq \sigma \leq 1$.

PROOF. The proof will be inferred from the following lemma.

LEMMA 2.1. *Let $1 \geq \eta_{1,n} \geq \cdots \geq \eta_{n,n} \geq -1$, $n \geq 1$, be an triangular array of numbers satisfying*

$$\sum_{i=1}^{n} \eta_{i,n} = 0, \tag{2.17}$$

$$\sum_{i=1}^{n} \eta_{i,n}^2 = 1, \tag{2.18}$$

$$\eta_{i,n} \to a_i, \;\; \eta_{n+1-i,n} \to -b_i \qquad \textit{as } n \to \infty, \tag{2.19}$$

where $1 \geq a_1 \geq a_2 \geq \cdots$, $1 \geq b_1 \geq b_2 \geq \cdots$, *are nonnegative constants satisfying for some* $0 \leq \sigma \leq 1$,

$$\sum_{i=1}^{\infty} a_i^2 + \sum_{i=1}^{\infty} b_i^2 = 1 - \sigma^2. \tag{2.20}$$

Now let $\{m_n\}_{n\geq 1}$ *be a sequence of positive integers such that for some* $0 < \lambda < \infty$,

$$m_n/n \to \lambda. \tag{2.21}$$

Define for each $n \geq 2$, *i.i.d. random variables* $X_{i,n}$, $i = 1.\dots, m_n$, *such that*

$$P\{X_{1,n} \leq x\} = \frac{1}{n}\#\{i : \eta_{i,n} \leq x\}, \qquad -\infty < x < \infty. \tag{2.22}$$

Then as $n \to \infty$,

$$\mathscr{L}\left\{\left(\sum_{i=1}^{m_n} X_{i,n}, \sum_{i=1}^{m_n} X_{i,n}^2\right)\right\} \to \mathscr{L}\left\{\left(W + \sigma\sqrt{\lambda}Z, S + \lambda\sigma^2\right)\right\}, \tag{2.23}$$

where $(W + \sigma\sqrt{\lambda}Z, S + \lambda\sigma^2)$ *is as in* (2.15).

PROOF. Set for $n \geq 1$,

$$(S_{n,1}, S_{n,2}) = \left(\sum_{i=1}^{m_n} X_{i,n}, \sum_{i=1}^{m_n} X_{i,n}^2\right).$$

Notice that $EX_{1,n} = 0$ and $Var X_{1,n} = 1$. Furthermore, the sequence $\{(S_{n,1}, S_{n,2})\}_{n\geq 1}$ is infinitesimal because by Markov's inequality,

$$\max_{1\leq i\leq m_n} P\{|X_{i,n}| > \delta\} = P\{X_{1,n}^2 > \delta^2\} = n^{-1}\sum_{i=1}^{n} I(\eta_{i,n}^2 > \delta^2) \leq \frac{1}{n\delta^2} \to 0.$$

Let μ denote the Lévy measure that places mass

$$\mu(B) = \lambda \sum_{i=1}^{\infty}\left\{\delta_{(a_i, a_i^2)}(B) + \delta_{(-b_i, b_i^2)}(B)\right\}$$

on measurable subsets B of $\mathbf{R}^2$, where δ_c denotes the point measure that places mass 1 at c. The measure μ is, in fact, a Lévy measure, since

$$\int_{\mathbf{R}^2} \min\{1, |x|^2\} d\backslash\mu(x) = \int_{\mathbf{R}^2} |x|^2 d\mu(x) < \infty,$$

with the choice of norm $|x| = \max\{|x_1|, |x_2|\}$.

Now in the notation of Araujo and Giné [1],

$$\mathscr{L}\left\{\left(\sum_{i=1}^{\infty} a_i(N_i - \lambda) - \sum_{i=1}^{\infty} b_i(N_i' - \lambda), \sum_{i=1}^{\infty} a_i^2(N_i - \lambda) + \sum_{i=1}^{\infty} b_i^2(N_i' - \lambda)\right)\right\}$$
$$=: cPois\ \mu,$$

and thus

$$\mathscr{L}\left\{\left(W+\sigma\sqrt{\lambda}Z,\, S+\lambda\sigma^2\right)\right\} = \delta_a * N(0,A) * cPois\ \mu,$$

where $a=(0,\lambda)$ and

$$A=\begin{pmatrix}\lambda\sigma^2 & 0\\ 0 & 0\end{pmatrix}.$$

See [1] for further explanation of the notation. So (2.23) can be restated as

$$\mathscr{L}\left\{\left(\sum_{i=1}^{m_n} X_{i,n},\, \sum_{i=1}^{m_n} X_{i,n}^2\right)\right\} \to \delta_a * N(0,A) * cPois\ \mu. \tag{2.24}$$

We shall show (2.24) by an application of the central limit theorem in $\mathbf{R}^2$ as, for example, in exercise 9(e) in [1], pages 67–68. [Note that the signs of c_n and c should be changed in part (d) of that exercise.] Toward this end, set for $i=1,\dots,m_n$ and any $\delta>0$, $X_{i,n}(\delta)=X_{i,n}$, if $|X_{i,n}|\le\delta$, and $=0$ otherwise. (We shall use the choice of $\alpha_n=0$ in our application of the result stated there.) To do this we must verify the following steps:

Step 1 : *Convergence of the shift* [*condition* (i) *of exercise* 9(*e*)]. Since the $X_{i,n}$ are bound by 1,

$$\begin{aligned} a_n &:= E\left\{\left(\sum_{i=1}^{m_n} X_{i,n}(1),\, \sum_{i=1}^{m_n} X_{i,n}^2(1)\right)\right\}\\ &= m_n E\left\{\left(X_{1,n},\, X_{1,n}^2\right)\right\} = \left(0,\, \frac{m_n}{n}\right) \to (0,\lambda)=a. \end{aligned}$$

Step 2 : *Convergence of the covariance* [*condition* (ii) *of exercise* 9(*e*)]. Since $EX_{1,n}=0$ and $|X_{1,n}|\le 1$, for all $\delta>0$,

$$\begin{aligned} \lim_{\delta\to0}\limsup_{n\to\infty} m_n|EX_{1,n}(\delta)|^2 &\le \lim_{\delta\to0}\limsup_{n\to\infty} m_n\left|P\left\{|X_{1,n}|>\delta\right\}\right|^2\\ &\le \lim_{\delta\to0}\limsup_{n\to\infty} m_n\left(\frac{1}{n\delta^2}\right)^2 = 0. \end{aligned}$$

Also,

$$\lim_{\delta\to0}\limsup_{n\to\infty} m_n|EX_{1,n}^3(\delta)| = \lim_{\delta\to0}\delta\ \limsup_{n\to\infty}\frac{m_n}{n}\sum_{i=1}^{n}\eta_{i,n}^2 I_{|\eta_{i,n}|\le\delta} \le \lambda\lim_{\delta\to0}\delta = 0,$$

$$\lim_{\delta\to0}\limsup_{n\to\infty} m_n EX_{1,n}^4(\delta) = \lim_{\delta\to0}\delta^2\limsup_{n\to\infty}\frac{m_n}{n}\sum_{i=1}^{n}\eta_{i,n}^2 I_{|\eta_{i,n}|\le\delta} \le \lambda\lim_{\delta\to0}\delta^2 = 0$$

and by (2.20),

$$\begin{aligned} \lambda\sigma^2 &= \lim_{\delta\to0}\limsup_{n\to\infty} m_n EX_{1,n}^2(\delta) = \lim_{\delta\to0}\limsup_{n\to\infty}\frac{m_n}{n}\sum_{i=1}^{n}\eta_{i,n}^2 I_{|\eta_{i,n}|\le\delta}\\ &= \lim_{\delta\to0}\liminf_{n\to\infty}\ m_n EX_{1,n}^2(\delta) = \lim_{\delta\to0}\liminf_{n\to\infty}\frac{m_n}{n}\sum_{i=1}^{n}\eta_{i,n}^2 I_{|\eta_{i,n}|\le\delta}. \end{aligned}$$

Therefore it follows that the covariance matrix of $\sum_{i=1}^{m_n}(X_{i,n}(\delta), X_{i,n}^2(\delta))$ converges to A as $n \to \infty$ and $\delta \searrow 0$.

Step 3 : *Convergence to* μ [*condition* (iii) *of exercise* 9(*e*)]. Since for all $n \geq 1$, $\mathscr{L}\left\{(X_{1,n}, X_{1,n}^2)\right\} = n^{-1}\sum_{i=1}^{n}\delta_{(\eta_{i,n},\eta_{i,n}^2)}$, we see from (2.19) and (2.21) that

$$m_n\mathscr{L}\left\{(X_{1,n}, X_{1,n}^2)\right\}|\{|x| > \delta\} \to_w \mu|\{|x| > \delta\}$$

for all but a countable number of $\delta > 0$.

This completes the proof of Lemma 2.1. $\square$

Denote the order statistics of

$$\frac{(X_i - \overline{X}_n)}{v_n} \qquad \text{for } 1 \leq i \leq n, \tag{2.25}$$

by

$$\eta_{1,n} \geq \cdots \geq \eta_{n,n}. \tag{2.26}$$

Clearly, with probability 1, for every subsequence $\{n_i'\}_{i\geq 1}$ of $\{n\}_{n\geq 1}$ there is a further subsequence $\{n_l\}_{l\geq 1}$ of $\{n_i'\}_{i\geq 1}$ such that for nonnegative constants $1 \geq a_1 \geq a_2 \geq \cdots$, $1 \geq b_1 \geq b_2 \geq \cdots$, some $\lambda_1 \leq \lambda \leq \lambda_2$ and $0 \leq \sigma \leq 1$, we have for each fixed $i \geq 1$,

$$\eta_{i,n_l} \to a_i,\ \eta_{n_l+1-i,n_l} \to -b_i, \text{ as } l \to \infty, \tag{2.27}$$

$$m_{n_l}/n_l \to \lambda, \text{ as } l \to \infty \tag{2.28}$$

and

$$\sum_{i=1}^{\infty} a_i^2 + \sum_{i=1}^{\infty} b_i^2 = 1 - \sigma^2. \tag{2.29}$$

Obviously in the notation of (2.22), conditioned on $\eta_{1,n}, \ldots, \eta_{n,n}$ fixed,

$$\mathscr{L}\left\{\left(\sum_{i=1}^{m_n} X_{i,n}, \sum_{i=1}^{m_n} X_{i,n}^2\right)\right\} = \mathscr{L}_n^*\{(W_n, S_n)\}.$$

Now since Lemma 2.1 obviously holds along subsequences, we can apply it to conclude that along the subsequence $\{n_l\}_{l\geq 1}$

$$\mathscr{L}_{n_l}^*\{(W_{n_l}, S_{n_l})\} \to \mathscr{L}\left\{\left(W + \sigma\sqrt{\lambda}Z, S + \lambda\sigma^2\right)\right\}, \tag{2.30}$$

which is clearly nondegenerate. This completes the proof of Proposition 2.1. $\square$

REMARK 2.1. The present proof of Proposition 2.1 was suggested to the authors by Evarist Giné. It replaces a somewhat longer direct proof.

PROPOSITION 2.2. *Let W, S and Z be as in Proposition* 2.1. *Whenever $0 \leq \sigma < 1$, for no choice of $0 \leq \tau < \infty$ does the possibly extended real valued random variable*

$$T := \frac{W + \tau Z}{\sqrt{S + \tau^2}}, \tag{2.31}$$

have a standard normal distribution.

First we require several lemmas.

LEMMA 2.2. *Let Y be a Poisson random variable with mean $\lambda > 0$. Then*

$$E\big((Y-\lambda)e^{cY}\big) = (\lambda e^c - \lambda)\exp\big(\lambda e^c - \lambda\big), \tag{2.32}$$

$$E\big((Y-\lambda)^2 e^{cY}\big) = \big((\lambda e^c - \lambda)^2 + \lambda e^c\big)\exp\big(\lambda e^c - \lambda\big) \tag{2.33}$$

for any real number c.

PROOF. The proof is obvious by computation. $\square$

LEMMA 2.3.

$$ET^2 = 1 \tag{2.34}$$

if and only if

$$a_i = b_i \qquad \textit{for all } i = 1, 2, \ldots, \tag{2.35}$$

where we assume that an infinite number of a_i or b_i are not zero in the case $\tau = 0$.

PROOF. Set for $i = 1, 2, \ldots,$

$$c_{2i-1} = a_i, \quad c_{2i} = -b_i, \quad Y_{2i-1} = N_i \quad \text{and } Y_{2i} = N_i',$$

and write

$$W = \sum_{i=1}^{\infty} c_i(Y_i - \lambda) \quad \text{and} \quad S = \sum_{i=1}^{\infty} c_i^2 Y_i.$$

Using the identity holding for all $\gamma > 0$,

$$\int_0^{\infty} e^{-t\gamma}\,dt = \gamma^{-1},$$

we have

$$
\begin{aligned}
ET^2 &= \int_0^\infty E\left\{(W+\tau Z)^2 e^{-t(S+\tau^2)}\right\} dt \\
&= \int_0^\infty E\left\{(\tau^2+W^2) e^{-t(S+\tau^2)}\right\} dt \\
&= \int_0^\infty E\left\{\left[\tau^2 + \sum_{i=1}^\infty c_i^2 (Y_i-\lambda)^2\right.\right. \\
&\qquad\qquad \left.\left. + \sum_{1\le i\ne j<\infty} c_i c_j (Y_i-\lambda)(Y_j-\lambda)\right] \exp\left(-t\tau^2 - t\sum_{l=1}^\infty c_l^2 Y_l\right)\right\} dt \\
&= \int_0^\infty \exp\left(-t\tau^2 + \lambda \sum_{l=1}^\infty (e^{-tc_l^2}-1)\right)\left\{\tau^2 + \sum_{i=1}^\infty c_i^2 \{\lambda e^{-tc_i^2} + (\lambda e^{-tc_i^2}-\lambda)^2\}\right. \\
&\qquad\qquad \left. + \sum_{1\le i\ne j<\infty} c_i c_j (\lambda e^{-tc_i^2}-\lambda)(\lambda e^{-tc_j^2}-\lambda)\right\} dt \\
&= \int_0^\infty \exp\left(-t\tau^2 + \lambda \sum_{l=1}^\infty (e^{-tc_l^2}-1)\right)\left\{\tau^2 + \lambda \sum_{i=1}^\infty c_i^2 e^{-tc_i^2}\right\} dt \\
&\quad + \int_0^\infty \exp\left(-t\tau^2 + \lambda \sum_{l=1}^\infty (e^{-tc_l^2}-1)\right)\left\{\lambda^2 \sum_{i=1}^\infty c_i^2 (e^{-tc_i^2}-1)^2\right. \\
&\qquad\qquad \left. + \lambda^2 \sum_{1\le i\ne j<\infty} c_i c_j (e^{-tc_i^2}-1)(e^{-tc_j^2}-1)\right\} dt \\
&= 1 + \lambda^2 \int_0^\infty \exp\left(-t\tau^2 + \lambda \sum_{l=1}^\infty (e^{-tc_l^2}-1)\right)\left\{\sum_{i=1}^\infty c_i (e^{-tc_i^2}-1)\right\}^2 dt\,.
\end{aligned}
$$

Thus we see that (2.34) holds if and only if

$$
\left\{\sum_{i=1}^\infty c_i (e^{-tc_i^2}-1)\right\}^2 = 0 \qquad \text{for all } t\ge 0. \tag{2.36}
$$

It is easy to see that (2.35) is a sufficient condition for (2.36). Now if (2.36) holds, we have

$$
\sum_{i=1}^\infty a_i (1-e^{-ta_i^2}) = \sum_{i=1}^\infty b_i (1-e^{-tb_i^2})
$$

for all $t \ge 0$, which is equivalent to

$$
\sum_{i=1}^\infty a_i^{2k+1} = \sum_{i=1}^\infty b_i^{2k+1} \qquad \text{for } k = 1, 2, \ldots. \tag{2.37}
$$

This implies that

$$a_1 = \max_i(a_i) = \lim_{k\to\infty}\left(\sum_{i=1}^{\infty} a_i^{2k+1}\right)^{1/(2k+1)}$$
$$= \lim_{k\to\infty}\left(\sum_{i=1}^{\infty} b_i^{2k+1}\right)^{1/(2k+1)} = b_1,$$

which together with (2.37) yields

$$\sum_{i=2}^{\infty} a_i^{2k+1} = \sum_{i=2}^{\infty} b_i^{2k+1} \qquad \text{for } k = 1, 2, \dots. \tag{2.38}$$

Therefore, by recurrence, (2.35) holds. □

LEMMA 2.4. *Let $N_1, N_1', N_2, N_2', \dots,$ be a sequence of i.i.d. Poisson random variables with mean λ, let Z be a standard normal random variable independent of this sequence, and let $\{a_i,\ i \ge 1\}$ be a sequence of real numbers with $0 < \sum_{i=1}^{\infty} a_i^2 < \infty$. Put*

$$W = \sum_{i=1}^{\infty} a_i(N_i - N_i'), \quad S = \sum_{i=1}^{\infty} a_i^2(N_i + N_i')$$

and

$$T = \frac{W + \tau Z}{\sqrt{S + \tau^2}},$$

where $0 \le \tau < 1$ and we assume that an infinite number of the a_i are not zero in the case $\tau = 0$. Then

$$ET^4 < EZ^4 = 3. \tag{2.39}$$

PROOF. Let E' denote the conditional expectation given the sequence $\{N_i + N_i'\}_{i\ge1}$. Notice that the conditional distribution of each $N_i - N_i'$ given $N_i + N_i' = m_i$ is that of

$$\sum_{k=1}^{m_i} s_k,$$

where $s_1, \dots, s_{m_i}$ are i.i.d. random variables with $P\{s_1 = 1\} = P\{s_1 = -1\} = 1/2$. Thus

$$E'\left[N_i - N_i'\right]^2 = m_i$$

and

$$E'\left[N_i - N_i'\right]^4 = 3m_i^2 - 2m_i.$$

One finds then that

$$E'W = 0, \quad E'\left[W^2\right] = S, \quad E'\left[W^3\right] = 0$$

and

$$E'\left[W^4\right] = 3S^2 - 2\sum_{i=1}^{\infty} a_i^4 \left(N_i + N_i'\right).$$

Thus

$$\begin{aligned} E'\left[(W+\tau Z)^4\right] &= E'\left[W^4\right] + 6\tau^2 S + 3\tau^4 \\ &= 3(S+\tau^2)^2 - 2\sum_{i=1}^{\infty} a_i^4 \left(N_i + N_i'\right). \end{aligned}$$

Hence

$$\begin{aligned} ET^4 &= E\left\{E'\left[T^4\right]\right\} \\ &= 3 - 2\sum_{i=1}^{\infty} a_i^4 E\left\{\frac{N_i + N_i'}{(S+\tau^2)^2}\right\} < 3. \end{aligned}$$ □

PROOF OF PROPOSITION 2.2. First consider the case when $\tau = 0$. In this case

$$T = \frac{W}{\sqrt{S}},$$

which, if it to be a standard normal random variable, it must be formed by a_i and b_i for which an infinite a_i or b_i are not zero: otherwise the event $S = 0$ would have a positive probability (recall we define $0/0 := 0$), which says T cannot be normal. Also, if T is to be standard normal we must have $ET^2 = 1$, which by Lemma 2.2 forces $a_i = b_i$ for all $i \geq 1$. However, when this happens, by Lemma 2.3 we have $ET^4 < 3 = EZ^4$, which again implies that T cannot be standard normal.

Nearly the same argument shows that whenever $\tau > 0$, T cannot be standard normal. This completes the proof of Proposition 2.2. □

We now have all the tools necessary to complete Theorem 2.1. Applying Proposition 2.1, we get that whenever along some subsequence $\{n'\}$ of $\{n\}$,

$$\mathscr{L}_{n'}^{*}\left(K_{n',m_{n'}}^{*}\right) \to \mathscr{L}(T), \tag{2.40}$$

for some random variable T, then T is necessarily of the form

$$T = \frac{W + \sigma\sqrt{\lambda}Z}{\sqrt{S + \lambda\sigma^2}}, \tag{2.41}$$

where $W + \sigma\sqrt{\lambda}Z$ and $S + \lambda\sigma^2$ are as in (2.15), (2.16) and (2.16). But by Proposition 2.2, if T is to be standard normal, we must have $\sigma = 1$, which forces (2.11) to hold. Thus by the arguments indicated in (2.8), (2.9), (2.10) through (2.11) above,

$$\mathscr{L}_{n}^{*}\left(K_{n,m_n}^{*}\right) \to_P \mathscr{L}(Z) \quad \text{implies } X \in DN.$$

To go the opposite direction,

$$X \in DN \quad \text{implies } \mathscr{L}_n^*\left(K_{n,m_n}^*\right) \to_P \mathscr{L}(Z), \tag{2.42}$$

note that from (2.9) we have with probability 1 for every subsequence $\{n_i'\}_{i\geq 1}$ of $\{n\}_{n\geq 1}$ there is a subsequence $\{n_l\}_{l\geq 1}$ of $\{n_i'\}_{i\geq 1}$ such that

$$\lim_{l\to\infty} \max_{1\leq i\leq n_l} \frac{\left(X_i - \overline{X}_{n_l}\right)^2}{\sum_{i=1}^n \{X_i - \overline{X}_{n_l}\}^2} = 0,$$

which in combination with the proof Proposition 2.1 forces all the a_i and b_i to be zero, which implies that (2.42) holds.

We now turn to the proof of the second part of Theorem 2.1. First assume that

$$\mathscr{L}_n^*\left(K_{n,m_n}^*\right) \to \mathscr{L}(Z) \qquad \text{as } n \to \infty, \tag{2.43}$$

but

$$\limsup_{n\to\infty} \max_{1\leq i\leq n} \frac{\left(X_i - \overline{X}_n\right)^2}{\sum_{i=1}^n \{X_i - \overline{X}_n\}^2} = c > 0 \qquad \text{a.s.} \tag{2.44}$$

Then we can use the proof of Proposition 2.1 to show that, with probability 1, along a subsequence $\{n'\}$,

$$\mathscr{L}_{n'}^*\left(K_{n',m_{n'}}^*\right) \to \mathscr{L}(Y), \tag{2.45}$$

where Y is a random variable, which by Proposition 2.2 is not standard normal. Thus, with probability 1,

$$\max_{1\leq i\leq n} \frac{\left(X_i - \overline{X}_n\right)^2}{\sum_{i=1}^n \{X_i - \overline{X}_n\}^2} \to 0 \qquad \text{as } n \to \infty. \tag{2.46}$$

must hold, which is equivalent to $EX^2 < \infty$ (see [12]).

Now assume that $EX^2 < \infty$, but (2.43) is not satisfied. Then with probability 1 we can find a subsequence $\{n_i'\}_{i\geq 1}$ of $\{n\}_{n\geq 1}$ such that (2.45) holds, where Y is a not a standard normal, which by Proposition 2.1 is necessarily of the form (2.41). Furthermore, by Propositions 2.2 this forces a_1 or b_1 to be nonzero, which by the proof of Proposition 2.1 and (2.46) cannot be the case. Thus we must have (2.43). This completes the proof of Theorem 2.1. □

REMARK 2.2. The use of the O'Brien [13] result (1.13) could have been avoided in the proof of Theorem 2.1 by showing via Propositions 2.1 and 2.2 that Theorem 2.1 is equivalent to the S. Csörgő and Mason [6] result (1.8). On the other hand, we should point out that Proposition 2.1, which is essential to the proof of Theorem 2.1, can also be used to in combination with (1.13) to establish (1.8). Moreover, our present proof based upon (1.13) is self-contained.

3. Supplementary results and remarks. Proposition 2.1, in combination with the equivalence of (2.3) and (2.4), says that whenever along a subsequence $\{n'\}$,

$$\mathscr{L}^*_{n'}\left(T^*_{n',n'}\right) \to \mathscr{L}(Y), \tag{3.1}$$

where Y is a nondegenerate random variable, then necessarily Y is of the form (2.41). Also from Proposition 2.1 we get that whenever

$$\mathscr{L}^*_n\left(\frac{\sqrt{m_n}\left\{\overline{X}^*_{n,m_n} - \overline{X}_n\right\}}{s_n}\right) \to \mathscr{L}(Y), \tag{3.2}$$

along a subsequence $\{n'\}$, where Y is a nondegenerate random variable, then necessarily Y is of the form $W + \sigma\sqrt{\lambda}Z$ as given in (2.15). From this it is easy to prove the S. Csörgő and Mason [6] result (2.15) using the O'Brien [13] result (1.13) and the fact $W + \sigma\sqrt{\lambda}Z$, being infinitely divisible, is standard normal if and only if $W = 0$ and $\sigma\sqrt{\lambda} = 1$.

The question naturally arises as what nondegenerate random variables Y are possible for which along the whole sequence $\{n\}$, one has

$$\mathscr{L}^*_n\left(T^*_{n,n}\right) \to_P \mathscr{L}(Y). \tag{3.3}$$

Hall [11] established the following result closely related to this question:

THEOREM (Hall [11]). *There exist measurable functions A_n and B_n of $X_1, \dots, X_n$ such that*

$$\mathscr{L}^*_n\left(\left\{\overline{X}^*_{n,n} - B_n\right\}/A_n\right) \to_P \mathscr{L}(Y)$$

where Y is a nondegenerate random variable if and only if

$$\begin{gathered} 1 - F \text{ is slowly varying at } \infty \text{ and} \\ P\{X < -x\}/P\{|X| > x\} \to 0 \qquad \text{as } x \to \infty, \end{gathered} \tag{3.4}$$

or

$$\begin{gathered} F \text{ is slowly varying at } -\infty \text{ and} \\ P\{X > x\}/P\{|X| > x\} \to 0 \qquad \text{as } x \to \infty, \end{gathered} \tag{3.5}$$

or

$$X \in DN. \tag{3.6}$$

Moreover, A_n and B_n can be chosen so that in situation (3.4), *Y is Poisson with mean* 1; *in situation* (3.5), *$-Y$ is Poisson with mean* 1; *and in situation* (3.6), *Y is standard normal.*

In particular, from Hall's theorem one can show that the only limiting random variables, which are possible in (3.2) along the entire sequence $\{n\}$, are Z, Y or $-Y$, where Y is Poisson with mean 1, and each of these cases is achieved. However, whenever (3.4) or (3.5) holds, (3.3) cannot be satisfied. To see this, note, for instance, that if we are in situation (3.4), then necessarily

$$\eta_{1,n} = \max_{1\le i\le n} \frac{X_i - \overline{X}_n}{\sqrt{\sum_{i=1}^n \{X_i - \overline{X}_n\}^2}} \to_P 1,$$

so that $\eta_{i+1,n} \to_P 0$ and $\eta_{n+1-i,n} \to_P 0$ for $i \ge 1$. (See, e.g., Section 4.5 of [8].) Then proceeding as in the proof of Proposition 2.1 we get that

$$\mathscr{L}_n^*\{(W_n, S_n)\} \to_P \mathscr{L}\{(N_1 - 1, N_1)\}.$$

But since $(N_1 - 1)/\sqrt{N_1} = -\infty$, with probability e^{-1}, we conclude that (3.3) cannot hold for any real valued random variable Y. In view of this and Hall's result, we conjecture the following:

THEOREM (Conjectured).

$$\mathscr{L}_n^*\left(T_{n,n}^*\right) \to_P \mathscr{L}(Y),$$

where Y is a nondegenerate random variable if and only if $X \in DN$, in which case Y is a standard normal random variable.

Crucial to treating this conjecture is to resolve the question of the uniqueness of the representation of the random variable appearing in (2.41) in terms of the parameters λ, $1 \ge a_1 \ge a_2 \ge \cdots$, $1 \ge b_1 \ge b_2 \ge \cdots$ and $0 \le \sigma \le 1$.

Another avenue of further investigation is to study the asymptotic distribution of the bootstrapped Student t-statistic when the bootstrap samples are taken at the rate

$$m_n \to \infty \text{ such } m_n/n \to 0 \qquad \text{as } n \to \infty. \tag{3.7}$$

Here is a result in this direction that can be easily inferred from Theorem 4.1 of Arcones and Giné [3] (see also their remark on the bottom of page 593).

THEOREM (Arcones and Giné [3]). *Whenever* (3.7) *is satisfied and F is in the domain of attraction of a stable law of index $1 < \alpha \le 2$, both*

$$\mathscr{L}_n^*\left(T_{n,m_n}^*\right) \to_P \mathscr{L}(Y) \quad \textit{and} \quad \mathscr{L}_n(T_n) \to \mathscr{L}(Y), \tag{3.8}$$

where Y is a nondegenerate random variable, depending on α, among other parameters.

For closely related work on the asymptotic distribution of the mean when sampling at the rate (3.7) consult Athreya [4], Arcones and Giné [2], [3] and Deheuvels, Mason and Shorack [7].

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