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LARGE DEVIATIONS UPPER BOUNDS FOR THE LAWS OF MATRIX-VALUED PROCESSES AND NON-COMMUNICATIVE ENTROPIES

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Using Itô's calculus, we study the large deviations properties of the law of the spectral measure of the Hermitian Brownian motion. We extend this strategy to the symmetric, unitary and Wishart processes. This dynamical approach is generalized to the study of the large deviations of the non-commutative laws of several independent Hermitian Brownian motions. As a consequence, we can bound from above entropies defined in the spirit of the microstates entropy introduced by Voiculescu.

1. Introduction. In this paper, we study the large deviations properties of non-commutative laws of large random matrices and related non-commutative entropies. In [2], the authors have studied the large deviations of the spectral measure of Wigner's matrices for the weak topology. They proved that the related rate function can be written as the sum of the non-commutative entropy defined by D. Voiculescu [27] and a Gaussian term. Similar results have also been proved for Wishart matrices in [20] and for the circular law in [3] and [21]. Since the spectral measure of a matrix describes its non-commutative law, it is natural to wish to generalize this result and seek for large deviation properties for the non-commutative law of several independent Wigner's matrices and to relate their rate functions with the free entropies defined by D. Voiculescu in [28] and [29]. In this direction, we shall precise what we mean by the large deviations for a non-commutative law and, in particular, what topology is involved. We shall describe these large deviations as large deviations of order II to underline the fact that they will imply large deviations statements (for the standard weak topology) for the spectral measure of any reasonable matrix-valued function of several independent Wigner's matrices.

Our approach is different from that of [2], [3] and [20] since it will not rely on the explicit knowledge of the law of the spectral measure. Indeed, this strategy is meaningless when we will consider the empirical law of m matrices, $m \geq 2$, where there is no spectral measure, except possibly for marginals. Our approach will be based on the study of matrix-valued processes with Brownian motion entries which we shall analyze via Itô's calculus and shall prove to be Markov processes with well-defined generators. Consequently, we can use the techniques developed in the field of hydrodynamics (see [23] for instance) to obtain large deviations bounds for the empirical processes. We will apply this

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strategy to study the Hermitian Brownian motion, the symmetric Brownian motion, the Wishart process, the Unitary Brownian motion and several independent Hermitian Brownian motions. It is then standard by the contraction principle to deduce large deviation estimates for the time marginals of these processes. In particular, since a Wigner's matrix can be seen as a Hermitian Brownian motion at time 1, we shall obtain large deviations estimates for the non-commutative law of several independent Wigner's matrices. We will compare the rate function governing these large deviations with the free entropy defined by D. Voiculescu via Free Fisher's information in [29]; it appears to be smaller or equal.

The ideas developed in hydrodynamics are very powerful to prove large deviations upper bounds, but more sophisticated to apply to obtain the corresponding large deviations lower bounds. This is why, except in the case of the Hermitian Brownian motion, we will only provide large deviations upper bounds. In the case of the Hermitian Brownian motion, we will prove a large deviation lower bound with a rate possibly greater than the rate we get for the upper bound; hence our result may not be sharp. However, it is proved in [8] that the rate function we get by contraction for the large deviation upper bound of one time marginal of the Hermitian Brownian motion is optimal since it agrees with the one found in [2] where a full large deviation principle was proven.

Let us describe more precisely in this introduction the general strategy we shall follow to obtain large deviation results for the Hermitian Brownian motion, and therefore for the Gaussian Wigner matrix. We shall then briefly state their generalization to several independent Hermitian Brownian motions.

The Hermitian Brownian motion is described on the space $\mathscr{H}_N$ of Hermitian matrices of dimension N as the Markov process $(H_N(t))_{t\in\mathbb{R}^+}$ with values in $\mathscr{H}_N$ and complex Brownian motions entries so that

$$E[H_N^{i,j}(t)H_N^{k,l}(s)] = \frac{t\wedge s}{N}\delta_i^l\delta_k^j.$$

More precisely, we can construct the entries $\{H_N^{i,j}(t), t\geq 0, (i,j)\in\{1,\dots,N\}\}$ via independent real valued Brownian motions $(\beta_{i,j},\beta'_{k,l})_{1\leq i\leq j\leq N}^{1\leq k<l\leq N}$ living on a probability space $(\Omega,\mathbb{P})$ by

$$\begin{aligned} H_N^{k,l} &= \frac{1}{\sqrt{2N}}(\beta_{k,l}+i\beta'_{k,l}) \quad \text{if } k<l,\\ &= \frac{1}{\sqrt{2N}}(\beta_{l,k}-i\beta'_{l,k}) \quad \text{if } k>l,\\ &= \frac{1}{\sqrt{N}}\beta_{l,l} \text{ if } k=l. \end{aligned}$$

Observe that, at any given time t, $H_N(t)$ is a Wigner's matrix of the Gaussian Unitary Ensemble (GUE). Hence, it is enough to study H_N on compact time intervals, say $[0,1]$, to study the GUE.

The idea to consider the process H_N to study its time marginals may sound at first bizarre, not to say stupid. However, the process may be studied thanks to Itô's differential calculus which will appear to be a very powerful tool here.

Indeed, let $(\lambda_i^{(N)}(t))_{1\le i\le N}$ be the (real-valued) eigenvalues of $H_N(t)$ and define the spectral empirical process by

$$\hat{\mu}^{(N)} : [0,1] \longrightarrow \mathscr{P}(\mathbb{R})$$
$$t \longrightarrow \hat{\mu}_t^{(N)} \equiv \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^{(N)}(t)}$$

where $\mathscr{P}(\mathbb{R})$ is the set of probability measures on the real line. Then, it turns out that, due to the independence between the eigenvectors and the eigenvalues of the GUE, $\hat{\mu}^{(N)}$ satisfies a stochastic differential equation with generator and martingale bracket which only depend on $\hat{\mu}^{(N)}$. More precisely, let us define, for $s \le t \in [0,1]$, $f \in \mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])$, and $\nu \in \mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$,

$$\begin{aligned} S^{s,t}(\nu, f) = {} & \int f(x,t)d\nu_t(x) - \int f(x,s)d\nu_s(x) \\ & - \int_s^t \int \partial_u f(x,u)d\nu_u(x)du \\ & - \frac{1}{2}\int_s^t \iint \frac{\partial_x f(x,u) - \partial_x f(y,u)}{x-y} d\nu_u(x)d\nu_u(y)du. \end{aligned}$$

Then, we shall prove (see Lemma 2.3) that:

LEMMA 1.1. *For any $f \in \mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])$, the process $Q_f^{(N)}$ given for $t \in [0,1]$ by*

$$Q_f^{(N)}(t) = S^{0,t}(\hat{\mu}^{(N)}, f)$$

is a martingale with bracket

$$\left\langle Q_f^{(N)} \right\rangle_t = \frac{1}{N^2}\int_0^t \hat{\mu}_s^{(N)}\big((\partial_x f)^2(x,s)\big)ds.$$

Hence, we can use the techniques developed in hydrodynamics (see [23]) which, at least for the upper bound, rely on classical bounds on exponential martingales, to prove large deviations results for the process $\hat{\mu}^{(N)}$. In fact, if we let

$$S^{0,1}(\nu) = \sup_{f\in\mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])} \left(S^{0,1}(\nu,f) - \tfrac{1}{2}\int_0^1 \int (\partial_x f(x,u))^2 d\nu_u(x)du \right),$$

we shall prove that, if $\hat{\mu}^{(N)}$ is viewed as an element of the space $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$ of continuous measure-valued processes furnished with the topology generated by the weak topology on $\mathscr{P}(\mathbb{R})$ and the uniform topology on $[0,1]$, then:

THEOREM 1.1. *The law of $\hat{\mu}^{(N)}$ satisfies a large deviation upper bound in the scale N^2 with good rate function S such that*

$$S(\nu) = \begin{cases} \infty, & \text{if } \nu_0 \neq \delta_0, \\ S^{0,1}(\nu), & \text{otherwise.} \end{cases}$$

In other words, for every closed subset $F \in \mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}\left(\hat{\mu}^{(N)} \in F\right) \leq -\inf_{\nu \in F} S(\nu).$$

Further, the law of $\hat{\mu}^{(N)}$ satisfies a large deviation lower bound in the scale N^2 which states as follows: for any open subset $O \in \mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$,

$$\liminf_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}\left(\hat{\mu}^{(N)} \in O\right) \geq -\inf_{\nu \in O \cap \mathscr{C}^\infty([0,1],\mathscr{P}(\mathbb{R}))} S(\nu)$$

where $\mathscr{C}^\infty([0,1], \mathscr{P}(\mathbb{R}))$ is the subset of $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$ of weak solutions ν such that $\nu_0 = \delta_0$ of the differential equation

$$S^{0,1}(\nu, f) = \int_0^1 \int \partial_x h_u(x) \partial_x f_u(x) d\nu_u(x) du$$

for any $f \in \mathscr{C}_b^{2,1}(\mathbb{R} \times [0,1])$, with $h \in \mathscr{M}\mathscr{F}_\infty \subset \mathscr{C}_b^{\infty,1}(\mathbb{R} \times [0,1])$ so that

$$\mathscr{M}\mathscr{F}_\infty = \Big\{ h \in \mathscr{C}_b^{\infty,1}(\mathbb{R} \times [0,1]) \cap \mathscr{C}([0,1], L^2(\mathbb{R}));$$
$$\exists C, \varepsilon \in (0,\infty); \sup_{t\in[0,1]} |\hat{h}_t(\lambda)| \leq C e^{-\varepsilon|\lambda|} \Big\}$$

where $\hat{h}_t$ stands for the Fourier transform of h_t.

Note here that the lower bound may not be sharp. However, very recent investigations of O. Zeitouni and one of the authors were able to improve into a full large deviation lower bound (see [17]).

As a consequence of Theorem 1.1, we get by the contraction principle:

COROLLARY 1.1. *The law of $\hat{\mu}_1^{(N)} \in \mathscr{P}(\mathbb{R})$ satisfies the following large deviation bounds:*

(i) *For every closed subset $F \in \mathscr{P}(\mathbb{R})$,*

$$\limsup_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}\left(\hat{\mu}_1^{(N)} \in F\right) \leq -\inf\{S(\nu), \nu_1 \in F\}.$$

(ii) *For any open subset $O \in \mathscr{P}(\mathbb{R})$,*

$$\liminf_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}\left(\hat{\mu}_1^{(N)} \in O\right) \geq -\inf\{S(\nu); \nu \in \mathscr{C}^\infty([0,1], \mathscr{P}(\mathbb{R})), \nu_1 \in O\}.$$

It is proved in [8] (see also [9]) that $I(\mu) = \inf\{S(\nu), \nu_1 = \mu\}$ is the rate function obtained in [2] for the full large deviation principle of the spectral measure of the GUE; the above upper bound is therefore sharp.

Consequently to the large deviation upper bound result of Theorem 1.1, we also obtain the exponentially fast convergence of $\hat{\mu}^{(N)}$ toward the semi-circular process.

COROLLARY 1.2. *The process $\hat{\mu}^{(N)}$ converges almost surely to the unique minimum of S which is described by the differential equation with initial data $\sigma_0 = \delta_0$,*

$$\sigma_t(f_t) - \sigma_0(f_0) = \int_0^t \sigma_s(\partial_s f_s) ds + \frac{1}{2}\int_0^t \sigma_s \otimes \sigma_s \left(\frac{\partial_x f_s(x) - \partial_x f_s(y)}{x - y}\right) ds \tag{1.1}$$

for every function $f \in \mathscr{C}_b^{2,1}(\mathbb{R} \times [0,1])$. Equivalently, σ_t is, for any $t \in [0,1]$, the semi-circular law

$$\sigma_t(dx) = \frac{1}{2\pi t}\sqrt{4t - x^2}\,1_{[-2\sqrt{t}, 2\sqrt{t}]}dx. \tag{1.2}$$

PROOF. Clearly, the law of $\hat{\mu}^{(N)}$ concentrates on the minimizers of the rate function $S^{0,1}$. Let $\nu \in \mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$ be a continuous $\mathscr{P}(\mathbb{R})$-valued process with initial data δ_0. If we denote for $(f, g) \in \mathscr{C}_b^{2,1}(\mathbb{R} \times [0,1])$, $0 \le s \le t \le 1$,

$$\ll f, g \gg_\nu^{s,t} \equiv \int_s^t \int \partial_x f(x,u) \partial_x g(x,u) d\nu_u(x) du,$$

$$\begin{aligned} S^{0,1}(\nu) &= \sup_{f \in \mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])} \left(S^{0,1}(\nu, f) - \frac{1}{2} \ll f, f \gg_\nu^{0,1} \right) \\ &= \sup_{f \in \mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])} \sup_{\lambda \in \mathbb{R}} \left(S^{0,1}(\nu, \lambda f) - \frac{1}{2} \ll \lambda f, \lambda f \gg_\nu^{0,1} \right) \\ &= \sup_{f \in \mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])} \frac{1}{2} \frac{(S^{0,1}(\nu, f))^2}{\ll f, f \gg_\nu^{0,1}}. \end{aligned} \tag{1.3}$$

In particular, $S^{0,1}(\nu)$ is null iff $S^{0,1}(\nu, f)$ is null for every $f \in \mathscr{C}_b^{2,1}(\mathbb{R} \times [0,1])$. In other words, the minimizers of $S^{0,1}$ are characterized by (1.1). It is not difficult to check that the semi-circular process defined by (1.2) satisfies this equation. Hence, the only point is to show that equation (1.1) admits a unique solution (existence is already a consequence of the convergence of $\hat{\mu}^{(N)}$ toward such solutions). This problem was solved by T. Chan for solutions with all their moments and in [7] when an extra diffusive term is added (i.e., when one considers Burger's equation). In fact, this equation is completely solvable as proved in [25]. An alternative proof of the uniqueness result is presented in Lemma 2.6. □

Turning to the study of large deviations for several Hermitian Brownian motions, we shall here illustrate our main theorem by describing two of its applications which hopefully will motivate the reader. More precise definitions and statements are given in section 4. For $N \in \mathbb{N}$, we shall denote $\mathscr{M}_N$ the space of $N \times N$ complex matrices and tr_N the normalized trace in $\mathscr{M}_N$ given for any $A \in \mathscr{M}_N$ by $\mathrm{tr}_N(A) = (1/N)\sum_{i=1}^N A_{ii}$. Let $m \in \mathbb{N}$ and $(H_N^k)_{1\le k\le m}$ be independent copies of H_N. To define functions of m Hermitian matrices $(X_k)_{1\le k\le m} \in \mathscr{H}_N^m$, $N \in \mathbb{N}$, recall that to every complex valued function f on $\mathbb{R}$, we can associate a function F_f from $\mathscr{H}_N$ into $\mathscr{M}_N$, $N \in \mathbb{N}$, so that, if $A \in \mathscr{H}_N$ is decomposed as $A = U^*DU$ for a diagonal matrix D and a unitary matrix U, $F_f(A) = U^*f(D)U$ where $f(D)$ stands for the diagonal matrix with entries $(f(D_{11}), \dots, f(D_{NN}))$. In particular, if $f(z) = (z-x)^{-1}$, for any $(\alpha_k)_{1\le k\le m} \in \mathbb{R}^m$ and $z \in \mathbb{C}\backslash\mathbb{R}$, we can define a function $\Phi_{z,\alpha}$ mapping $\mathscr{H}_N^m$ into $\mathscr{M}_N$ for any $N \in \mathbb{N}$, so that for any $N \in \mathbb{N}$, $(X_k)_{1\le k\le m} \in \mathscr{H}_N^m$,

$$\Phi_{z,\alpha}(X) = \left(z - \sum_{k=1}^m \alpha_k X_k\right)^{-1} = F_f\left(\sum_{k=1}^m \alpha_k X_k\right).$$

We denote by $\overrightarrow{\prod}$ the non-commutative product. Let $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ be the complex vector space generated by the functions mapping, for any $N \in \mathbb{N}$, $\mathscr{H}_N^m$ into $\mathscr{M}_N$, given by

$$ST(\mathbb{C}) = \left\{ F(X_k, 1 \le k \le m) = \overrightarrow{\prod_{1\le i\le n}} \left(z_i - \sum_{k=1}^m \alpha_i^k X_k\right)^{-1}; \right.$$
$$\left. z_i \in \mathbb{C}\backslash\mathbb{R}, \alpha_i^k \in \mathbb{R}, n \in \mathbb{N} \right\}$$

and $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$ be its restriction to the functions F mapping $\mathscr{H}_N^m$ into $\mathscr{H}_N$ for any $N \in \mathbb{N}$. We let $\overline{\mathscr{M}_1^=}$ be the subset of the topological dual of $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ with real valued restriction to $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$ satisfying some natural assumptions of boundedness, positiveness and total mass described in section 4.3. $\mathscr{C}([0,1], \overline{\mathscr{M}_1^=})$ will be the space of continuous $\overline{\mathscr{M}_1^=}$ valued processes. We shall describe a good rate function $\overline{S}$ on the space of continuous non-commutative laws-valued processes $\mathscr{C}([0,1], \overline{\mathscr{M}_1^=})$, furnished with the $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$-topology on the $\overline{\mathscr{M}_1^=}$ variable and the uniform topology on the time variable, so that:

COROLLARY 1.3. *For any integer number n, any times $(t_j)_{j=1}^n \in [0,1]^n$, any family $(F_j)_{j=1}^n \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})$, for any real constants $(a_j, b_j, a_j \le b_j)_{j=1}^n$,*

$$\limsup_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}\left(\bigcap_{j=1}^n \{\mathrm{tr}_N\left(F_j(H_N^k(t_j), 1 \le k \le m)\right) \in [a_j, b_j]\}\right)$$
$$\le -\inf\left\{\overline{S}(\nu) \ : \ \nu_{t_j}\left(F_j(X_k, 1 \le k \le m)\right) \in [a_j, b_j]\ \forall j \in \{1, \dots, n\}\right\}.$$

The $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$-topology will be seen to be compatible with the standard weak topology so that the contraction principle will give:

PROPERTY 1.1. *For any function $F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})$, the law of the spectral measure of $(F(H_N^k(t), 1 \leq k \leq m))_{t\in[0,1]} \in \mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$ satisfies a large deviation upper bound in the scale N^2 in the weak topology with good rate function*

$$\mathscr{S}_F(\nu) = \inf\{\overline{S}(\tilde{\nu}), \nu = \tilde{\nu} \circ F^{-1}\}$$

where $\tilde{\nu} \circ F^{-1}$ is the measure valued-process so that, for any function $g \in \mathscr{C}_b(\mathbb{R})$, for any time $t \in [0,1]$,

$$\tilde{\nu}_t \circ F^{-1}(g) \equiv \tilde{\nu}_t(g(F)).$$

In this sense, our result can be seen as large deviation of order II. The full statement of our theorem is given in Theorem 4.1. The comparison of our free entropy and Voiculescu's follows it. Also, our large deviations theorem will entail the convergence of the non-commutative law of $(H_N^k, 1 \leq k \leq m)$ toward the law of m free Brownian motions.

Our paper is organized as follows. In Section 2, we provide a detailed study of the large deviations results for the Hermitian Brownian motion, and also as a key step toward it, we describe its generator. In Section 3, we generalize our approach to other matrices ensembles: the symmetric Brownian motion, the Wishart process and the Unitary Brownian motion. In Section 4, we introduce a topology on the non-commutative law of several independent Hermitian Brownian motions and prove the corresponding large deviation result.

In this paper, whenever we state a large deviation result for a spectral measure valued-process, the topology under study is in the space $\mathscr{C}([0,1], \mathscr{P}(\Sigma))$ of continuous processes on $[0,1]$ endowed with the uniform topology with values in the Polish space of probability measures $\mathscr{P}(\Sigma)$ on a Polish alphabet space Σ. Let d denote the Wasserstein distance on $\mathscr{P}(\Sigma)$ given for any $(\mu, \nu) \in \mathscr{P}(\Sigma)^2$ by

$$d(\mu, \nu) = \sup \left| \int f d\mu - \int f d\nu \right| \tag{1.4}$$

where the supremum is taken over all the Lipschitz functions on Σ, equipped with the distance l, with Lipschitz constant

$$||f||_{\mathscr{L}} = \sup_{x\in\mathbb{R}} |f(x)| + \sup_{x,y\in\Sigma} \frac{|f(x) - f(y)|}{l(x,y)}$$

bounded by one. The above topology on $\mathscr{C}([0,1], \mathscr{P}(\Sigma))$ is compatible with the distance given, for any $(\mu, \nu) \in \mathscr{C}([0,1], \mathscr{P}(\Sigma))$, by

$$\mathscr{D}(\mu, \nu) = \sup_{t\in[0,1]} d(\mu_t, \nu_t). \tag{1.5}$$

We shall denote $\mathscr{C}_0(\mathbb{R})$ [resp. $\mathscr{C}_c(\mathbb{R})$] the set of continuous functions on $\mathbb{R}$ going to zero at infinity (resp. compactly supported), and, for $p \in \mathbb{N}$, $\mathscr{C}_b^{p,1}(\mathbb{R} \times [0,1])$

the set of continuously differentiable functions of the time, with values in $\mathscr{C}_b^p(\mathbb{R})$, the set of p-times bounded continuously differentiable functions on $\mathbb{R}$ with bounded space-continuous time derivatives. The non-commutative law of several matrices will be seen as a subset $\overline{\overline{\mathscr{M}_1}}$ of the topological dual $\overline{\mathscr{M}}$ of the vector space $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$. The large deviations for the non-commutative laws-valued processes will be in the space $\mathscr{C}([0,1], \overline{\overline{\mathscr{M}_1}})$ of continuous $\overline{\overline{\mathscr{M}_1}}$-valued processes. An extra tightness criterium will be added to strengthen the $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$-topology.

2. Large deviations results and entropy for the Hermitian Brownian motion. To study the Markov process $(H_N(t))_{t\in[0,1]}$ we shall first describe its generator, key step toward large deviations estimates for the measure-valued process $\hat{\mu}^{(N)}$. This can be done thanks to Itô's formula (see Lemma 1.1).

2.1. *Stochastic calculus.* To state Itô's formula, we need to introduce a few extra notations due to the absence of commutativity. In particular, we shall introduce the notion of bi-process. Namely, if we denote $\mathscr{F}_t^{(N)}$ the natural filtration of $(H_N(s), s \le t)$, a bi-process is a random process, $\mathscr{F}_t^{(N)}$-adapted, with value in $\mathscr{M}_N \otimes \mathscr{M}_N$. For instance, if we consider two $\mathscr{H}_N$-valued $\mathscr{F}_t^{(N)}$-adapted processes $(P(s))_{s\ge 0}$ and $(Q(s))_{s\ge 0}$, $(Y(s) = P(s) \otimes Q(s))_{s\ge 0}$ is a bi-process. The integral of Y against H_N is defined as the matrix

$$\int_0^t Y(s)\sharp dH_N(s) \equiv \int_0^t P(s)dH_N(s)Q(s) = \left(\sum_{k,l=1}^N \int_0^t p_{i,k}(s)q_{l,j}(s)dH_N^{k,l}(s)\right)_{1\le i,j\le N}$$

This definition extends naturally to more general bi-process as follows:

$$\int_0^t Y(s)\sharp dH_N(s) \equiv \left(\sum_{1\le k,l\le N} \int_0^t Y_{i,k,l,j}(s)dH_N^{k,l}(s)\right)_{1\le i,j\le N}.$$

Multi-dimensional Itô's calculus yield, for any adapted processes A, B, C and D,

$$\begin{aligned}
&\int_0^t (A_s\otimes B_s)\sharp dH_N(s)\int_0^t (C_s\otimes D_s)\sharp dH_N(s)\\
&\quad = \int_0^t \left(A_s \otimes \left[B_s \int_0^s (C_u\otimes D_u)\sharp dH_N(u)\right]\right)\sharp dH_N(s)\\
&\qquad + \int_0^t \left(\left[\int_0^s (A_u\otimes B_u)\sharp dH_N(u) C_s\right]\otimes D_s\right)\sharp dH_N(s)\\
&\qquad + \int_0^t A_s \mathrm{tr}_N(B_sC_s)D_s ds.
\end{aligned} \tag{2.1}$$

To write Itô's formula, we need to introduce two linear maps on the set of smooth test functions, here reduced to the set of polynomial functions, which are the analogue of classical derivation and Laplacian:

$$D_0 : \mathbb{C}[X] \longrightarrow \mathbb{C}[X] \otimes \mathbb{C}[X]$$
$$X^n \longmapsto \sum_{i=0}^{n-1} X^i \otimes X^{n-1-i}$$

$$L_0 : \mathbb{C}[X] \longrightarrow \mathbb{C}[X] \otimes \mathbb{C}[X]$$
$$X^n \longmapsto \sum_{i=1}^{n-1} i X^{i-1} \otimes X^{n-1-i}$$

REMARKS. (i) $L_0 = (\partial_x \otimes \mathrm{Id}) \circ D_0$ where ∂_x denotes the classical derivation.

(ii) If a polynomial function f is considered as a complex function, our formulae read

$$D_0 f(x, y) = \frac{f(x) - f(y)}{x - y},$$
$$L_0 f(x, y) = \frac{1}{x - y}\left(\partial_x f(x) - \frac{f(x) - f(y)}{x - y}\right) = \int_0^1 \partial_x^2 f\big(ux + (1-u)y\big) u\,du.$$

Note that

$$\lim_{y \to x} D_0 f(x, y) = \partial_x f(x) \quad \text{and} \quad \lim_{y \to x} L_0 f(x, y) = \tfrac{1}{2}\partial_x^2 f(x).$$

Denote $\mathscr{C}_b^1([0,1], \mathbb{C}[X])$ the set of polynomial functions with bounded continuously differentiable coefficients. We have:

LEMMA 2.1. *Itô's formula for H_N: for every $f \in \mathscr{C}_b^1([0,1], \mathbb{C}[X])$,*

$$\begin{aligned} f(H_N(t), t) = f(H_N(0), 0) &\int_0^t D_0 f(H_N(s), H_N(s); s) \sharp dH_N(s) \\ &+ \int_0^t \partial_s f(H_N(s), s) ds \\ &+ \int_0^t (Id_N \otimes \mathrm{tr}_N) \circ (L_0 f)(H_N(s), H_N(s); s) ds \end{aligned}$$

where $Id_N \otimes \mathrm{tr}_N$ is the linear operator on $\mathscr{M}_N \otimes \mathscr{M}_N$ so that for any $A, B \in \mathscr{M}_N$, $Id_N \otimes \mathrm{tr}_N(A \otimes B) = \mathrm{tr}_N(B)A$.

PROOF. Even though this point is a direct consequence of multidimensional Itô's formula, let us detail the computation for $f(x) = x^k$ with an integer number $k \in \mathbb{N}$. Then, for any $(i, j) \in \{1, \dots, N\}$, Itô's calculus gives

$$\begin{aligned} d(H_N(t)^k)_{ij} = &\sum_{l=0}^{k-1} \sum_{p,n=1}^{N} (H_N(t)^l)_{ip} d(H_N(t))_{pn} (H_N(t)^{k-l-1})_{nj} \\ &+ \frac{1}{N} \sum_{l+m=0}^{k-2} \sum_{p,n=1}^{N} (H_N(t)^l)_{ip} (H_N(t)^m)_{nn} (H_N(t)^{k-l-m-2})_{pj} dt \end{aligned}$$

$$= \sum_{l=0}^{k-1} \left(H_N(t)^l dH_N(t) H_N(t)^{k-l-1} \right)_{ij}$$
$$+ \sum_{l=0}^{k-2} (k-l-1) \mathrm{tr}_N (H_N(t)^l)(H_N(t)^{k-l-2})_{ij} dt$$

This also proves the first formula of Lemma 2.1 for more general polynomial functions by linearity. Let us finally compute the martingale bracket of the normalized trace of the above martingale. We have

$$\left\langle \int_0^{\cdot} \sum_{l=0}^{k-1} \mathrm{tr}_N \left(H_N(s)^l dH_N(s) H_N(s)^{k-l-1} \right) \right\rangle_t$$
$$= \frac{1}{N^2} k^2 \sum_{ij} \sum_{mn} \left\langle \int_0^{\cdot} (H_N(s)^{k-1})_{ij} d(H_N(s))_{ij}, \right.$$
$$\left. \int_0^{\cdot} (H_N(s)^{k-1})_{mn} d(H_N(s))_{mn} \right\rangle_t$$
$$= \frac{1}{N^2} k^2 \sum_{ij} \sum_{mn=ji} \left\langle \int_0^{\cdot} (H_N(s)^{k-1})_{ij} d(H_N(s))_{ij}, \right.$$
$$\left. \int_0^{\cdot} (H_N(s)^{k-1})_{mn} d(H_N(s))_{mn} \right\rangle_t$$
$$= \frac{1}{N^2} k^2 \mathrm{tr}_N \left(\int_0^t (H_N(s)^{2(k-1)}) ds \right).$$

Similar computations give the bracket of more general polynomial functions. □

We shall generalize this formula to smooth bounded functions. To this end, let us recall that, for any real-valued function f, for any $N \in \mathbb{N}$, we can define f on $\mathscr{H}_N$ by

$$f(A) = U^* f(D) U$$

if $A = U^* D U$ with a diagonal matrix D and a unitary matrix U and $f(D)$ is the diagonal matrix with entries $(f(D_{1,1}), \dots, f(D_{N,N}))$. For $f \in \mathscr{C}_b^1(\mathbb{R})$, we let, for any $N \in \mathbb{N}$ and any $A, B \in \mathscr{H}_N$, $A = UDU^*$, $B = \tilde{U}\tilde{D}\tilde{U}^*$,

$$D_0 f(A, B) = \int_0^1 U \otimes \tilde{U} f'(\alpha D \otimes (1-\alpha)\tilde{D}) U^* \otimes \tilde{U}^* d\alpha,$$

with for any $(i, j, k, l) \in \{1, \dots, N\}^4$,

$$f'(\alpha D \otimes (1-\alpha)\tilde{D})_{i,j,k,l} = \delta_{i=j} \delta_{k=l} f'(\alpha D_{i,i} + (1-\alpha)\tilde{D}_{k,k})$$

and

$$(U \otimes \tilde{U})_{i,j,k,l} = U_{i,j} \tilde{U}_{k,l}.$$

Similarly, for any $N \in \mathbb{N}$ and any $A, B \in \mathscr{H}_N$, $A = UDU^*$, $B = \tilde{U}\tilde{D}\tilde{U}^*$,

$$L_0 f(A, B) = \int_0^1 \alpha U \otimes \tilde{U} f''(\alpha D \otimes (1-\alpha)\tilde{D}) U^* \otimes \tilde{U}^* d\alpha.$$

With these natural extensions of D_0 and L_0, we have:

LEMMA 2.2. *Any $f \in \mathscr{C}_b^{2,1}(\mathbb{R} \times [0, 1])$ satisfies the conclusions of Lemma 2.1.*

PROOF. To prove Lemma 2.2, we shall first consider functions $f_\lambda(x) = e^{i\lambda x}$ for a real number x [or equivalently, $cos(\lambda x)$ or $sin(\lambda x)$]. We shall approximate f_λ by the polynomial functions

$$f_\lambda^n(x) = \sum_{k=0}^n \frac{(i\lambda x)^k}{k!}.$$

Set

$$M_{f_\lambda}(t) \equiv f_\lambda(H_N(t), t) - f_\lambda(H_N(0), 0) - \int_0^t \partial_s f_\lambda(H_N(s), s) ds$$
$$- \int_0^t (Id_N \otimes \mathrm{tr}_N) \circ (L_0 f_\lambda)(H_N(s), H_N(s); s) ds.$$

$M_{f_\lambda^n}$ converges point-wise toward M_{f_λ} since f_λ^n (and its derivatives) converges point-wise toward f_λ(and its derivatives). Let

$$\bar{M}_{f_\lambda}(t) = \int_0^t D_0 f_\lambda(H_N(s), H_N(s); s) \sharp dH_N(s).$$

To prove that the matrix valued martingale $\bar{M}_{f_\lambda^n}$ converges almost surely toward the matrix valued martingale $\bar{M}_{f_\lambda}$, it is enough to prove that for any $t \geq 0$,

$$\lim_{n\to\infty} ||\bar{M}_{f_\lambda^n}(t) - \bar{M}_{f_\lambda}(t)||_2 = 0 \tag{2.2}$$

where $||\ ||_2$ is the natural norm defined for any random matrix A in $\mathscr{H}_N$ by

$$||A||_2^2 = \mathbb{E}[\mathrm{tr}_N(AA^*)].$$

Then, since for every $n \in \mathbb{N}$, $M_{f_\lambda^n} = \bar{M}_{f_\lambda^n}$, we deduce $M_{f_\lambda} = \bar{M}_{f_\lambda}$.

But

$$\begin{aligned} &||\bar{M}_{f_\lambda^n}(t) - \bar{M}_{f_\lambda}(t)||^2 \\ &\quad = \frac{1}{N} \sum_{1\leq i,j\leq N} \mathbb{E}\left[\left(\int_0^t D_0(f_\lambda - f_\lambda^n)(H_N(s), H_N(s); s)\sharp dH_N(s)\right)_{i,j}^2\right] \\ &\quad = \mathbb{E}\left[\int_0^t \mathrm{tr}_N \otimes \mathrm{tr}_N (D_0(f_\lambda - f_\lambda^n))^2 (H_N(s), s) ds\right] \\ &\quad = \mathbb{E}\left[\int_0^t \hat{\mu}_s^{(N)} \otimes \hat{\mu}_s^{(N)}\left[\left(\int_0^1 \partial_x(f_\lambda - f_\lambda^n)(ux + (1-u)y) du\right)^2\right] ds\right] \end{aligned} \tag{2.3}$$

where the second line was obtained by use of (2.1). Moreover, observe that Taylor's formula reads

$$(f_\lambda - f_\lambda^n)(x) = \frac{(i\lambda x)^{n+1}}{n!}\int_0^1 (1-t)^n f_\lambda(tx)dt$$

so that with (2.3) we conclude

$$\begin{aligned}
&\|\bar{M}_{f_\lambda^n}(t) - \bar{M}_{f_\lambda}(t)\|_2^2 \\
&\quad \le \frac{(|\lambda|)^{2(n+1)}}{n!^2}\mathbb{E}\left[\int_0^t \hat{\mu}_s^{(N)} \otimes \hat{\mu}_s^{(N)} \left(\int_0^1 |ux + (1-u)y|^n du\right)^2\right] \\
&\quad \le \frac{(|\lambda|)^{2(n+1)}}{n!^2}\mathbb{E}\left[\int_0^t \operatorname{tr}_N((H_N(s))^{2n})ds\right] \qquad (2.4)\\
&\quad \le \lambda^2 \frac{(2n)!}{4^n(n!^2)}\mathbb{E}\left[\int_0^t \operatorname{tr}_N\left(\frac{(2\lambda)^{2n}}{2n!}(H_N(s))^{2n}\right)ds\right]
\end{aligned}$$

where the convexity of $x \to x^n$ and of $x \to x^2$ was used to get the second line. To control the above bound, let us invoke the result of [19] which says that for any integer number N there exists a polynomial function B_N so that for any $y \in \mathbb{C}$,

$$\mathbb{E}\left[\sum_{n=0}^{\infty} \frac{y^n}{n!}\operatorname{tr}_N(H_N^n(t))\right] = \exp\left(\frac{y^2 t}{2N}\right) B_N(y^2 t).$$

Hence, for any $s \in [0, 1]$, any $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left[\operatorname{tr}_N(\frac{(2\lambda)^{2n}}{2n!}(H_N(s))^{2n})\right]$$

goes to zero as n goes to infinity. It is also uniformly bounded [for instance, by $\exp(\frac{\lambda^2}{2N})\sup_{t\in[0,1]} B_N(\lambda^2 t)$]. Hence, dominated convergence theorem results with (2.2), giving the almost sure convergence of $\bar{M}_{f_\lambda^n}$. Since $M_{f_\lambda^n} = \bar{M}_{f_\lambda^n}$ converges everywhere toward M_{f_λ}, we conclude that $M_{f_\lambda} = \bar{M}_{f_\lambda}$.

Now, if f is continuously differentiable, with bounded derivative and so that:

1. For every $t \in [0, 1]$, $f(x, t) = \int_{\mathbb{R}} e^{ix\lambda}\mu_t(d\lambda)$ with a complex measure μ_t with finite mass.
2. For every $t \in [0, 1]$, $\partial_t f(x, t) = \int_{\mathbb{R}} e^{ix\lambda}\mu_t'(d\lambda)$ with a complex measure μ_t' with finite mass.
3. $\sup_{0\le s\le 1}\int_{\mathbb{R}} \lambda^2|\mu_s|(d\lambda) < \infty$ and $\sup_{0\le s\le 1}|\mu_s'|(\mathbb{R}) < \infty$.

Then we can apply Fubini's theorem to conclude that Lemma 2.2 holds for such functions.

Finally, if $f \in \mathscr{C}_b^{2,1}(\mathbb{R}, [0, 1])$, set $f_\alpha(x, t) = \frac{f(x,t)}{1+\alpha x^2}$. Then $f_\alpha(., t)$ satisfies the hypotheses above with $\mu_t^\alpha(d\lambda) = \hat{f}_\alpha(\lambda, t)d\lambda$ and $(\mu_t^\alpha)'(d\lambda) = (\widehat{\partial_t f_\alpha})(\lambda, t)d\lambda$.

In fact, it belongs to $L^2(\mathbb{R})$ and its Fourier transform belong to $L^1(\mathbb{R})$ for any $t \in [0, 1]$ so that the inversion theorem results with

$$f_\alpha(x, t) = \int_{\mathbb{R}} e^{-i\lambda x} \hat{f}_\alpha(\lambda, t) d\lambda$$

where $\hat{f}_\alpha$ is the Fourier transform of f_α. Since

$$\begin{aligned}\int_{\mathbb{R}} \lambda^2 |\hat{f}_\alpha(\lambda, t)| d\lambda \le \left(\int_{|\lambda|>1} |\lambda^2 \hat{f}_\alpha(\lambda, t)|^2 d\lambda \right)^{\frac{1}{2}} \left(\int_{|\lambda|>1} \frac{1}{|\lambda|^2} d\lambda \right)^{\frac{1}{2}} \\ + \sqrt{2} \left(\int_{[-1,1]} |\hat{f}_\alpha(\lambda, t)|^2 d\lambda \right)^{\frac{1}{2}}\end{aligned} \tag{2.5}$$

with Plancherel's theorem resulting with

$$\int_{|\lambda|>1} |\lambda^2 \hat{f}_\alpha(\lambda, t)|^2 d\lambda = \int_{|\lambda|>1} |\widehat{\partial_x^2 f}_\alpha(\lambda, t)|^2 d\lambda \le \int_{\mathbb{R}} |\partial_x^2 f(x, t)|^2 dx$$

we get

$$\int_{\mathbb{R}} \lambda^2 |\hat{f}_\alpha(\lambda, t)| d\lambda \le \sqrt{2} \left\{ ||\partial_x^2 f_\alpha||_2 + ||f_\alpha||_2 \right\}.$$

Hence, $\int \lambda^2 |\mu_s^\alpha|(d\lambda) = \int_{\mathbb{R}} \lambda^2 |\hat{f}_\alpha(\lambda, t)| d\lambda$ is uniformly bounded for $\alpha > 0$ and $f \in \mathscr{C}_b^{2,1}(\mathbb{R} \times [0, 1])$. Similarly, one can prove that

$$\int_{\mathbb{R}} \left| \frac{(\widehat{\partial_t f_\alpha})(\lambda, t)}{2\pi} \right| d\lambda$$

is uniformly bounded.

Now, as in the first part of the proof, the martingale valued process M_{f_α} converges point-wise to M_f as $\alpha \downarrow 0$. Hence, to conclude, we need to prove that the martingale valued process $\int_0^t D_0 f_\alpha(H_N(s), H_N(s); s) \sharp dH_N(s)$ converges toward $\int_0^t D_0 f(H_N(s), H_N(s); s) \sharp dH_N(s)$ as α goes to zero, that is $||\bar{M}_{f_\alpha}(t) - \bar{M}_f(t)||_2$ goes to zero as α goes to zero. As above, this point boils down to proving that

$$\mathbb{E}\left[\mathrm{tr}_N((\partial_x(f_\alpha - f))^2(H_N(t), t)) \right]$$

goes to zero as N goes to infinity. But we find a finite constant c so that

$$\mathbb{E}\left[\mathrm{tr}_N((\partial_x(f_\alpha - f))^2(H_N(t), t) \right] \le c\alpha(||f||_\infty + ||\partial_x f||_\infty)^2 \mathbb{E}[\mathrm{tr}_N(H_N(t)^2)]$$

which goes to zero as α decreases to zero. Hence, the formula is verified for every $f \in \mathscr{C}_b^{2,1}(\mathbb{R} \times [0, 1])$ and the proof of the lemma is complete. □

As a consequence of Lemma 2.2, the spectral process $\hat{\mu}_t^{(N)}$ satisfies:

LEMMA 2.3. *For any $f \in \mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])$, the process $Q_f^{(N)}$ defined by*

$$Q_f^{(N)}(t) = \hat{\mu}_t^{(N)}(f(x,t)) - \hat{\mu}_0^{(N)}(f(x,0)) - \int_0^t \hat{\mu}_s^{(N)}(\partial_s f(x,s))ds - \int_0^t (\hat{\mu}_s^{(N)} \otimes \hat{\mu}_s^{(N)}) \circ (L_0 f)(s)ds$$

is a martingale with bracket

$$\left\langle Q_f^{(N)} \right\rangle_t = \frac{1}{N^2}\int_0^t \hat{\mu}_s^{(N)}\left((\partial_x f)^2(x,s)\right)ds$$

This result is a direct consequence of Lemma 2.2 once one takes the normalized trace tr_N and realizes that for any adapted processes A and B,

$$\left\langle \mathrm{tr}_N\left(\int_0^{\cdot} A_s dH_s^N\right), \mathrm{tr}_N\left(\int_0^{\cdot} B_s dH_s^N\right)\right\rangle_t = \frac{1}{N^2}\int_0^t \mathrm{tr}_N(A_s B_s)ds.$$

Lemma 2.3 is reminiscent of mean field interacting particles systems; the law of $\hat{\mu}_{\cdot}^{(N)}$ is described via a generator depending (non linearly) on $\hat{\mu}_{\cdot}^{(N)}$. However, a crucial difference lies in the fact that the non linear dependence on $\hat{\mu}_{\cdot}^{(N)}$ is contained in the diffusive part rather than in the drift part. We can still follow the ideas developed in hydrodynamics (see [23]) to prove large deviation results.

The proof of Theorem 1.1 follows the usual scheme: first, we prove that S is a good rate function, then we show that $\hat{\mu}_{\cdot}^{(N)}$ is exponentially tight and that a weak large deviation upper bound holds. Finally, we study the large deviation lower bound.

2.2. *S is a good rate function.*

LEMMA 2.4. *S is a good rate function, that is that S is a non-negative function such that for any $M \in \mathbb{R}^+$, $\{\nu \in \mathscr{C}([0,1], \mathscr{P}(\mathbb{R})); S(\nu) \le M\}$ is compact.*

PROOF. $S^{0,1}$ is non-negative according to (1.4). As a supremum of continuous functions on $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$, S is lower semi-continuous. Hence, the only point is to show that the level sets of S are included into relatively compact sets. Following Lemma 5.4 of [13] and Lemma 1.3 of [16], the relatively compact subsets of $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$ are included into compact sets of the form

$$\mathscr{K} = \mathscr{K}_{\mathscr{M}} \bigcap \left(\bigcap_{n\in\mathbb{N}} \mathscr{K}_n\right) \tag{2.6}$$

with

$$\mathscr{K}_{\mathscr{M}} = \{\nu \in \mathscr{C}([0,1], \mathscr{P}(\mathbb{R})), \nu_t \in K_M \ \forall t \in [0,1]\} \tag{2.7}$$

and

$$\mathscr{K}_n = \{\nu \in \mathscr{C}([0,1], \mathscr{P}(\mathbb{R})), (t \to \nu_t(f_n)) \in K_n\} \tag{2.8}$$

where K_M and K_n are compact subsets of $\mathscr{P}(\mathbb{R})$ and $\mathscr{C}([0,1], \mathbb{R})$ and $(f_n)_{n\in\mathbb{N}}$ a family of bounded continuous functions dense in $\mathscr{C}_c(\mathbb{R})$. Indeed, the elements of $\bigcap_{n\in\mathbb{N}} \mathscr{K}_n$ can easily be seen to be tight by a standard diagonalization procedure with limit points in $\mathscr{C}([0,1], \mathscr{C}_c(\mathbb{R})')$ if $\mathscr{C}_c(\mathbb{R})'$ denotes the algebraic dual of $\mathscr{C}_c(\mathbb{R})$. If they also belong to $\mathscr{K}_{\mathscr{M}}$, their limits can be seen to be in $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$. We shall assume further that the f_n's belong to $\mathscr{C}_b^2(\mathbb{R})$. According to Prohorov's theorem, we may take K_M as

$$K_M(L) = \bigcap_{m\in\mathbb{N}} \left\{\mu \in \mathscr{P}(\mathbb{R}) : \mu(|x| \geq L_m) \leq \frac{1}{m}\right\}$$

with a positive real valued sequence $L = (L_m)_{m\in\mathbb{N}}$.

Moreover, since for every integer number n, $t \to \nu_t(f_n)$ is uniformly bounded on $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$ - because f_n is bounded -, we deduce from Arzéla-Ascoli's theorem that we can take K_n of the form

$$K(\delta) = \bigcap_{m\in\mathbb{N}} \left\{g \in \mathscr{C}([0,1], \mathbb{R}), \sup_{|t-s|\leq\delta_m} |g(t) - g(s)| \leq \frac{1}{m}\right\}$$

for a positive sequence $\delta = (\delta_m)_{m\in\mathbb{N}}$.

Following the above description of relatively compact subsets of $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$, to achieve our proof, we need to show that, for any $M > 0$:

(i) For any $m \in \mathbb{N}^*$, there is a positive real number L_m^M so that for any $\nu \in \{S \leq M\}$,

$$\sup_{0\leq s\leq 1} \nu_s(|x| \geq L_m^M) \leq \frac{1}{m}.$$

(ii) For any $m \in \mathbb{N}^*$ and $f \in \mathscr{C}_b^2(\mathbb{R})$, there exists a positive real number δ_m^M so that for any $\nu \in \{S \leq M\}$,

$$\sup_{|t-s|\leq\delta_m^M} |\nu_t(f) - \nu_s(f)| \leq \frac{1}{m}.$$

To show these two points, let us first remark that, as in [13], we may write

$$S^{0,1}(\nu) = \sup_{0\leq s\leq t\leq 1} S^{s,t}(\nu)$$

with

$$S^{s,t}(\nu) = \sup_{f\in\mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])} \left(S^{s,t}(\nu, f) - \tfrac{1}{2}\int_s^t \int (\partial_x f(x,u))^2 d\nu_u(x) du\right)$$

To prove (i), we take for $\delta > 0$, $f(x) = f_\delta(x) = x^2(1 + \delta x^2)^{-1} \in \mathscr{C}^{2,1}(\mathbb{R}\times[0,1])$ in the following inequality:

$$S^{0,t}(\nu, f) - \tfrac{1}{2} \ll f, f \gg_\nu^{0,t} \leq S^{0,t}(\nu) \leq S^{0,1}(\nu) \tag{2.9}$$

for $t \in [0,1]$. We then get

$$\nu_t(f_\delta) \le 8t + 8\int_0^t \nu_s\left(\frac{x^2}{(1+\delta^2 x^2)^2}\right) ds + S^{0,1}(\nu) \le 8t + 8\int_0^t \nu_s(f_\delta) ds + S^{0,1}(\nu).$$

With f_δ uniformly bounded, Gronwall's lemma results with

$$\sup_{t\in[0,1]} \nu_t(f_\delta(x)) \le (8 + S^{0,1}(\nu))e^8. \tag{2.10}$$

We can now let $\delta \downarrow 0$ to get by monotone convergence theorem that, if $S(\nu) \le M$,

$$\sup_{t\in[0,1]} \nu_t(x^2) \le (8+M)e^8$$

and hence conclude by Chebyshev's inequality.

To establish (ii), note that (1.4) implies

$$|S^{s,t}(\nu, f)| \le \sqrt{2S^{0,1}(\nu) \ll f, f \gg_\nu^{s,t}}$$

and hence, if $S^{0,1}(\nu) \le M$, that

$$|\nu_t(f) - \nu_s(f)| \le \sqrt{2M}\|\partial_x f\|_\infty \sqrt{|t-s|} + \tfrac{1}{2}\|\partial_x^2 f\|_\infty |t-s|$$

which is enough to conclude. □

2.3. *Exponential tightness.* Here, we shall prove that:

THEOREM 2.1. *For any integer number L, there exists a finite integer number $N_0 \in \mathbb{N}$ and a compact set $\mathscr{K}_L$ in $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$ so that $\forall N \ge N_0$,*

$$\mathbb{P}(\hat{\mu}^{(N)} \in \mathscr{K}_L^c) \le \exp\{-LN^2\}$$

In view of the previous description of the relatively compact subsets of $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$, we need to show that:

LEMMA 2.5. (a) *For every positive real numbers L and m, there is an $N_0 \in \mathbb{N}$ and a positive real number $M_{L,m}$ so that $\forall N \ge N_0$,*

$$\mathbb{P}\left(\sup_{0\le t\le 1} \hat{\mu}_t^{(N)}(|x| \ge M_{L,m}) \ge \frac{1}{m}\right) \le \exp(-LN^2).$$

(b) *For any $f \in \mathscr{C}_b^2(\mathbb{R})$, for any positive real numbers L and m, there exist an $N_0 \in \mathbb{N}$ and a positive real number $\delta_{L,m,f}$ such that $\forall N \ge N_0$,*

$$\mathbb{P}\left(\sup_{|t-s|\le \delta_{L,m,f}} |\hat{\mu}_t^{(N)}(f) - \hat{\mu}_s^{(N)}(f)| \ge \frac{1}{m}\right) \le \exp(-LN^2).$$

Indeed, if $\mathscr{K}_L$ is a compact set of type (2.6), (a) allows us to choose $\mathscr{K}_{\mathscr{M}}$ of the form (2.7) such that

$$\mathbb{P}(\hat{\mu}^{(N)} \in (\mathscr{K}_M(L))^c) \leq \exp(-2LN^2) \tag{2.11}$$

whereas (b) shows that for any $n \in \mathbb{N}$ we can choose $\delta_n = \delta_n(L)$ so that, with the notations of (2.8),

$$\mathbb{P}(\hat{\mu}^{(N)} \in (\mathscr{K}_n)^c) \leq \exp(-2LN^2 n)$$

and thus

$$\mathbb{P}\left(\hat{\mu}^{(N)} \in \left(\bigcap_n \mathscr{K}_n\right)^c\right) \leq \sum_{n\in\mathbb{N}} \mathbb{P}\left(\hat{\mu}^{(N)} \in (\mathscr{K}_n)^c\right) \leq \exp(-2LN^2). \tag{2.12}$$

Inequalities (2.11) and (2.12) give Theorem 2.1.

PROOF OF LEMMA 2.5. To prove (a), it is enough to find two real positive numbers η and m so that

$$\mathbb{E}\left[\exp(\eta N^2 \sup_{0\leq t\leq 1} \int x^2 d\hat{\mu}_t^{(N)}(x))\right] \leq \exp(mN^2) \tag{2.13}$$

Indeed, (2.13) and Chebyshev's inequality yields

$$\mathbb{P}\left(\sup_{0\leq t\leq 1} \hat{\mu}_t^{(N)}(|x| \geq \sqrt{M_{L,m}}) \geq \frac{1}{m}\right) \leq \mathbb{P}\left(\sup_{0\leq t\leq 1} \int x^2 d\hat{\mu}_t^{(N)}(x) \geq \frac{M_{L,m}}{m}\right)$$
$$\leq \exp(-LN^2)$$

with $L = \eta \frac{M_{L,m}}{m} - m$. To show (2.13), notice that

$$\sup_{0\leq t\leq 1} \int x^2 d\hat{\mu}_t^{(N)}(x) = \sup_{0\leq t\leq 1} \frac{1}{N} \sum_{i,j} |H_N^{i,j}(t)|^2 \leq \frac{1}{N} \sum_{i,j=1}^{N} \sup_{0\leq t\leq 1} |H_N^{i,j}(t)|^2.$$

By Désiré-André reflection principle, we get, for $\eta < (1/2)$,

$$\mathbb{E}[\exp(\eta N^2 \sup_{0\leq t\leq 1} \int x^2 d\hat{\mu}_t^{(N)}(x))] \leq 2^{N^2}(1-2\eta)^{-\frac{N^2}{2}}.$$

We turn to the proof of (b). It is a consequence of Lemma 2.2. Indeed, for any $f \in \mathscr{C}_b^2(\mathbb{R})$ and N large enough,

$$\begin{aligned} &\sup_{|t-s|<\delta} \left|\int f(x) d\hat{\mu}_t^{(N)}(x) - \int f(x) d\hat{\mu}_s^{(N)}(x)\right| \\ &\quad \leq \sup_{|t-s|<\delta} \left|Q_f^{(N)}(t) - Q_f^{(N)}(s)\right| + \tfrac{1}{2} \left\|\partial_x^2 f\right\|_\infty |t-s| \end{aligned} \tag{2.14}$$

Noticing that

$$\sup_{|t-s|<\delta} \left|Q_f^{(N)}(t) - Q_f^{(N)}(s)\right| \leq 2 \max_{i\in\mathbb{N}, i\leq[1/\delta]} \sup_{t\in[\delta i, \delta(i+2)]} \left|Q_f^{(N)}(t) - Q_f^{(N)}(\delta i)\right|,$$

it is not hard to deduce (b) from (2.14) by Chebyshev's inequality if we can show that for any $L \in \mathbb{R}^+$, any $s \in [0,1]$, N large enough and $\delta \in (0, 1-s)$ small enough,

$$\mathbb{E}\left[\exp\left(LN^2 \sup_{t\in[s,s+\delta]} |Q_f^{(N)}(t) - Q_f^{(N)}(s)|\right)\right] \le e^{N^2}. \tag{2.15}$$

Since $e^x + e^{-x} \ge e^{|x|}$, it is enough to bound $\mathbb{E}\big[\sup_{t\in[s,s+\delta]}\{\exp(LN^2(Q_f^{(N)}(t) - Q_f^{(N)}(s)))\}\big]$ up to change f into $-f$. To this end, we use Doob's inequality to find a finite constant c so that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[s,s+\delta]}\left\{\exp\left(LN^2\left(Q_f^{(N)}(t) - Q_f^{(N)}(s)\right)\right)\right\}\right]\\
&\quad\le \mathbb{E}\left[\sup_{t\in[s,s+\delta]}\left\{\exp\left(LN^2\left(Q_f^{(N)}(t) - Q_f^{(N)}(s)\right)\right.\right.\right.\\
&\qquad\qquad\left.\left.\left. -\tfrac{1}{2}L^2N^4\left(\left\langle Q_f^{(N)}\right\rangle_t - \left\langle Q_f^{(N)}\right\rangle_s\right)\right)\right\}^2\right]^{\frac{1}{2}}\\
&\qquad\times\mathbb{E}\left[\exp\left(L^2N^4(\left\langle Q_f^{(N)}\right\rangle_{s+\delta} - \left\langle Q_f^{(N)}\right\rangle_s)\right)\right]^{\frac{1}{2}}\\
&\quad\le c\mathbb{E}\left[\left\{\exp\left(LN^2\left(Q_f^{(N)}(s+\delta) - Q_f^{(N)}(s)\right)\right.\right.\right.\\
&\qquad\qquad\left.\left.\left. -\tfrac{1}{2}L^2N^4\left(\left\langle Q_f^{(N)}\right\rangle_{s+\delta} - \left\langle Q_f^{(N)}\right\rangle_s\right)\right)\right\}^2\right]^{\frac{1}{2}}\\
&\qquad\times\mathbb{E}\Big[\exp(L^2N^4(\langle Q_f^{(N)}\rangle_{s+\delta} - \langle Q_f^{(N)}\rangle_s))\Big]^{\frac{1}{2}}.
\end{aligned}$$

But the Cauchy-Schwarz inequality and super-martingales inequality yield

$$\begin{aligned}
&\mathbb{E}\left[\left\{\exp\left(LN^2\left(Q_f^{(N)}(s+\delta) - Q_f^{(N)}(s)\right) - \tfrac{1}{2}L^2N^4\left(\left\langle Q_f^{(N)}\right\rangle_{s+\delta} - \left\langle Q_f^{(N)}\right\rangle_s\right)\right)\right\}^2\right]\\
&\quad\le \mathbb{E}\left[\exp\left(6L^2N^4\left(\left\langle Q_f^{(N)}\right\rangle_{s+\delta} - \left\langle Q_f^{(N)}\right\rangle_s\right)\right)\right]^{\frac{1}{2}}
\end{aligned}$$

so that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[s,s+2\delta]}\left\{\exp\left(LN^2\left(Q_f^{(N)}(t) - Q_f^{(N)}(s)\right)\right)\right\}\right]\\
&\qquad\le c\mathbb{E}\left[\exp\left(6L^2N^4\left(\left\langle Q_f^{(N)}\right\rangle_{s+\delta} - \left\langle Q_f^{(N)}\right\rangle_s\right)\right)\right]^{\frac{1}{3}}.
\end{aligned}\tag{2.16}$$

Since

$$\left\langle Q_f^{(N)}\right\rangle_{s+\delta} - \left\langle Q_f^{(N)}\right\rangle_s = \frac{1}{N^2}\int_s^{s+\delta}\int(\partial_x f(x))^2 d\hat{\mu}_u^{(N)} du \leq \frac{1}{N^2}\delta\|\partial_x f\|_\infty^2,$$

taking $2L^2\delta\|\partial_x f\|_\infty^2 < 1$ in (2.16) gives (2.15) and thus completes the proof. □

2.4. *Weak large deviation upper bound.* To achieve the proof of the upper bound in Theorem 1.1, and thanks to the exponential tightness result of the previous section, it is enough (see [14], Theorem 4.1.11) to prove a weak large deviation upper bound, that is that for every compact subset K of $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$,

$$\limsup_{N\to\infty}\frac{1}{N^2}\ln\mathbb{P}(\hat{\mu}^{(N)}\in K) \leq -\inf_{\nu\in K} S(\nu).$$

This last result can easily be seen to be equivalent to

THEOREM 2.2. *For every process ν in $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$, if $B_\delta(\nu)$ denotes the open ball with center ν and radius δ for the distance $\mathcal{D}$, then*

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\ln\mathbb{P}\big(\hat{\mu}^{(N)}\in B_\delta(\nu)\big) \leq -S(\nu).$$

The end of this section is therefore dedicated to the proof of Theorem 2.2.

PROOF. Note first that $\mathbb{P}(\hat{\mu}_0^{(N)} = \delta_0) = 1$ so that

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\ln\mathbb{P}(\hat{\mu}^{(N)}\in B_\delta(\nu)) = -\infty$$

if $\nu_0 \neq \delta_0$. Hence, we shall assume hereafter that $\nu_0 = \delta_0$. We shall follow the ideas developed in [23]. To this end, we define a family of positives supermartingales $\{\zeta_f^{(N)}, f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])\}$, equal to 1 at $t=0$, thanks to Lemma 2.3:

$$\begin{aligned}\zeta_f^{(N)}(t) &= \exp\left(N^2 Q_f^{(N)}(t) - \frac{N^4}{2}\left\langle Q_f^{(N)}\right\rangle_t\right)\\ &= \exp\left(N^2(S^{0,t}(\hat{\mu}^{(N)}, f) - \frac{1}{2}\ll f, f\gg_{\hat{\mu}^{(N)}}^{0,t})\right)\end{aligned}$$

Let $\nu\in\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$ and $f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$; then

$$\begin{aligned}\mathbb{P}\left(\hat{\mu}^{(N)}\in B(\nu,\delta)\right) &= \mathbb{E}\left[1_{\hat{\mu}^{(N)}\in B(\nu,\delta)}\frac{\zeta_f^{(N)}(1)}{\zeta_f^{(N)}(1)}\right]\\ &\leq \sup_{\nu'\in B(\nu,\delta)}\exp\left(-N^2\left(S^{0,1}(\nu', f) - \frac{1}{2}\ll f, f\gg_{\nu'}^{0,1}\right)\right)\\ &= \exp\left(-N^2\inf_{\nu'\in B(\nu,\delta)}\left(S^{0,1}(\nu', f) - \frac{1}{2}\ll f, f\gg_{\nu'}^{0,1}\right)\right).\end{aligned}$$

Notice that if f belongs to $\mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])$, the function $\nu' \to S^{0,1}(\nu', f) - \frac{1}{2} \ll f, f \gg_{\nu'}^{0,1}$ is continuous. Thus, for any function $f \in \mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])$,

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\ln\mathbb{P}\left(\hat{\mu}^{(N)}\in B(\nu,\delta)\right)\leq -\left(S^{0,1}(\nu,f)-\frac{1}{2}\ll f,f\gg_{\nu}^{0,1}\right)$$

We conclude by taking the supremum over f that

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\ln\mathbb{P}\left(\hat{\mu}^{(N)}\in B(\nu,\delta)\right)$$
$$\leq -\sup_{f\in\mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])}\left(S^{0,1}(\nu,f)-\frac{1}{2}\ll f,f\gg_{\nu}^{0,1}\right) \qquad \square$$

2.5. *Large deviation lower bound.* In this section, we provide a large deviation lower bound for the Hermitian Brownian motion which is unfortunately not sharp since its rate function is shown to be equal to S only at nice processes μ, even though we believe it is equal to S everywhere. In fact, large deviation lower bounds are related to a uniqueness statement for the weak solutions of

$$S^{s,t}(\nu,f)=\int_s^t\int\partial_x h_u(x)\partial_x f_u(x)d\nu_u(x)du = \ll h,f\gg_{\nu}^{s,t} \tag{2.17}$$

for $f\in\mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])$, $0\leq s\leq t\leq 1$ and for a large class of magnetic fields h. Unfortunately, we have only been able to prove such a uniqueness property for very smooth fields h.

Let us first recall the second part of Theorem 1.1, that is, if $\mathscr{C}^\infty([0,1],\mathscr{P}(\mathbb{R}))$ denotes the subset of $\mathscr{C}([0,1],\mathscr{P}(\mathbb{R}))$ of weak solutions $\nu\in\mathscr{C}([0,1],\mathscr{P}(\mathbb{R}))$ satisfying

$$S^{0,1}(\nu,f)=\int_0^1\int\partial_x h_u(x)\partial_x f_u(x)d\nu_u(x)du$$

for any $f\in\mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])$, with $h\in\mathscr{MF}_\infty\subset\mathscr{C}_b^{\infty,1}(\mathbb{R}\times[0,1])$ so that

$$\mathscr{MF}_\infty=\Big\{h\in\mathscr{C}_b^{\infty,1}(\mathbb{R}\times[0,1])\cap\mathscr{C}([0,1],L^2(\mathbb{R}));$$
$$\exists(C,\varepsilon)\in(0,\infty);\ \sup_{t\in[0,1]}|\hat{h}_t(\lambda)|\leq Ce^{-\varepsilon|\lambda|}\Big\}$$

where $\hat{h}_t$ stands for the Fourier transform of h_t,

PROPERTY 2.1. *The law of $\hat{\mu}^{(N)}$ satisfies a large deviation lower bound in the scale N^2 so that for any open subset O of $\mathscr{C}([0,1],\mathscr{P}(\mathbb{R}))$,*

$$\liminf_{N\to\infty}\frac{1}{N^2}\ln\mathbb{P}\left(\hat{\mu}^{(N)}\in O\right)\geq -\inf_{\nu\in O\cap\mathscr{C}^\infty([0,1],\mathscr{P}(\mathbb{R}))}S(\nu).$$

To prove Property 2.1, we shall first recall the general strategy to demonstrate such a result and then study the uniqueness property of the solutions of (2.17) to conclude.

One first begin with the observation that for any open subset O of $\mathscr{C}([0,1],\mathscr{P}(\mathbb{R}))$, for any $\nu \in O$,

$$\liminf_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}\left(\hat{\mu}^{(N)} \in O\right) \geq \liminf_{\delta\to 0} \liminf_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}\left(\hat{\mu}^{(N)} \in B(\nu,\delta)\right) \tag{2.18}$$

where $B(\nu,\delta)$ stands for an open ball in the metric space $\mathscr{C}([0,1],\mathscr{P}(\mathbb{R}))$ with radius δ. Hence, the only point is to bound from below $P(\hat{\mu}^{(N)} \in B(\nu,\delta))$ for sufficiently small δ's and $\nu \in \mathscr{C}([0,1],\mathscr{P}(\mathbb{R}))$ with finite entropy S. Since by (1.4) any $\nu \in \mathscr{C}([0,1],\mathscr{P}(\mathbb{R}))$ with finite entropy S satisfies $\nu_0 = \delta_0$ and

$$S^{0,1}(\nu,f)^2 \leq 2S^{0,1}(\nu) \ll f,f \gg_\nu^{0,1},$$

$f \to S^{0,1}(\nu,f)$ is a bounded linear form in $H_\nu^{0,1}$, the closure of $\mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])$ by the norm induced by the scalar product $\ll .,. \gg_\nu^{0,1}$. Hence, there exists a function $h \in H_\nu^{0,1}$ such that ν is solution of (2.17) by Riesz's theorem. In the following, we shall restrict ourselves to magnetic fields $h \in \mathscr{MF}_\infty$, that is to measure-valued processes $\nu \in \mathscr{C}^\infty([0,1],\mathscr{P}(\mathbb{R}))$.

The strategy to prove the lower bound is then to center our probability measure $\mathbb{P}$ around ν via Girsanov's theorem, namely, consider

$$\zeta_h^{(N)}(1) = \exp\left(N^2\left(S^{0,1}(\hat{\mu}^{(N)},h) - \tfrac{1}{2} \ll h,h \gg_{\hat{\mu}^{(N)}}^{0,1}\right)\right)$$

and $\mathbb{P}_h(\hat{\mu}_.^{(N)} \in .) \equiv \mathbb{P}(1_{\hat{\mu}_.^{(N)}\in .}\zeta_h^{(N)}(1))$. We have the following bound:

$$\begin{aligned}
&\mathbb{P}\left(\hat{\mu}_.^{(N)} \in B(\nu,\delta)\right)\\
&\quad = \mathbb{P}_h\Big(1_{\hat{\mu}_.^{(N)}\in B(\nu,\delta)} \exp\Big\{-N^2(S^{0,1}(\hat{\mu}^{(N)},h) - \tfrac{1}{2} \ll h,h \gg_{\hat{\mu}^{(N)}}^{0,1})\Big\}\Big)\\
&\quad \geq \exp\Big\{-N^2 \sup_{\mu\in B(\nu,\delta)} (S^{0,1}(\mu,h) - \tfrac{1}{2} \ll h,h \gg_\mu^{0,1})\Big\}\mathbb{P}_h(\hat{\mu}_.^{(N)} \in B(\nu,\delta)).
\end{aligned} \tag{2.19}$$

Since $h \in \mathscr{MF}_\infty$, h belongs to $\mathscr{C}^{2,1}(\mathbb{R}\times[0,1])$ which results with

$$\lim_{\delta\downarrow 0} \sup_{\mu\in B(\nu,\delta)} \left(S^{0,1}(\mu,h) - \tfrac{1}{2} \ll h,h \gg_\mu^{0,1}\right) = S^{0,1}(\nu,h) - \tfrac{1}{2} \ll h,h \gg_\nu^{0,1}. \tag{2.20}$$

Further, under $\mathbb{P}_h$, it is not hard to check that, according to Girsanov's formula, for $(i,j) \in \{1,.,N\}^2$, the canonical entries $X^{i,j}$ satisfy

$$dX_t^{i,j} = \frac{1}{\sqrt{N}} dB_t^{i,j} + (\partial_x h_t(X))_{i,j}\, dt.$$

Following the lines of the two previous subsections, one can see that the law of $\hat{\mu}_.^{(N)}$ under $\mathbb{P}_h$ is exponentially tight (by exponential tightness under $\mathbb{P}$ and a uniform bound on the Girsanov density) and that its limit points (again by Itô's calculus) satisfy the weak equation (2.17). Thus, if we can prove that this equation admits a unique solution, therefore equal to ν, we will have proved that $\hat{\mu}_{\cdot}^{(N)}$ converges under $\mathbb{P}_h$ toward ν and in particular that for any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_h(\hat{\mu}_{\cdot}^{(N)} \in B(\nu, \delta)) = 1. \tag{2.21}$$

Equations (2.19)–(2.21) result, once one let N going to infinity and then δ decreasing to zero, with

$$\begin{aligned}\liminf_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\ln\mathbb{P}\Big(\hat{\mu}^{(N)}\in B(\nu,\delta)\Big) &\geq -\Big(S^{0,1}(\nu,h)-\frac{1}{2}\ll h,h\gg_{\nu}^{0,1}\Big)\\ &= -S^{0,1}(\nu)\end{aligned} \tag{2.22}$$

where the last equality follows from the definition of $S^{0,1}$ and (2.17). Since (2.22) is proved below only for $\nu \in \mathscr{C}^{\infty}([0,1], \mathscr{P}(\mathbb{R}))$, taking the supremum on $\nu \in \mathscr{C}^{\infty}([0,1], \mathscr{P}(\mathbb{R})) \cap O$ provides the desired estimate of Property 2.1. To complete our proof, we shall show the following:

LEMMA 2.6. *For any $h \in \mathscr{M}\mathscr{F}_{\infty}$, (2.17) admits a unique weak solution.*

PROOF. Note first that the existence is already clear since it can be constructed as the limit point of $\hat{\mu}_{\cdot}^{(N)}$ under $\mathbb{P}_h$. Hence, we shall concentrate on the uniqueness statement. To this end, we take $f(x) = e^{i\lambda x}$ and denote $\mathscr{L}_t(\lambda) = \int e^{i\lambda x} d\nu_t(x)$ the Fourier transform of ν_t. Note that by the inversion theorem and the continuity of h_t,

$$h_t(x) = \int e^{-i\lambda x}\hat{h}_t(\lambda)d\lambda = \int e^{i\lambda x}\hat{h}_t(-\lambda)d\lambda$$

everywhere and, according to the tail of $\hat{h}_t$ for $h \in \mathscr{M}\mathscr{F}_{\infty}$,

$$\partial_x h_t(x) = i\int e^{-i\lambda x}\lambda\hat{h}_t(-\lambda)d\lambda.$$

Define $m_t(\lambda) = i\lambda\hat{h}_t(-\lambda)$ and note that it decreases exponentially fast to infinity, $|m_t(\lambda)| \leq Ce^{-2\varepsilon|\lambda|}$ for a positive ε and a positive finite constant C and for any time $t \in [0,1]$ when $h \in \mathscr{M}\mathscr{F}_{\infty}$. We then have

$$\begin{aligned}\mathscr{L}_t(\lambda) = 1 &- \frac{\lambda^2}{2}\int_0^t\int_0^1 \mathscr{L}_s(\alpha\lambda)\mathscr{L}_s((1-\alpha)\lambda)d\alpha ds\\ &+ i\lambda\int_0^t\int \mathscr{L}_s(\lambda+\lambda')m_s(\lambda')d\lambda' ds.\end{aligned} \tag{2.23}$$

Multiplying both sides of this equality by $e^{-\varepsilon|\lambda|}$ gives, with $\mathscr{L}_t^{\varepsilon}(\lambda) = e^{-\varepsilon|\lambda|}\mathscr{L}_t(\lambda)$,

$$\begin{aligned}\mathscr{L}_t^{\varepsilon}(\lambda) = e^{-\varepsilon|\lambda|} &- \frac{\lambda^2}{2}\int_0^t\int_0^1 \mathscr{L}_s^{\varepsilon}(\alpha\lambda)\mathscr{L}_s^{\varepsilon}((1-\alpha)\lambda)d\alpha ds\\ &+ i\lambda\int_0^t\int \mathscr{L}_s^{\varepsilon}(\lambda+\lambda')e^{\varepsilon|\lambda+\lambda'|-\varepsilon|\lambda|}m_s(\lambda')d\lambda' ds.\end{aligned} \tag{2.24}$$

Note by the way here that $\mathscr{L}_t^{\varepsilon}$ is the Fourier transform of $P_{\varepsilon} * \nu_t$ where P_{ε} is a Cauchy's law and that the stability of the first term of our equation by the multiplication by $e^{-\varepsilon|\lambda|}$ reflects the fact that our non-commutative derivation is stable by convolution by these laws; this is a non-trivial fact since the non-commutative derivation involves a quadratic term. The convolution by Cauchy laws is the only classical convolution for which such a property is satisfied.

Assume we have two solutions $\mathscr{L}^\varepsilon$ and $\tilde{\mathscr{L}}^\varepsilon$ to equation (2.24) with exponentially decreasing tails and denote $\Delta_t^\varepsilon(\lambda) = |\mathscr{L}_t^\varepsilon(\lambda) - \tilde{\mathscr{L}}_t^\varepsilon(\lambda)|$. Then, we obtain

$$\begin{aligned}\Delta_t^\varepsilon(\lambda) \le \lambda^2 \int_0^t \sup_{|z|\le|\lambda|} \Delta_s^\varepsilon(z) \int_0^1 (|\mathscr{L}_s^\varepsilon(\alpha\lambda)| + |\tilde{\mathscr{L}}_s^\varepsilon(\alpha\lambda)|) d\alpha ds \\ + |\lambda| \int_0^t \int \Delta_s^\varepsilon(\lambda+\lambda')|m_s(\lambda')| e^{\varepsilon|\lambda+\lambda'|-\varepsilon|\lambda|} d\lambda' ds.\end{aligned} \tag{2.25}$$

Since by definition $|\mathscr{L}_s^\varepsilon(\alpha\lambda)|$ and $|\tilde{\mathscr{L}}_s^\varepsilon(\alpha\lambda)|$ are bounded by $e^{-\varepsilon\alpha|\lambda|}$,

$$\int_0^1 (|\mathscr{L}_s^\varepsilon(\alpha\lambda)| + |\tilde{\mathscr{L}}_s^\varepsilon(\alpha\lambda)|) d\alpha \le \frac{2}{\varepsilon|\lambda|}. \tag{2.26}$$

Further, for any $R \in \mathbb{R}^+$,

$$\begin{aligned}&\int \Delta_s^\varepsilon(\lambda+\lambda')|m_s(\lambda')| e^{\varepsilon|\lambda+\lambda'|-\varepsilon|\lambda|} d\lambda' \\ &\quad \le \left(\sup_{|z|\le R} \Delta_s^\varepsilon(z) + 2e^{-\varepsilon R}\right) \int |m_s(\lambda')| e^{\varepsilon|\lambda+\lambda'|-\varepsilon|\lambda|} d\lambda'\end{aligned} \tag{2.27}$$

where the last term in the right hand side of (2.27) is bounded, according to our assumption, by

$$\int |m_t(\lambda')| e^{\varepsilon|\lambda+\lambda'|-\varepsilon|\lambda|} d\lambda' \le C \int e^{\varepsilon|\lambda+\lambda'|-\varepsilon|\lambda|-2\varepsilon|\lambda'|} d\lambda' \le \frac{2C}{\varepsilon}. \tag{2.28}$$

Inequalities (2.25)–(2.28) result with, if we denote $\bar{\Delta}_s^\varepsilon(R) = \sup_{|z|\le R} \Delta_s^\varepsilon(z)$,

$$\bar{\Delta}_t^\varepsilon(R) \le \frac{2R}{\varepsilon}(1+C) \int_0^t \bar{\Delta}_s^\varepsilon(R) ds + 2tRe^{-\varepsilon R}$$

so that Gronwall's lemma yields, for $t \le 1$,

$$\bar{\Delta}_t^\varepsilon(R) \le 2Re^{-\varepsilon R} e^{\frac{2R}{\varepsilon}(1+C)t}.$$

As a consequence, for $t < \tau \equiv (\varepsilon^2/2(1+C))$, we can let R going to infinity and conclude

$$\bar{\Delta}_t^\varepsilon(\infty) = 0.$$

Thus, $\mathscr{L}_t$ is uniquely defined until time τ. Starting now at time $(3\tau/4)$ with two solutions with same initial data at time $(3\tau/4)$, we can proceed as before to see that $\mathscr{L}_t$ is uniquely defined until time $(7/4)\tau$. Proceeding by induction, we finally get that $\mathscr{L}_t$ is uniquely defined on the interval $[0, 1]$. Since the measure valued-process ν is uniquely described by its Fourier transform, the proof of the lemma is complete. □

3. Generalizations to other matrices ensembles. In this section, we follow the ideas of the previous part to develop large deviations upper bounds for different matrices ensembles; first, we consider the symmetric Brownian motion, then Wishart matrices and finally the unitary Brownian motion.

3.1. *The symmetric Brownian motion.* We consider the process defined on the space $\mathscr{S}_N$ of symmetric real matrices and with Brownian entries. More precisely, we can construct the entries $\{S_N^{i,j}(t), t \geq 0, (i,j) \in \{1,\dots,N\}\}$ via independent real valued Brownian motions $(\beta_{i,j})_{1\leq i\leq j\leq n}$ by

$$S_N^{k,l} = \frac{\sqrt{1+\delta_{k=l}}}{\sqrt{N}}\beta_{k\wedge l, k\vee l}.$$

We let $(\tilde{\lambda}_i^{(N)}(t), 1 \leq i \leq N)$ be the eigenvalues of $S_N(t)$ and $\tilde{\mu}^{(N)}$ be their empirical process. We shall prove the following result:

THEOREM 3.1. *$\tilde{\mu}^{(N)}$ satisfies a large deviation upper bound in the scale N^2 with good rate function $S_s = \frac{1}{2}S$ with S as defined in Theorem* 1.1. *In other words, for every closed subset $F \in \mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$,*

$$\limsup_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}(\tilde{\mu}^{(N)} \in F) \leq -\inf_{\nu\in F} S_s(\nu).$$

Further, the following large deviations lower bounds holds; for any open subset O of $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}))$,

$$\liminf_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}(\tilde{\mu}^{(N)} \in O) \geq -\inf_{\nu\in O\cap\mathscr{C}^\infty([0,1],\mathscr{P}(\mathbb{R}))} S_s(\nu).$$

As a consequence, the law of $\tilde{\mu}^{(N)}$ converges exponentially fast toward a Dirac measure at the semi-circular process defined by (1.2). The strategy of the proof of Theorem 3.1 follows that of the previous section; the main point is to prove that Itô's calculus can be developed for $\tilde{\mu}_{\cdot}^{(N)}$. In fact, we can see that:

LEMMA 3.1. *Itô's formula for S_N: for every $f \in \mathscr{C}_b^1([0,1], \mathbb{C}[X])$,*

$$\begin{aligned} f(S_N(t),t) &= f(S_N(0),0) + \int_0^t D_0 f(S_N(s), H_N(s); s)\sharp dS_N(s) + \int_0^t \partial_s f(S_N(s),s)ds \\ &\quad + \int_0^t (Id\otimes \mathrm{tr}_N)\circ(L_0 f)(S_N(s),S_N(s);s)ds + \frac{1}{2N}\int_0^t \partial_x^2 f(S_N(s),s)ds \end{aligned}$$

Furthermore, the martingale bracket for $\mathrm{tr}_N(\int_0^t D_0 f(S_N(s), S_N(s); s)\sharp dS_N(s))$ is given by

$$\left\langle \mathrm{tr}_N\left(\int_0^{\cdot} D_0 f(S_N(s),S_N(s);s)\sharp dS_N(s)\right)\right\rangle_t = \frac{2}{N^2}\int_0^t \int (\partial_x f)^2(x,s) d\tilde{\mu}_s^{(N)}(x)ds.$$

The proof is a direct consequence of multi-dimensional Itô's formula and follows the proof of Lemma 2.1. The generalization to $f \in \mathscr{C}_b^{2,1}(\mathbb{R},[0,1])$ is a copy of the proof of Lemma 2.2. We leave it to the reader. The proof of Theorem 3.1 now follows exactly the scheme described in the proof of Theorem 1.1.

3.2. *The Wishart process.* We will here present the matrix-valued Wishart process. The first results concerning the convergence of its marginals appeared in the 1970's ([31] and [24]). Large deviations for time marginals of this process were obtained in [20]. It is defined by:

DEFINITION 3.1. We will call Wishart process the matrix-valued process $(P_{N,M}(t))_{t\in\mathbb{R}^+}$, with value in $\mathscr{H}_N$, and defined by $P_{N,M} = G_{N,M}G_{N,M}^*$ where $(G_{N,M}(t))_{t\in\mathbb{R}^+}$ is the $\mathscr{M}_{N\times M}$-valued process with independent complex Brownian entries which satisfy

$$E\left[G_{N,M}^{i,j}(t)G_{N,M}^{k,l}(s)\right]=0 \quad \text{and} \quad E\left[G_{N,M}^{i,j}(t)\bar{G}_{N,M}^{k,l}(s)\right]=\frac{t\wedge s}{M}\delta_i^k\delta_j^l.$$

Stochastic calculus can be developed as well for this process. To state it, let us introduce a new operator given by, for any smooth complex function f,

$$L_\alpha f = \alpha(X\otimes 1 + 1\otimes X)L_0 f,$$

that is,

$$L_\alpha f(x,y)=\alpha\frac{x+y}{x-y}\left(\partial_x f(x)-\frac{f(x)-f(y)}{x-y}\right)=\alpha\int_0^1 u(x+y)\partial_x^2 f(ux+(1-u)y)\,du.$$

Then:

LEMMA 3.2.

1. $dP_{N,M}(t) = (G_{N,M}dG_{N,M}^* + dG_{N,M}G_{N,M}^*)(t) + Id_N dt$.
2. *For every matrix A, B, C and D of $M_N(\mathbb{C})$,*

$$\begin{aligned}&\langle A\,dP_{N,M}(t)\,B, C\,dP_{N,M}(t)\,D\rangle\\&\quad=\frac{N}{M}\Big(AD\mathrm{tr}_N\big(BCP_{N,M}(t)\big)+AP_{N,M}(t)D\mathrm{tr}_N\big(BC\big)\Big)dt.\end{aligned}$$

3. *For every function $f\in\mathscr{C}_b^1([0,1],\mathbb{C}[X])$,*

$$\begin{aligned}&f\left(P_{N,M}(t),t\right)\\&\quad=f\left(P_{N,M}(0),0\right)\\&\qquad+\int_0^t D_0 f(P_{N,M}(s),P_{N,M}(s);s)\sharp(G_{N,M}dG_{N,M}^*+dG_{N,M}G_{N,M}^*)(s)\\&\qquad+\int_0^t\left(\partial_s f\left(P_{N,M}(s),s\right)+\partial_x f\left(P_{N,M}(s),s\right)\right)ds\\&\qquad+\int_0^t Id_N\otimes\mathrm{tr}_N\left[L_{\frac{N}{M}}f\left(P_{N,M}(s),P_{N,M}(s);s\right)\right]ds.\end{aligned}$$

4. *For every matrix A and B of* $M_N(\mathbb{C})$,

$$\langle \mathrm{tr}_N A dG_{N,M}(t), \mathrm{tr}_N B dG^*_{N,M}(t)\rangle = \frac{1}{NM}\mathrm{tr}_N(AB)dt.$$

5. *For every function* f, g,

$$\begin{aligned} &d\left\langle \mathrm{tr}_N f\left(P_{N,M}(t)\right), \mathrm{tr}_N g\left(P_{N,M}(t)\right)\right\rangle \\ &\quad = \frac{2}{NM}\mathrm{tr}_N\left(P_{N,M}\partial_x f\left(P_{N,M}\right)\partial_x g\left(P_{N,M}\right)\right)(t)dt. \end{aligned}$$

6. *Let* $\hat{\mu}_t^{(M,N)}$ *be the spectral measure of* $P_{N,M}(t)$; *for every function* $f \in \mathscr{C}_b^1([0,1], \mathbb{C}[X])$, *The process* $Q_f^{(N,M)}$ *defined by*

$$\begin{aligned} Q_f^{(N,M)}(t) &= \hat{\mu}_t^{(M,N)}(f(x,t)) - \hat{\mu}_0^{(M,N)}(f(x,0)) - \int_0^t \hat{\mu}_s^{(M,N)}(\partial_s f(x,s))ds \\ &\quad - \int_0^t \hat{\mu}_s^{(M,N)}(\partial_x f(x,s))ds - \int_0^t \hat{\mu}_s^{(M,N)} \otimes \hat{\mu}_s^{(M,N)}(L_{\frac{N}{M}} f(x,s))ds \end{aligned}$$

is a martingale with bracket

$$\left\langle Q_f^{(N,M)}\right\rangle_t = \frac{2}{NM}\int_0^t \hat{\mu}_s^{(M,N)}(x(\partial_x f(x,s))^2)ds.$$

The proof of this lemma for polynomial functions follows the ideas of Lemma 2.1. The generalization to smooth functions follows the proof of Lemma 2.2.

We can now state our large deviation result for $\hat{\mu}^{(M,N)}$:

$$\begin{aligned} \hat{\mu}^{(M,N)} &: \mathbb{R}^+ \to \mathscr{P}(\mathbb{R}^+) \\ &\quad t \;\to\; \hat{\mu}_t^{(M,N)} \end{aligned}$$

considered as a measure-valued process of $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}^+))$ when the ratio (N/M) converges toward a constant $\alpha \in \mathbb{R}^{+*}$. Set, for any $\alpha \in \mathbb{R}^{+*}$, any $s,t \in [0,1]$, any functions f and g of $\mathscr{C}_b^{2,1}(\mathbb{R}\times[0,1])$ and every process $\nu \in \mathscr{C}([0,1], \mathscr{P}(\mathbb{R}^+))$,

$$\begin{aligned} S^{s,t}(\nu, f; \alpha) &= \int f(x,t)d\nu_t(x) - \int f(x,s)d\nu_s(x) \\ &\quad - \int_s^t \int \big(\partial_u f(x,u) + \partial_x f(x,u)\big)d\nu_u(x)du \\ &\quad - \alpha\int_s^t \iint \Big(\frac{x+y}{2}\Big)\left(\frac{\partial_x f(x,u) - \partial_x f(y,u)}{x-y}\right) d\nu_u(x)d\nu_u(y)du, \end{aligned}$$

$$\ll f, g \gg_{\alpha,\nu}^{s,t} = 2\alpha \int_s^t \int x\partial_x f(x,u)\partial_x g(x,u)d\nu_u(x)du,$$

$$S^{s,t}(\nu;\alpha) = \sup_{f\in\tilde{\mathscr{C}}_b^{2,1}(\mathbb{R}\times[0,1])} \left\{S^{s,t}(\nu,f;\alpha) - \tfrac{1}{2} \ll f, f \gg_{\alpha,\nu}^{s,t}\right\}$$

where $\tilde{\mathscr{C}}_b^{2,1}(\mathbb{R}^+ \times [0,1])$ is the subset of $\mathscr{C}_b^{2,1}(\mathbb{R}^+ \times [0,1])$ of functions so that

$$\sup_{t\in[0,1]} \sup_{x\in\mathbb{R}^+} |x\partial_x^2 f(x,t)| < \infty \text{ and } \sup_{x\in\mathbb{R}^+} |x(\partial_x f(x,t))^2| < \infty.$$

Then:

THEOREM 3.2. *If N/M converges toward $\alpha \in \mathbb{R}^{+*}$ when N goes to infinity, then the law $\hat{\mu}^{(M,N)}$, as a probability measure on $\mathscr{C}([0,1], \mathscr{P}(\mathbb{R}^+))$, satisfies a large deviation upper bound in the scale N^2, with good rate function $S(.;\alpha)$ given by*

$$S(\nu;\alpha) = \begin{cases} \infty, & \text{if } \nu_0 \neq \delta_0, \\ S^{0,1}(\nu;\alpha), & \text{otherwise}. \end{cases}$$

The proof of this theorem follows the same steps as the proof of Theorem 1.1 once one notices that $L_\alpha f(x,y)$ and $x(\partial_x f(x,t))^2$ are bounded continuous for $f \in \tilde{\mathscr{C}}_b^{2,1}(\mathbb{R}^+ \times [0,1])$. The only point which should be noticed concerns the proof of the exponential tightness; even though we can not control the exponential moments of $\int x^2 d\hat{\mu}_t^{(M,N)}(x)$ (which involves exponential moments of fourth moments of Brownian motions), we can still control exponential moments of $\int x d\hat{\mu}_t^{(M,N)}(x)$ (which relies on the exponential moments of square Brownian motion) which is enough because our eigenvalues are non-negative. In the same lines, one can check that the level sets of $S(\nu;\alpha)$ are relatively compact by controlling $\sup_{t\in[0,1]} \int x(1+\eta x)^{-1} d\nu_t(x)$ on the level sets of $S(\nu;\alpha)$ (note that $f_t(x) \equiv x(1+\eta x)^{-1} \in \tilde{\mathscr{C}}_b^{2,1}(\mathbb{R} \times [0,1])$) uniformly on the level sets of the rate function and conclude by monotone convergence theorem.

We deduce the following corollary from Theorem 3.2.

COROLLARY 3.1. *$\hat{\mu}^{(M,N)}$ converges almost surely toward the solution $\pi_\alpha(t)$ of the differential equation*

$$\pi_\alpha(t)(f) - \pi_\alpha(0)(f) = \int_0^t \pi_\alpha(s)(\partial_s f + \partial_x f)ds + \int_0^t \pi_\alpha(s) \otimes \pi_\alpha(s)(L_\alpha f)ds$$

for every test function f. $\pi_\alpha(t)$ is more explicitly given by the Pastur-Marchenko law given if $\alpha \le 1$ by

$$\pi_\alpha(t)(dx) = \pi_\alpha^1(t)(dx) = 1_{[t(\sqrt{\alpha}-1)^2, t(\sqrt{\alpha}+1)^2]} \frac{\sqrt{4\alpha t^2 - (x - t(1+\alpha))^2}}{2\pi\alpha t x} dx$$

and $\pi_\alpha(t)(dx) = \pi_\alpha^1(t)(dx) + (1-\alpha)\delta_0(dx)$ if $\alpha \ge 1$.

PROOF. Again, the only difficult point is to prove uniqueness for the above differential equation. We generalize the strategy followed in [25] to prove this point. Taking $f(x) = (1/z - x)$ for z with non-zero imaginary part and letting

$$G_t^\alpha(z) = \int \frac{1}{z-y} d\pi_\alpha(t)(y)$$

for a solution π_α of this equation, we find

$$\partial_t G_t^\alpha(z) = -\partial_z G_t^\alpha(z) + \alpha(\partial_z G_t^\alpha(z) - 2zG_t^\alpha(z)\partial_z G_t^\alpha(z) - G_t(z)^2).$$

The associated characteristic function satisfies

$$\partial_t z_t^\alpha = 2\alpha z_t^\alpha G_t^\alpha(z_t^\alpha) + (1-\alpha). \tag{3.1}$$

If the imaginary part y_0 of z_0^α is large enough, and say its real part x_0 belongs to $[0,1]$, there is a unique solution z^α to (3.1) and it stays away from the real axis. Indeed, since G^α is bounded Lipschitz away from the real axis, uniqueness of such a solution is clear by a standard Gronwall's argument. Hence, the only point is to prove existence of such a solution. Assume $y_0 \geq 0$ to simplify the notations. Considering the sequence constructed by $\partial_t z_t^{n+1} = 2\alpha z_t^n G_t^\alpha(z_t^n) + (1-\alpha)$, $z_0^{n+1} = z_0^\alpha$, $z_t^0 \equiv z_0^\alpha$, one finds that, if $z_t^n = x_t^n + iy_t^n$ and $\rho_t^n = (x_t^n/y_t^n)$, since $|\Re(G)(z)| \leq |2\Im(z)|^{-1}$ and $|\Im(G(z))| \leq |\Im(z)|^{-1}$,

$$1 - \alpha - \alpha|\rho_t^n| \leq \partial_t x_t^{n+1} \leq \alpha|\rho_t^n| + 1 + \alpha,$$
$$-2\alpha|\rho_t^n| - \alpha \leq \partial_t y_t^{n+1} \leq 2\alpha|\rho_t^n| + \alpha.$$

Letting $\rho_*^n = \sup_{t\in[0,1]} |\rho_t^n|$, we see that for ρ_0 small enough (i.e., y_0 large enough),

$$\frac{1-\alpha+x_0-\alpha\rho_*^n}{y_0+2\alpha\rho_*^n+\alpha} \leq \rho_*^{n+1} \leq \frac{1+x_0-\alpha-\alpha\rho_*^n}{y_0-2\alpha\rho_*^n-\alpha}$$

so that ρ_*^n remains in the interval $[0, \rho^*]$ with

$$\rho^* = \frac{1}{4\alpha}\left(-3\alpha + y_0 - \sqrt{(3\alpha - y_0)^2 - 8\alpha(1+\alpha+x_0)}\right).$$

As a consequence, y_t^n is bounded from below by $y^* = y_0 - 2\alpha\rho^* - \alpha > 0$. Since G^α is bounded Lipschitz in this region, Gronwall's lemma gives the convergence of the sequence toward a solution of our characteristic function living in $\{z \in \mathbb{C}; \mathrm{Im}(z) \geq y^*\}$. Now, $u_t \equiv G_t^\alpha(z_t^\alpha)$ satisfies

$$\partial_t u_t = -\alpha u_t^2 \qquad \text{with } u_0 = (1/z_0),$$

that is,

$$u_t = \frac{1}{z_0 + \alpha t}.$$

Plugging this result in (3.1) gives

$$\partial_t z_t^\alpha = 2\alpha \frac{z_t^\alpha}{z_0 + \alpha t} + (1-\alpha) \tag{3.2}$$

and therefore

$$z_t^\alpha = \left(1 + \frac{t}{z_0}\right)(z_0 + \alpha t). \tag{3.3}$$

Hence, z_0 is a root of the above second degree polynomial function such that $(z_0/\mathrm{Im}(z_t^\alpha))$ converges to one as $\mathrm{Im}(z_t^\alpha)$ goes to infinity. We find

$$z_0 = f_t(z_t^\alpha) = \frac{1}{2}\left(-(1+\alpha)t + z_t^\alpha + \sqrt{(-(1+\alpha)t + z_t^\alpha)^2 - 4\alpha t^2}\right).$$

so that $G_t^\alpha(z_t^\alpha) = (1/f_t(z_t^\alpha) + \alpha t)$ is uniquely determined on the values taken by z_t^α when z_0 has a sufficiently large imaginary part. This set obviously contains limit points. Since G^α is analytic on $\mathrm{Im}(z) > 0$ (see Theorem 10.7 in [26]), it is uniquely determined in $\mathrm{Im}(z) > 0$ (see Theorem 10.18 in [26]). Similarly, one can see that G_t^α is uniquely determined in $\mathrm{Im}(z) < 0$. Finally, Stieljes inversion formula shows that the uniqueness of G_t^α gives the uniqueness of $\pi_\alpha(t)$. □

3.3. *The unitary Brownian motion.* As we have seen in the previous sections, our techniques extend to every case where we can obtain an Itô formula for the spectral measure of the matrix-valued process under study. Here is a last case where such formulae will be proven; the case of the Unitary Brownian motion. It is defined by

DEFINITION 3.2. We will call Unitary Brownian motion the matrix-valued process $(U_N(t))_{t\in\mathbb{R}^+}$, with value in the set of $N \times N$ unitary matrices $\mathscr{U}_N$, and defined by the stochastic differential formula

$$dU_N(t) = i dH_N(t) U_N(t) - \tfrac{1}{2} U_N(t) dt$$

and $U_0 = I$ (see [5]).

Stochastic calculus can be developed as in Lemma 2.1 for this process and one finds that:

LEMMA 3.3. (i) *For every adapted matrix-valued processes A, B, C and D, we have*

$$\left\langle \int_0^\cdot A(s) dU_N(s) B(s), \int_0^\cdot C(s) dU_N(s) D(s) \right\rangle_t$$
$$= -\int_0^t \mathrm{tr}_N(U_N(t)B(t)C(t)) A(s) U_N(s) D(s) ds.$$

(ii) *For every polynomial function f, we have*

$$\begin{aligned} df(U_N(t)) = {} & i\,(D_0 f(U_N(t), U_N(t)) \times Id_N \otimes U_N(t))\,\sharp dH_N(t) \\ & -(Id_N \otimes \mathrm{tr}_N)\,(L_0 f(U_N(t), U_N(t)) \times U_N(t) \otimes U_N(t))\,dt \\ & -\tfrac{1}{2} U_N(t) \partial_z f(U_N(t)) dt \end{aligned}$$

(iii) *For every polynomial functions* (f, g), *we have, if* $\bar{g}(z) = g(\bar{z}) = g(1/z)$ *for* $|z| = 1$,

$$d\langle \mathrm{tr}_N f(U_N(.)), \mathrm{tr}_N \bar{g}(U_N(.))ds\rangle_t = -\frac{1}{N^2}\mathrm{tr}_N\big(U_N^2(t)\partial_z f(U_N(t))\partial_z \bar{g}(U_N(t))\big)dt$$

$$= \frac{1}{N^2}\mathrm{tr}_N\big(\partial_z f(U_N(t))\overline{\partial_z g}(U_N(t))\big)dt.$$

REMARK 3.1. Since $U_N^{-1} = U_N^*$, the conclusions of Lemma 3.3 remains valid for any polynomial functions f and g of (X^{-1}, X), that is any functions (f, g) of the form $\sum_{p=-n}^{n} a_p x^p$ for a finite integer number n and real numbers $(a_p)_{p\in\{-n,n\}}$.

Note that in fact, it is enough to consider real-valued test functions f on $\bar{s} = \{z = x + iy, x^2 + y^2 = 1\}$ since any complex-valued function can be decomposed into the sum of two real-valued such functions. In the following, we shall denote $\mathscr{C}_b^{2,1}(\bar{s}\times[0,1], \mathbb{R})$ the set of bounded twice space-continuously differentiable and time-continuously differentiable real-valued functions on $\bar{s}$. Note that for any $f \in \mathscr{C}_b^{2,1}(\bar{s}\times[0,1], \mathbb{R})$, $f(U_N(t)) \in \mathscr{H}_N$.

We can extend the conclusions of Lemma 3.3 as follows:

LEMMA 3.4. *For any functions* $(f, g) \in \mathscr{C}_b^{2,1}(\bar{s}\times[0,1], \mathbb{R})$, *the conclusions of Lemma* 3.3 *are valid*.

The proof of this result is even easier than that of Lemma 2.2 (since the random variables are uniformly bounded) and is left to the reader.

We can now define the rate function governing our large deviation upper bound; for $f \in \mathscr{C}_b^{2,1}(\bar{s}\times[0,1], \mathbb{R})$ and $\nu \in \mathscr{C}([0,1], \mathscr{P}(\bar{s}))$, we set

$$\begin{aligned} U^{s,t}(f,\nu) = {} & \int f(u,t)d\nu_t(u) - \int f(u,0)d\nu_0(u) - \int_0^t \int \partial_s f(u,s)d\nu_s(u)ds \\ & + \frac{1}{2}\int_0^t \int \left(uv\frac{\partial_z f(u,s) - \partial_z f(v,s)}{u-v}\right) d\nu_s(u)d\nu_s(v)ds \\ & + \frac{1}{2}\int_0^t \int u\partial_z f(u,s)d\nu_s(u)ds, \end{aligned}$$

and set, if $|\partial_z f|^2 = \partial_z f\overline{\partial_z f}$,

$$U^{0,1}(\nu) = \sup_{f\in\mathscr{C}^{2,1}(\bar{s}\times[0,1])} \left\{U^{0,1}(f,\nu) - \tfrac{1}{2}\textstyle\int_0^1 \int |\partial_z f(u,s)|^2 d\nu_s(u)du\right\}.$$

Then, we can prove as in the previous sections that:

THEOREM 3.3. *The law of the empirical process* $\bar{\nu}_{\cdot}^{(N)}$ *with values in* $\mathscr{C}([0,1], \mathscr{P}(\bar{s}))$ *satisfies a large deviation upper bound with good rate function which is infinite if* $\nu_0 \neq \delta_1$ *and otherwise equal to* $U^{0,1}(\nu)$.

As a consequence:

COROLLARY 3.2. *The empirical process $\bar{\nu}^{(N)}_{\cdot}$ converges almost surely to the unique solution in $\mathscr{C}([0,1],\mathscr{P}(\bar{s}))$ of*

$$\begin{aligned}\int f(u,t)d\nu_t(u) &= \int f(u,0)d\nu_0(u) + \int_0^t \int \partial_s f(u,s)d\nu_s(u)ds \\ &\quad - \frac{1}{2}\int_0^t \int \left(uv\frac{\partial_z f(u,s)-\partial_z f(s,v)}{u-v}\right)d\nu_s(u)d\nu_s(v)dt \\ &\quad - \frac{1}{2}\int_0^t \int u\partial_z f(u,s)d\nu_s(u)ds,\end{aligned} \tag{3.4}$$

for every test function $f \in \mathscr{C}_b^{2,1}(\bar{s}\times[0,1],\mathbb{R})$ and with $\nu_0 = \delta_1$.

PROOF. Again, the point is to show uniqueness of the solutions of this equation. We here proceed as in Lemma 2.6; take $f(u) = u^n$ with $n \in \mathbb{N}$. We find, if $m_t(n) = \int u^n d\nu_t(u)$ with a solution ν of (3.5), that it must satisfy

$$m_t(n) = 1 - \tfrac{1}{2}n\sum_{l=1}^{n-1}\int_0^t m_s(l)m_s(n-l)ds - \tfrac{1}{2}n\int_0^t m_s(n)ds. \tag{3.5}$$

Letting $\tilde{m}_t(n) = \int u^n d\tilde{\nu}_t(u)$ for another solution of (3.5) and denoting, for $n \in \mathbb{N}$,

$$\Delta_t(n) \equiv \sup_{m\le n}|m_t(m) - \tilde{m}_t(m)|,$$

we find, since $m_t(n)$ and $\tilde{m}_t(n)$ are uniformly bounded by 1, that

$$\Delta_t(n) \le (n^2+n)\int_0^t \Delta_s(n)ds$$

so that Gronwall's lemma shows that $\Delta_t(n)$ is identically null. Since the moments $(m_t(n))_{t\in[0,1]}$ characterize the compactly supported measure ν, we conclude to the uniqueness of the solutions of (3.5). This unique solution ν was already identified as the law of the free unitary Brownian motion in [5]. □

4. Large deviations for several matrices and Voiculescu's non commutative entropies. In this last part, our dynamical strategy will be the key to understand large deviations and related entropies for non-commutative random variables.

In the former part, we always consider only one matrix at a time, enabling ourselves to diagonalize this matrix and therefore to overlook most of the non-commutative framework of our matrices valued random variables.

Here, we shall consider several independent Hermitian Brownian motions and establish a large deviation upper bound for the process of the law (in a non-commutative sense) of the time marginals of these processes. One should remark that, since the spectral measure of one matrix defines its non commutative law, the previous large deviation results of Section 2 for the spectral measure can be seen as large deviations statements for the law (in the non-commutative sense) of the time marginals of a single Hermitian Brownian motion.

By the contraction principle, we get a large deviations upper bound for the law of several independent Wigner's matrices. The rate function for this large deviation result defines a natural entropy for the non-commutative law of several variables. We can compare this entropy with one of the two entropies defined by D. Voiculescu in [28] and [29]. To be able to define such an entropy and derive related large deviations results was the major object of our present research.

It is important to notice that in such a framework, there is no way to use any explicit formula for the law of the eigenvalues of each of the Hermitian Brownian motions since they can not be diagonalized simultaneously. In particular, the strategy followed in [2] (see also [3] and [20]) to prove a large deviation principle for the spectral measure of Wigner's matrices is useless when considering more than one matrix.

To motivate our readers, we shall first state our results somehow heuristically since they necessitate, to be properly set, the definition of a nice topology (and related test functions) on the space of the laws of several non-commutative variables. In fact, such a definition lacking in the actual non-commutative field, we shall detail it in Subsections 4.2 and 4.3. We will finally prove our results in Subsection 4.4.

4.1. *Statement of a large deviation result for the process of the joint law of the times marginals of several Hermitian Brownian motions and related results.* In the non-commutative formalism, the law μ_A of an operator A is usually defined by its moments, that is by $(\tau(A^p))_{p\in\mathbb{N}}$ where τ is the trace state of the underlying algebra. In other words, μ_A is the positive linear form on $\mathbb{C}[X]$ such that $\mu_A(x^n) = \tau(A^n)$ for every $n \in \mathbb{N}$. From this point of view, the law of the time marginals of the Hermitian Brownian motion H_N is described by the family $(\mathrm{tr}_N H_N^p(t))_{p\in\mathbb{N},t\in[0,1]}$. But the associated topology is not the weak topology. To obtain results for the spectral measure in the weak topology, we were driven in the previous part to consider instead the family $(\mathrm{tr}_N f(H_N(t)))_{f\in\mathscr{C}_b(\mathbb{R}),t\in[0,1]}$, that is to consider $\hat{\mu}_t^{(N)}$, $t \in [0,1]$, as a positive linear form on $\mathscr{C}_b(\mathbb{R})$ rather than on $\mathbb{C}[X]$.

Almost-surely,the non-commutative law of the time marginals of $(H_N^k, 1 \le k \le m)$ of m independent Hermitian Brownian motions is described by the family

$$\Big(\mathrm{tr}_N P\big(H_N^k(t), 1 \le k \le m\big)\Big)_{P\in\mathscr{P}(X_k, 1\le k\le m), t\in[0,1]}$$

where $\mathscr{P}(X_k, 1 \le k \le m)$ denotes the set of non-commutative polynomial functions of m variables. Again, this is not the right topology for our purpose. It is more appropriate to consider instead the topology generated by the non-commutative cylinder functions

$$F(X_k, 1 \le k \le m) = \overrightarrow{\prod_{1\le i\le n}} f_i\Big(\sum_{k=1}^m \alpha_i^k X_k\Big) \tag{4.1}$$

where $(f_i)_{i=1}^n$ belong to $\mathscr{C}_b(\mathbb{R})$ and $(\alpha_i^k, 1 \le k \le m)_{i=1}^n$ to $(\mathbb{R}^m)^n$. $\overrightarrow{\prod}$ denotes the non-commutative product. More precisely, we shall consider the non-commutative functions belonging to the complex vector space $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ generated by the set $ST(\mathbb{C})$ of non-commutative Stieljes transforms

$$F(X_k, 1 \le k \le m) = \overrightarrow{\prod_{1\le i\le n}} \frac{1}{z_i - \sum_{k=1}^m \alpha_i^k X} \tag{4.2}$$

where $(z_i)_{1\le i\le n}$ belong to $\mathbb{C}\backslash\mathbb{R}$ and $(\alpha_i^k, 1 \le k \le m)_{i=1}^n$ to $(\mathbb{R}^m)^n$. $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})$, the closure of $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ with respect to the uniform operator norm (see (4.15)), is a separable Banach space for this norm.

The non-commutative law of m non-commutative variables $(X_k, 1 \le k \le m)$ will be defined as elements of the algebraic dual $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})'$ of $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})$, with real-valued restriction to $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$, the real vector space of Hermitian valued functions of $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})$ (henceforth isomorphic to $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})'$), satisfying properties of positiveness, boundedness and total mass 1 which will be described in the next sections. We will denote $\overline{\mathscr{M}}_{=}^{1}$ this set.

The topology under study (called $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$-topology) is such that a family $(\tau_n)_{n\in\mathbb{N}}$ converges toward τ iff for every $f \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$,

$$\lim_{n\to\infty} \tau_n(f) = \tau(f).$$

$\overline{\mathscr{M}}_1^{=}$, equipped with this topology, is a compact metric space, hence a Polish space. This topology generalizes the usual topology inherited from compactly supported functions. To recover the usual weak topology, we shall add a tightness criterium under which our topology will be equivalent to that inherited from the test functions of (4.1). For this strengthened topology, the relatively compact subsets of $\overline{\mathscr{M}}_1^{=}$ will then be included in

$$\mathscr{K}_1^{=}(A) = \{\mu \in \overline{\mathscr{M}}_1^{=}; \max_{1\le k\le m} \mu(X_k^2) \le A\}$$

for some $A > 0$. We let $\mathscr{K}_1^{=}(\infty) = \bigcup_{A\in\mathbb{N}} \mathscr{K}_1^{=}(A)$.

We denote $\mathscr{C}\mathscr{C}_{st}(\mathbb{R}\times[0,1])$ the space of continuously differentiable $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$-valued non-commutative functions and $\mathscr{C}([0,1], \overline{\mathscr{M}}_1^{=})$ the set of continuous $\overline{\mathscr{M}}_1^{=}$-valued processes. The topology on the time variable remains the uniform topology.

To describe the rate function of our large deviation principle, let us first introduce the definitions of a few differential operators. For $l \in \{1, \dots, m\}$, we define the following extension D_{X_l} from $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ into $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})\otimes\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ of the differential operator D_0 defined in the previous section so that for any $n \in \mathbb{N}$, any $(z_i)_{1\le i\le n} \in (\mathbb{C}\backslash\mathbb{R})^n$, any $(\alpha_i^k, 1\le k\le m)_{1\le i\le n} \in (\mathbb{R}^m)^n$,

$$D_{X_l}\left(\prod_{1\le j\le n}^{\rightarrow} \frac{1}{z_j - \sum_{i=1}^m \alpha_j^i X_i}\right)$$
$$= \sum_{i=1}^n \alpha_i^l \left(\prod_{1\le j\le i}^{\rightarrow} \frac{1}{z_j - \sum_{i=1}^m \alpha_j^i X_i}\right) \otimes \left(\prod_{i\le j\le n}^{\rightarrow} \frac{1}{z_j - \sum_{i=1}^m \alpha_j^i X_i}\right)$$

Further, if m^* denotes the map from $\mathscr{C}\mathscr{C}_{st}(\mathbb{C}) \otimes \mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ into $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ so that for any $(F,G) \in (\mathscr{C}\mathscr{C}_{st}(\mathbb{C}))^2$, any $(X_k, 1\le k\le m) \in \mathscr{H}_N^m$, $N\in\mathbb{N}$,

$$m^*(F(X_k, 1\le k\le m) \otimes G(X_k, 1\le k\le m))$$
$$= G(X_k, 1\le k\le m) F(X_k, 1\le k\le m),$$

we set, for $l \in \{1,..,m\}$, $\mathscr{D}_{X_l}$ to be the cyclic derivative w.r.t X_l,

$$\mathscr{D}_{X_l} = m^* \circ D_{X_l}.$$

If $.^*$ denotes the natural extension of the involution on $(\mathscr{M}_N, N\in\mathbb{N})$ to $\mathscr{M} = \bigcup_{N\in\mathbb{N}} \mathscr{M}_N$, for any non-commutative Stieljes function $F \in ST(\mathbb{C})$, we let

$$\mathscr{D}_{X_l}^*\big(F(X_k, 1\le k\le m)\big) = \Big(\mathscr{D}_{X_l}\big(F(X_k, 1\le k\le m)\big)\Big)^*.$$

We are now in position to define our rate function. For $0\le s\le t\le 1$, $F, G \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R}\times[0,1])$, $\nu \in \mathscr{C}([0,1], \overline{\mathscr{M}_1^=})$, let

$$\overline{S}^{s,t}(\nu, F) = \nu_t(F_t) - \nu_s(F_s) - \int_s^t \nu_u(\partial_u F_u)du$$
$$-\tfrac{1}{2}\int_s^t \nu_u \otimes \nu_u \left(\sum_{l=1}^m D_{X_l} \circ \mathscr{D}_{X_l} F_u\right) du,$$

$$\langle\langle\langle F, F\rangle\rangle\rangle_\nu^{s,t} = \sum_{l=1}^m \int_s^t \nu_u(\mathscr{D}_{X_l} F_u \mathscr{D}_{X_l}^* G_u) du,$$

$$\overline{S}^{s,t}(\nu) = \sup_{F\in\mathscr{C}\mathscr{C}_{st}(\mathbb{R}\times[0,1])} \left(\overline{S}^{s,t}(\nu, F) - \tfrac{1}{2}\langle\langle\langle F, F\rangle\rangle\rangle_\nu^{s,t}\right).$$

Let $\delta_0^{\star m}$ be the non-commutative law such that $\delta_0^{\star m}(F(X_k, 1\le k\le m)) = \mathrm{tr}_1 F(0,\dots,0)$ for every $F\in\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$. We set

$$\overline{S}(\nu) = \begin{cases} +\infty, & \text{if } \nu_0 \ne \delta_0^{\star m} \text{or } \nu \notin \mathscr{C}([0,1], \overline{\mathscr{M}_1^=}),\\ \overline{S}^{0,1}(\nu), & \text{otherwise.}\end{cases}$$

Then, we shall prove:

THEOREM 4.1. *Let $\hat\mu^{(N)}$ be the continuous $\overline{\mathscr{M}_1^=}$-valued process so that for any $F\in\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$, we have $\hat\mu_t^{(N)}(F) = \mathrm{tr}_N(F(H_N^k(t), 1\le k\le m))$; the law $\hat\mu^{(N)}$ satisfies a large deviation upper bound in the scale N^2 with good rate function*

$\overline{S}$. *More precisely*:

1. *The level sets of* $\overline{S}$ *are compact in* $\mathscr{C}([0,1],\overline{\mathscr{M}_1^{=}})$.
2. *For any* $M \in \mathbb{R}^+$, *there exists* $A \in \mathbb{R}^+$ *so that the level set* $E_M \equiv \{\mu \in \mathscr{C}([0,1],\overline{\mathscr{M}_1^{=}}), \bar{S}(\mu) \le M\}$ *satisfies*
$$E_M \subset \mathscr{C}([0,1],\mathscr{K}_1^{=}(A)) \equiv \left\{\mu \in \mathscr{C}([0,1],\overline{\mathscr{M}_1^{=}}), \mu_t \in \mathscr{K}_1^{=}(A),\ \forall t \in [0,1]\right\}.$$
Hence,
$$\left\{\mu \in \mathscr{C}\left([0,1],\overline{\mathscr{M}_1^{=}}\right), \bar{S}(\mu) < \infty\right\} \subset \bigcup_{A\in\mathbb{N}} \mathscr{C}([0,1],\mathscr{K}_1^{=}(A)).$$
3. *For any closed subset* F *of* $\mathscr{C}([0,1],\overline{\mathscr{M}_1^{=}})$,
$$\limsup_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}\left(\hat{\mu}^{(N)} \in F\right) \le -\inf_{\nu\in F} \overline{S}(\nu)$$

If $\overline{\overline{\mathscr{C}\mathscr{C}}}_{st}(\mathbb{R})$ is the set of non-commutative functions F mapping $\mathscr{H}_N \times \mathscr{H}_N$ into $\mathscr{H}_N$ for any $N \in \mathbb{N}$ and so that there exists a family $(F_n)_{n\in\mathbb{N}} \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})$, so that

$$|F - F_n|(X_k, 1 \le k \le m) \le \frac{1}{n}\left(\sum_{k=1}^m X_k^2 + 1\right)$$

where $|F| = \sqrt{FF^*}$, $\le$ is to be understood in the sense of quadratic forms, then, Theorem 4.1 can be strengthened to the $\overline{\overline{\mathscr{C}\mathscr{C}}}_{st}(\mathbb{R})$-topology as follows:

COROLLARY 4.1. (i) *For any integer number* n, *any times* $(t_j)_{j=1}^n \in [0,1]^n$, *any family* $(F_j)_{j=1}^n \in \overline{\overline{\mathscr{C}\mathscr{C}}}_{st}(\mathbb{R})$, *for any real constants* $(a_j, b_j, a_j \le b_j)_{j=1}^n$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}\left(\bigcap_{j=1}^m \{\mathrm{tr}_N\left(F_j(H_N^k(t_j), 1 \le k \le n)\right) \in [a_j, b_j]\}\right)$$
$$\le -\inf\left\{\overline{S}(\nu)\ :\ \nu_{t_j}\left(F_j(X_k, 1 \le k \le m)\right) \in [a_j, b_j]\ \forall j \in \{1,\dots,n\}\right\}$$

(ii) *For any function* $F \in \overline{\overline{\mathscr{C}\mathscr{C}}}_{st}(\mathbb{R})$, *the law* $\hat{\mu}_F^{(N)}$ *of the spectral measure of* $(F(H_N^k(t), 1 \le k \le m))_{t\in[0,1]} \in \mathscr{C}([0,1],\mathscr{P}(\mathbb{R}))$ *satisfies a large deviation upper bound in the scale* N^2 *with good rate function*

$$\mathscr{S}_F(\nu) = \inf\left\{\overline{S}(\tilde{\nu}), \nu = \tilde{\nu} \circ F^{-1}\right\}$$

where $\tilde{\nu} \circ F^{-1}$ *is the measure valued-process so that, for any function* $g \in \mathscr{C}_b(\mathbb{R})$, *for any time* $t \in [0,1]$,

$$\tilde{\nu}_t \circ F^{-1}(g) \equiv \tilde{\nu}_t(g(F)).$$

PROOF. Corollary 4.1 is a consequence of the contraction principle; indeed, by Property 4.2, Theorem 4.1 and Theorem 4.2.1 of [14], the spectral measure $\hat{\mu}_F^{(N)}$ of $(F(H_N^k(t), 1 \le k \le m))_{t\in[0,1]}$ satisfies a large deviations upper bound with good rate function $\mathscr{S}_F$ for any $F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})$. Further, by Lemma 4.2, the laws of $(\hat{\mu}^{(N)}_{F_n})_{n\in\mathbb{N}}$ are exponentially good approximations of the law of $\hat{\mu}^{(N)}_F$ since for any $\delta > 0$, if $\mathscr{D}$ is given by (1.5),

$$\mathbb{P}\left(\mathscr{D}(\hat{\mu}^{(N)}_{F_n}, \hat{\mu}^{(N)}_F) > \delta\right) \le \mathbb{P}\left(\sup_{t\in[0,1]} \mathrm{tr}_N(\sum_{1\le k\le m} (H^k_N(t))^2 + 1) \ge n\delta\right)$$

so that (2.10) and Chebishev's inequality result with

$$\limsup_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\mathscr{D}(\hat{\mu}^{(N)}_{F_n}, \hat{\mu}^{(N)}_F) > \delta\right) = -\infty$$

for all $\delta > 0$. Thus, Theorem 4.1 (2) and Theorem 4.2.23 of [14] show that the law of $\hat{\mu}^{(N)}_F$ satisfies a large deviations upper bound with rate function $\mathscr{S}_F$ for the weak topology. This completes the proof of the second point. For the first point, we can proceed similarly by noticing first that if the functions F_j belong to $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$, the contraction principle immediately implies the result and that the extension can be obtained by exponential approximations. □

Properties 1.3 and 1.1 are direct consequences of Corollary 4.1.

Here, we considered Hermitian Brownian motions. However, it is not hard to generalize the results to symmetric Brownian motions, Wishart processes or Unitary Brownian motions following Section 3. Also, viewing unitary matrices as functions of Wigner's matrices we can deduce a large deviation upper bound for the non-commutative law of independent unitary matrices following Haar measure on the unitary ensemble. More precisely, let $(U^N_1, \dots, U^N_m)$ be m $N \times N$ i.i.d matrices uniformly distributed on $U(N)$. $(U^N_1, \dots, U^N_m)$ has the same law as

$$\begin{aligned} &F\left((H^N_l(1))_{1\le l\le 2m}\right) \\ &\quad = \left(|H^N_{2l-1}(1) + iH^N_{2l}(1)|^{-1}\left(H^N_{2l-1}(1) + iH^N_{2l}(1)\right)\right)_{1\le l\le m} \end{aligned} \tag{4.3}$$

with $(H^N_l(t), 1 \le l \le 2m, t \in [0,1])$ i.i.d Hermitian Brownian motions and

$$\begin{aligned} &|H^N_{2l-1}(1) + iH^N_{2l}(1)| \\ &\quad = \left((H^N_{2l-1}(1) + iH^N_{2l}(1))(H^N_{2l-1}(1) + iH^N_{2l}(1))^*\right)^{\frac{1}{2}} \\ &\quad = \left(H^N_{2l-1}(1)^2 + H^N_{2l}(1)^2 + i(H^N_{2l}(1)H^N_{2l-1}(1) - H^N_{2l-1}(1)H^N_{2l}(1))\right)^{\frac{1}{2}}. \end{aligned}$$

As a consequence, if we consider the joint law $\bar{\mu}^{(N)}$ of $(U^N_1, \dots, U^N_m)$ as an element of the topological dual $\mathscr{P}(U_l, 1 \le l \le m)^*$ of the set $\mathscr{P}(U_l, 1 \le l \le m)$ of non-commutative polynomial functions of m variables,

$$\bar{\mu}^{(N)}(P) = \mathrm{tr}_N P(U^N_1, \dots, U^N_m), \qquad P \in \mathscr{P}(U_l, 1 \le l \le m),$$

furnished with the weak topology compatible with the distance

$$\hat{\mathscr{D}}(\mu, \nu) = \sum_{k\in\mathbb{N}} \frac{1}{2^k} \sup_{\sigma:\{1,\dots,k\}\to\{1,\dots,m\}^k} \left| \mu\left(\overrightarrow{\prod_{1\le i\le k}} U_{\sigma(i)}\right) - \nu\left(\overrightarrow{\prod_{1\le i\le k}} U_{\sigma(i)}\right) \right|,$$

Theorem 4.1 results with:

COROLLARY 4.2. *There exists a good rate function* $\bar{\mathscr{U}}$ *on* $\mathscr{P}(U_l, 1 \le l \le m)^*$ *so that for any closed subset* F *of* $\mathscr{P}(U_l, 1 \le l \le m)^*$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\bar{\mu}^{(N)} \in F\right) \le -\inf_F \bar{\mathscr{U}}.$$

If, for $n \in \mathbb{N}$, *we let*

$$F_n(X_l, 1 \le l \le 2m) = \left(\left(|X_{2l-1} + iX_{2l}|^2 + \frac{1}{n}\right)^{-\frac{1}{2}} (X_{2l-1} + iX_{2l})\right)_{1\le l\le m},$$

$\bar{\mathscr{U}}$ *is given by*

$$\bar{\mathscr{U}}(\mu) = \liminf_{n\to\infty} \inf\left\{\overline{S}(\nu), \nu_1 \circ F_n^{-1} = \mu\right\}.$$

Here, $\nu \circ F_n^{-1}$ denotes the image by F_n of the non commutative measure ν given, for any polynomial function P of m non-commutative variables, by $\nu\circ F_n^{-1}(P) = \nu(P(F_n))$. The proof of this corollary is provided in Subsection 4.5.

Another corollary of Theorem 4.1 gives the convergence of the non-commutative law of $(H_N^k, 1 \le k \le m)$

PROPERTY 4.1. *The non-commutative law of* $(H_N^k, 1 \le k \le m)$ *converges almost surely toward the law of* m *free Brownian motions, unique solution in* $\mathscr{C}([0,1], \mathscr{K}_1^{=}(\infty))$ *of the differential equation*

$$\nu_t(F_t) = \nu_s(F_s) + \int_s^t \nu_u(\partial_u F_u) du + \tfrac{1}{2}\int_s^t \nu_u \otimes \nu_u \left(\sum_{l=1}^m D_{X_l} \circ \mathscr{D}_{X_l} F_u\right) du$$

for any $F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R} \times [0,1])$, *any* $0 \le s \le t \le 1$, *with initial data* $\delta_0^{\star m}$.

This property is proven in Section 4.6.

Another application of the contraction principle and of Theorem 4.1 is to obtain large deviations estimates for the time marginals of our process. Namely:

COROLLARY 4.3. *For any time* $t \in [0,1]$, *the law, in the non-commutative sense, of* $(H_N^k(t), 1 \le k \le m)$ *satisfies a large deviation upper bound in the scale* N^2 *with good rate function given, for any* $\nu \in \overline{\mathscr{M}_1^{=}}$ *by*

$$\overline{S}_t(\nu) = \inf\{\overline{S}(\tilde{\nu}), \tilde{\nu}_t = \nu\}.$$

We shall define:

DEFINITION 4.1. We shall call non-commutative entropy of order II the function on $\overline{\mathscr{M}_1^{=}}$ given by

$$\bar{I}(\mu) = \inf_{\nu\in\mathscr{C}([0,1],\mathscr{K}_1^{=}(\infty))} \{\overline{S}(\nu), \nu_1 = \mu\}.$$

This entropy can in fact be compared with the second entropy defined by Voiculescu (see [29]) up to a proper interpretation of our entropy in his formalism. Let's be given a non-commutative probability space $(\mathscr{A}, \tau)$, that is a von Neumann algebra $\mathscr{A}$ endowed with a trace state τ, and a family $(A_1, \ldots, A_m)$ of self-adjoint elements in $\mathscr{A}$. The associated law μ is then defined as the element of $\mathscr{M}_1^{=}$ such that

$$\forall F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R}),\ \mu(F) = \tau\left(F(A_1, \ldots, A_m)\right).$$

The definition of the free information of μ given by Voiculescu is based on the adjoints $(\mathscr{J}^{\mu}_{X_l}, 1 \le l \le m)$ in $L^2(\mu)$ of D_{X_l}, $l \in \{1, \ldots, m\}$, satisfying, if they exist, for any non-commutative polynomial function F,

$$\mu \otimes \mu(D_{X_l} F) = \mu(F \mathscr{J}^{\mu}_{X_l}), \qquad 1 \le l \le m.$$

The free information is then infinite if $(\mathscr{J}^{\mu}_{X_l}, 1 \le l \le m)$ are not well-defined and otherwise given by

$$\Phi^*(A_1, \ldots, A_m) = \sum_{l=1}^{m} \mu(|\mathscr{J}^{\mu}_{X_l}|^2) \equiv \Phi^*(\mu).$$

Replacing the polynomial functions with non-commutative functionals of $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$, we have

$$\Phi^*(\mu) \ge 2 \sup_{F_l \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})} \left\{ \mu \otimes \mu\left(\sum_{l=1}^{m} D_{X_l} F_l \right) - \frac{1}{2} \mu\left(\sum_{l=1}^{m} F_l^2 \right) \right\}$$

and equality if we further assume that the $(\mathscr{J}^{\mu}_{X_l}, 1 \le l \le m)$ can be approximated by functions of $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$. To define Voiculescu's free entropy χ^*, we need to introduce first the following map on $\overline{\mathscr{M}_1^{=}}$; for $F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})$, $t \in [0, 1]$, we set

$$\mu_t^b(F) = \tau\left(F(tA_l + \sqrt{t(1-t)} S_l, 1 \le l \le m) \right)$$

where $(S_l, 1 \le l \le m)$ are free semicircular variables, free with $(A_l, 1 \le l \le m)$. $(\mu_t^b)_{t \in [0,1]}$ is the law of a m-dimensional free Brownian bridge between $\delta_0^{\star m}$ and μ (that is a non commutative process with initial law $\delta_0^{\star m}$ and law μ at time one). Then, we let

$$\chi^*(\mu) = \frac{1}{2} \int_0^1 \left(\frac{m}{t} - \Phi^*(\mu_t^b) \right) dt + \frac{m}{2} \ln(2\pi e).$$

The reader can check that this corresponds to the more common definition given by Voiculescu in [29]:

$$\chi^*(\mu) = \frac{1}{2} \int_0^\infty \left(\frac{m}{1+t} - \Phi^*(X_1 + \sqrt{t} S_1, \ldots, X_m + \sqrt{t} S_m) \right) dt + \frac{m}{2} \ln(2\pi e).$$

by changing the time variable $t = (1+s)^{-1}$ and recalling that for any $\lambda \in \mathbb{R}$, $\Phi^*(\lambda A_1, \ldots, \lambda A_m) = \lambda^{-2} \Phi^*(A_1, \ldots, A_m)$.

In comparison, if we consider our entropy $\bar{I}$, then, we first perform an infimum over the processes with marginals $\delta_0^{\star m}$ at time 0 and μ at time 1. Because it is natural that such an infimum should be taken at the free Brownian bridge, we bound $\bar{I}$ by

$$\bar{I}(\mu) \le \bar{S}(\mu^b). \tag{4.4}$$

One can check that μ^b verifies the differential equation

$$\begin{aligned} \mu_t^b(F_t) - \mu_s^b(F_s) = &\int_s^t \mu_u^b(\partial_u F_u)du \\ &-\frac{1}{2}\int_s^t \mu_u^b \otimes \mu_u^b \left(\left(\sum_{l=1}^m D_{X_l} \circ \mathscr{D}_{X_l}\right) F_u\right) du \\ &+\int_s^t \mu_u^b \left(\sum_{l=1}^m \frac{X_l}{u} \mathscr{D}_{X_l} F_u\right) du \end{aligned} \tag{4.5}$$

for every test functions $F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R} \times [0, 1])$ and $\mu_0 = \delta_0^{\star m}$, $\mu_1 = \mu$. With (4.4), we find

$$\bar{I}(\mu) \le \int_0^1 \frac{\bar{J}(\tilde{\mu}_u^b)}{u} du \tag{4.6}$$

where $\tilde{\mu}_u^b$ is the image of μ_u^b by the homothety of ratio $(1/\sqrt{u})$ and

$$\bar{J}(\mu) = \sup_{F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})} \left\{ \mu \otimes \mu \left(\sum_{l=1}^m D_{X_l} \circ \mathscr{D}_{X_l} F\right) - \mu\left(\sum_{l=1}^m X_l \mathscr{D}_{X_l} F\right) - \tfrac{1}{2}\sum_{l=1}^m \mu(|\mathscr{D}_{X_l}, F|^2)\right\}.$$

$\bar{J}$ stands for our definition of the free Wigner's information. It is straightforward, by the translation $F(X_l, 1 \le l \le m) \to F(X_l, 1 \le l \le m) - (1/2)\sum_{l=1}^m X_l^2$ [which is possible even though $\sum_{l=1}^m X_l^2$ are not bounded because $\mu(\sum_{l=1}^m X_l^2) = \tau(\sum_{l=1}^m A_l^2)$ is finite] that

$$\begin{aligned} \bar{J}(\mu) &= \sup_{F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})} \left\{ \mu \otimes \mu \left(\sum_{l=1}^m D_{X_l} \circ \mathscr{D}_{X_l} F\right) - \tfrac{1}{2}\sum_{l=1}^m \mu(|\mathscr{D}_{X_l} F|^2)\right\} \\ &\quad + \tfrac{1}{2}\mu\left(\sum_{l=1}^m X_l^2\right) - \frac{m}{2} \\ &\le \tfrac{1}{2}(\Phi^*(\mu) - m) + \tfrac{1}{2}\sum_{l=1}^m (\mu(X_l^2) - 1). \end{aligned} \tag{4.7}$$

where the second term naturally shows up since we are considering the Gaussian ensemble. Because, in view of [29], page 214, $\Phi^*(\tilde{\mu}_t^b) = t\Phi^*(\mu_t^b)$, we conclude that, up to a Gaussian term, our entropy is dominated by Voiculescu's. More precisely,

$$\bar{I}(\mu) \le -\chi^*(\mu) + \frac{1}{2}\sum_{l=1}^m (\mu(X_l^2) - 1) + \frac{m}{2}\log(2\pi e).$$

Let us finally remark that we believe (4.6) to be an equality but that we have not make our mind whether (4.7) should be an equality or not in general, which amounts to wonder if $(\mathcal{J}^{\mu}_{X_l}, 1 \le l \le m)$ can be approximated by functions of the cyclic gradient space $\{(\mathcal{D}_{X_l} F, 1 \le l \le m), F \in \mathcal{C}\mathcal{C}_{st}(\mathbb{R})\}$. Two very recent works suggest a positive answer; D. Voiculescu [30] has shown that the non-commutative Hilbert transform $(\mathcal{J}^{\mu}_{X_l}, 1 \le l \le m)$ is in the cyclic gradient space as soon as it is polynomial and we could show [9] that any law μ with finite χ^* entropy can be approximated by laws μ^n with non-commutative Hilbert transform in the cyclic gradient space. A more detailed study of the relations between χ^* and $\bar{I}$ is actually the subject of the work in progress [9].

4.2. *Definitions and Itô's formulæ.* In this section, we introduce our sets of Stieltjes test functions and precise the definitions of the differential operators introduced in the previous part. We then provide an Itô's formula.

Let us first give a few definitions. We shall consider the following spaces

$$\mathcal{M} = \bigcup_{N\in\mathbb{N}} \mathcal{M}_N \qquad \mathcal{H} = \bigcup_{N\in\mathbb{N}} \mathcal{H}_N \qquad \mathcal{H}_{\times m} = \bigcup_{N\in\mathbb{N}} \mathcal{H}_N^m$$

$\mathcal{M}$ is furnished with the involution *, extension of the usual involution on each $\mathcal{M}_N$, $N \in \mathbb{N}$. The set of non-commutative functions $\mathcal{E}(\mathbb{C})$ is the subset of the functions $F : \mathcal{H}_{\times m} \to \mathcal{M}$ such that for any $N \in \mathbb{N}$, for any $(A_l, 1 \le l \le m) \in \mathcal{H}_N$, $F(A_l, 1 \le l \le m)$ belongs to the algebra generated by $(A_l, 1 \le l \le m)$. In particular, $F(\mathcal{H}_N^m) \subset \mathcal{M}_N$.

REMARK 4.1. In fact, in view of the following, we could consider more generally $\mathcal{E}(\mathbb{C})$ as functions on von Neumann algebras so that for any von Neumann algebra M, any $(A_l, 1 \le l \le m) \in M^m$, $F(A_l, 1 \le l \le m)$ belongs to the algebra generated by $(A_l, 1 \le l \le m)$, and thus in particular to M. Furnishing von Neumann algebras with an involution * and an operator norm $\|\ \|_\infty$, we can extend the rest of this article to this set up. This approach is more common in free probability. However, we found that it might confuse the reader unnecessarily in this paper.

We shall denote $\mathcal{E}(\mathbb{R})$ the subset of $\mathcal{E}(\mathbb{C})$ of Hermitian matrices-valued functions. $(X_l, 1 \le l \le m)$ will denote the canonical coordinates in $\mathcal{H}_{\times m}$ and $(A_l, 1 \le l \le m)$ some element of $\mathcal{H}_{\times m}$.

REMARKS 4.1.

1. If f is a real function, we can define the functions
$$F_l(A_l, 1 \le l \le m) = f(A_l), \qquad 1 \le l \le m.$$
It is straightforward that F_l, $1 \le l \le m$, belong to $\mathcal{E}(\mathbb{R})$.
2. It is not hard to verify that $\mathcal{E}(\mathbb{C})$ is a non-commutative algebra, but that $\mathcal{E}(\mathbb{R})$ is not an algebra.
3. There exists a partial order on $\mathcal{E}(\mathbb{R})$. If $(F, G) \in \mathcal{E}(\mathbb{R})$, $F \le G$ iff $\forall N \in \mathbb{N}$, $\forall (A_l)_{1\le l\le m} \in \mathcal{H}_N^m$,
$$G(A_l, 1 \le l \le m) - F(A_l, 1 \le l \le m) \text{ is a non negative matrix of } \mathcal{H}_N.$$

4. Remark that any function $F \in \mathscr{E}(\mathbb{C})$ can be decomposed into the sum $F_1 + iF_2$ of two functions $(F_1, F_2) \in \mathscr{E}(\mathbb{R})$ with $F_1 = \frac{1}{2}(F+F^*)$ and $F_2 = \frac{1}{2}(-iF + iF^*)$. Here, F^* is defined by $F^*(A_l, 1 \le l \le m) = (F(A_l, 1 \le l \le m))^*$ for any $(A_l, 1 \le l \le m) \in \mathscr{H}_N^m$, $N \in \mathbb{N}$.

For any integer number $n \in \mathbb{N}$, any real numbers $\underline{\alpha} = (\alpha_p)_{1 \le p \le n} \in (\mathbb{R}^m)^n$, and any complex numbers $\underline{z} = (z_k)_{1 \le p \le n}$ with non zero imaginary part, we introduce the non commutative Stieljes function

$$\Phi_{\underline{\alpha},\underline{z}}(X_l, 1 \le l \le m) \equiv \overrightarrow{\prod_{1 \le p \le n}} \frac{1}{z_p - \sum_{l=1}^m \alpha_p^l X_l}$$

and let

$$ST(\mathbb{C}) \equiv \{\Phi_{\underline{\alpha},\underline{z}}; (\underline{\alpha}, \underline{z}) \in (\mathbb{R}^m \times (\mathbb{C}\backslash\mathbb{R}))^n, n \in \mathbb{N}\} \subset \mathscr{E}(\mathbb{C})$$

be the non-commutative Stieljes basis. We let $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ be the complex vector space generated by $ST(\mathbb{C})$;

$$\mathscr{C}\mathscr{C}_{st}(\mathbb{C}) \equiv \left\{ \sum_{i=1}^p t_j F_j; (t_j)_{1 \le j \le p} \in \mathbb{C}^p, (F_j)_{1 \le j \le p} \in ST(\mathbb{C})^p, p \in \mathbb{N} \right\}.$$

$\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ contains the identity.

To extend the definitions of the differential operators encountered in the previous section to $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$, we first describe the notion of non-commutative Stieljes bi-functions. Note first that for any $(F, G) \in \mathscr{E}(\mathbb{C})$, $F \otimes G$ is well-defined as the function on $\mathscr{H}_{\times m}$ so that for any $N \in \mathbb{N}$, any $(A_l, 1 \le l \le m) \in \mathscr{H}_N^m$,

$$F \otimes G(A_l, 1 \le l \le m) = F(A_l, 1 \le l \le m) \otimes G(A_l, 1 \le l \le m)$$

where $F(A_l, 1 \le l \le m) \otimes G(A_l, 1 \le l \le m)$ stands for the standard tensor product in $\mathscr{M}_N \otimes \mathscr{M}_N$. We can therefore set

$$ST \otimes ST(\mathbb{C}) \equiv \Big\{ \Phi_{\underline{\alpha},\underline{z}} \otimes \Phi_{\underline{\tilde{\alpha}},\underline{\tilde{z}}}; (\underline{\alpha}, \underline{z}) \in (\mathbb{R}^m \times (\mathbb{C}\backslash\mathbb{R}))^n,$$
$$(\underline{\tilde{\alpha}}, \underline{\tilde{z}}) \in (\mathbb{R}^m \times (\mathbb{C}\backslash\mathbb{R}))^{\tilde{n}}, (n, \tilde{n}) \in \mathbb{N} \Big\}.$$

We denote $\mathscr{C}\mathscr{C}_{st} \otimes \mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ the complex vector space generated by $ST \otimes ST(\mathbb{C})$. We can now describe the following extensions of the derivations appearing in the previous section; we set D_{X_l}, $l \in \{1, \dots, m\}$ to be the linear operator on $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ satisfying for every $F, G \in ST(\mathbb{C})$ and $f(x) = (1/z - x)$, $z \in \mathbb{C}\backslash\mathbb{R}$, $(\alpha_k, 1 \le k \le m) \in \mathbb{R}^m$, any $(A_k, 1 \le k \le m) \in \mathscr{H}_{\times m}$ by

$$D_{X_l}(FG) = D_{X_l}(F) 1 \otimes G + F \otimes 1 D_{X_l}(G),$$

$$D_{X_l}\left(f\left(\sum_{k=1}^m \alpha_k X_k\right)\right)(A_k, 1 \le k \le m) = \alpha_l (D_0 f)\left(\sum_{k=1}^m \alpha_k A_k, \sum_{k=1}^m \alpha_k A_k\right) \tag{4.8}$$

where D_0 was defined in Section 2. More explicitly,

$$D_{X_l}\left(f\left(\sum_{k=1}^m \alpha_k X_k\right)\right)(A_k, 1\le k\le m) = \alpha_l \frac{1}{z-\sum_{k=1}^m \alpha_k A_k}\otimes \frac{1}{z-\sum_{k=1}^m \alpha_k A_k}. \tag{4.9}$$

From (4.8) and (4.9), it is straightforward that for any $n\in\mathbb{N}$, any $(\underline{\alpha},\underline{z})\in(\mathbb{R}^m\times(\mathbb{C}\backslash\mathbb{R}))^n$,

$$\begin{aligned} D_{X_l}\Phi_{\underline{\alpha},\underline{z}}(A_k, 1\le k\le m) &= \sum_{j=1}^n \alpha_j^l \Phi_{\underline{\alpha}_j,\underline{z}_j}(A_k,1\le k\le m)\Phi_{\alpha_j,z_j}(A_k,1\le k\le m)\\ &\quad \otimes \Phi_{\alpha_j,z_j}(A_k,1\le k\le m)\Phi_{\underline{\alpha}^j,\underline{z}^j}(A_k,1\le k\le m)\end{aligned} \tag{4.10}$$

with $(\underline{\alpha}_j,\underline{z}_j)=(\alpha_k,z_k)_{k<j}$ (resp. $(\underline{\alpha}^j,\underline{z}^j)=(\alpha_k,z_k)_{k>j}$). In particular, for any $l\in\{1,\dots,m\}$, $D_{X_l}(\mathscr{C}\mathscr{C}_{st}(\mathbb{C}))\subset\mathscr{C}\mathscr{C}_{st}\otimes\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$. Further, if m^* denotes the linear map from $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})\otimes\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ into $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ so that for any $(F,G)\in(ST(\mathbb{C}))^2$, any $(A_k,1\le k\le m)\in\mathscr{H}_{\times m}$,

$$m^*(F\otimes G)(A_k,1\le k\le m)=G(A_k,1\le k\le m)F(A_k,1\le k\le m),$$

we let the cyclic derivative $\mathscr{D}_{X_l}$, $l\in\{1,\dots,m\}$, be the linear operator from $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ into $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ given by

$$\mathscr{D}_{X_l}\equiv m^*\circ D_{X_l}. \tag{4.11}$$

More precisely, for any $n\in\mathbb{N}$, $(\underline{\alpha},\underline{z})\in(\mathbb{R}^m,(\mathbb{C}\backslash\mathbb{R}))^n$, (4.11) yields

$$\mathscr{D}_{X_l}\Phi_{\underline{\alpha},\underline{z}}(A_k,1\le k\le m)=\sum_{j=1}^n \alpha_j^l\Phi_{\underline{\alpha}(j),\underline{z}(j)}(A,B) \tag{4.12}$$

with $(\underline{\alpha}(j),\underline{z}(j))\in(\mathbb{R}^m,(\mathbb{C}\backslash\mathbb{R}))^{n+1}$ given by

$$\begin{aligned}(\underline{\alpha}(j),\underline{z}(j))_p &= (\alpha_{j+p-1},z_{j+p-1}) \quad p\in[1,n-j+1]\\ &= (\alpha_{p-n+j-1},z_{p-n+j-1}) \quad p\in[n-j+2,n+1].\end{aligned}$$

In the sequel, we shall denote $(\underline{\alpha}(j),\underline{z}(j))_l=(\underline{\alpha},\underline{z})_{\sigma_j(l)}$ for a map σ_j from $\{1,\dots,n\}$ into $\{1,\dots,n+1\}$, $n\in\mathbb{N}$, defined by the above formulae. Note that (4.12) implies the stability property $\mathscr{D}_{X_l}(\mathscr{C}\mathscr{C}_{st}(\mathbb{C}))\subset\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$, $1\le l\le m$. We finally define, for $l\in\{1,\dots,m\}$, the linear second order operator L_{X_l} from $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ into $\mathscr{C}\mathscr{C}_{st}\otimes\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ by

$$L_{X_l}\equiv \tfrac{1}{2}D_{X_l}\circ\mathscr{D}_{X_l}. \tag{4.13}$$

L_{X_l} can also be defined as

$$L_{X_l}=\tfrac{1}{2}\tilde{m}^*\big((1\otimes D_{X_l}+D_{X_l}\otimes 1)\circ D_{X_l}\big)$$

if $\tilde{m}^*$ is the linear map from $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})^{\otimes 3}$ into $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})^{\otimes 2}$ so that for any $(F,G,H) \in \mathscr{C}_{st}(\mathbb{C})^3$,

$$\tilde{m}^*(F\otimes G\otimes H)=G\otimes(HF).$$

We also have the more explicit formula for any integer number $n\in\mathbb{N}$, any $(\underline{\alpha},\underline{z})\in(\mathbb{R}^m,(\mathbb{C}\backslash\mathbb{R}))^n$, any $(A_k,1\le k\le m)\in\mathscr{H}_{\times m}$,

$$\begin{aligned} &L_{X_l}\Phi_{\underline{\alpha},\underline{z}}(A_k,1\le k\le m)\\ &\quad=\sum_{j=1}^{n}\sum_{l=1}^{n+1}\alpha_j^l\alpha_{\sigma_j(l)}\Phi_{(\underline{\alpha}_{\sigma_j})_l,(\underline{z}_{\sigma_j})_l}\Phi_{\alpha_{\sigma_j(l)},z_{\sigma_j(l)}}(A_k,1\le k\le m)\\ &\qquad\otimes\Phi_{\alpha_{\sigma_j(l)},z_{\sigma_j(l)}}\Phi_{(\underline{\alpha}_{\sigma_j})^l,(\underline{z}_{\sigma_j})^l}(A_k,1\le k\le m).\end{aligned}\tag{4.14}$$

We can define the adjoint of $\mathscr{D}_{X_l}$ so that for any $F\in\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ and any $(A_k,1\le k\le m)\in\mathscr{H}_{\times m}$,

$$\mathscr{D}_{X_l}^*F(A_k,1\le k\le m)\equiv(\mathscr{D}_{X_l}F(A_k,1\le k\le m))^*.$$

We shall note, for any $F\in\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$, $l\in\{1,\dots,m\}$,

$$|\mathscr{D}_{X_l}F|^2\equiv\mathscr{D}_{X_l}F\mathscr{D}_{X_l}^*F.$$

Note for further purpose that $|\mathscr{D}_{X_l}(\mathscr{C}\mathscr{C}_{st}(\mathbb{C}))|^2\subset\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$.

If $\mathscr{C}\mathscr{C}_{st}(\mathbb{R}\times[0,1])$ denotes the space of continuously differentiable functions from $[0,1]$ into $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})=\mathscr{C}\mathscr{C}_{st}(\mathbb{C})\cap\mathscr{E}(\mathbb{R})$, we can state the following:

THEOREM 4.2. *Itô's formula: let $F\in\mathscr{C}\mathscr{C}_{st}(\mathbb{R}\times[0,1])$; then the process on $[0,1]$ given by*

$$\begin{aligned} Q_F^{(N)}(t)\equiv{}&\mathrm{tr}_N F(H_N^k(t),1\le k\le m;t)-\delta_0^{\star m}(F(.;0))\\ &-\int_0^t\mathrm{tr}_N\partial_sF(H_N^k(s),1\le k\le m;s)ds\\ &-\int_0^t\mathrm{tr}_N\otimes\mathrm{tr}_N\biggl(\biggl(\sum_{l=1}^m L_{X_l}\biggr)(F)(H_N^k(s),1\le k\le m;s)\biggr)ds\end{aligned}$$

is a real-valued local martingale for the canonical filtration of $(H_N^k,1\le k\le m)$ with martingale bracket

$$\Bigl\langle Q_F^{(N)},Q_F^{(N)}\Bigr\rangle_t=\frac{1}{N^2}\int_0^t\mathrm{tr}_N\biggl(\sum_{l=1}^m|\mathscr{D}_{X_l}F|^2(H_N^k(s),1\le k\le m;s)\biggr)ds.$$

PROOF. The proof of the formula is analogue to that of Lemma 2.1 and can be seen to be a consequence of Itô's formula. $Q_F^{(N)}(t)$ can be seen to be real-valued since, as $F\in\mathscr{C}\mathscr{C}_{st}(\mathbb{R}\times[0,1])$, for any time $t\in[0,1]$,

$$\begin{aligned}\mathrm{tr}_NF(H_N^k(t),1\le k\le m;t)&=\mathrm{tr}_NF^*(H_N^k(t),1\le k\le m;t)\\ &=\overline{\mathrm{tr}_NF(H_N^k(t),1\le k\le m;t)}\end{aligned}$$

and hence $Q_F^{(N)}(t)=Q_{F^*}^{(N)}(t)=\overline{Q_F^{(N)}(t)}$. □

So far, we have exhibited a vector space $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$ generated by a family $ST(\mathbb{C})$ of test functions which is stable under the diverse differential operators showing up when we perform Itô's calculus. In the next section, we shall endow this set with a uniform topology and study the associated topology inherited on the space of non-commutative probability measures by duality.

4.3. *Topology.* We can endow the space of non-commutative functions $\mathscr{E}(\mathbb{C})$ with the following norm $||\cdot||_{\infty,\infty}$. For any $F\in\mathscr{E}(\mathbb{C})$, any $N\in\mathbb{N}$, set

$$\|F\|_{\infty,N}=\sup\{\|F(A_k,1\le k\le m)\|_\infty:(A_k,1\le k\le m)\in\mathscr{H}_N^m\}$$

with $\|\cdot\|_\infty$ the operator norm given, for any $A\in\mathscr{M}_N$, any $N\in\mathbb{N}$,

$$||A||_\infty=\sup_{||u||_N=1}\langle u,A^*Au\rangle_N^{\frac{1}{2}}$$

if $<u,v>_N=\sum_{i=1}^N u_i\bar{v}_i$ and $||\quad||_N$ the associated norm. Note that if A is Hermitian, $||A||_\infty$ is the spectral radius of A but that it is greater in general. Then, we define, for any $F\in\mathscr{E}(\mathbb{C})$,

$$\|F\|_{\infty,\infty}=\sup_{N\ge1}\|F\|_{\infty,N}. \tag{4.15}$$

REMARK 4.2. This norm generalizes the usual supremum norm on $\mathbb{R}$ since, if $F(A_k,1\le k\le m)=f(A_l)$ for $f\in\mathscr{C}_b(\mathbb{R})$, we have $\|F\|_{\infty,\infty}=\|f\|_\infty$.

Let $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})$ (resp. $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$) be the closure of $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ (resp. $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$) for the norm $||\quad||_{\infty,\infty}$. $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})$ and $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$ are separable Banach spaces. Note that:

LEMMA 4.1. (a) *For any* $F\in\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})$, $||F||_{\infty,\infty}<\infty$.
(b) *For any* $F\in\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$, *any* $f\in\mathscr{C}_b(\mathbb{R})$, $f\circ F\in\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$.

PROOF. The first point boils down to show that any $F\in ST(\mathbb{R})$ has finite norm. Note that for any $N\in\mathbb{N}$, any $A,B\in\mathscr{M}_N\times\mathscr{M}_N$, $||AB||_\infty\le||A||_\infty||B||_\infty$, so that for any $F,G\in ST(\mathbb{C})$,

$$||FG||_{\infty,\infty}\le||F||_{\infty,\infty}||G||_{\infty,\infty}.$$

Hence, it is enough to bound the norm of $(z-\sum_{l=1}^m\alpha_lX_l)^{-1}$ for $z\in\mathbb{C}\backslash\mathbb{R}$ and $\alpha_k\in\mathbb{R}$ to conclude, or equivalently that for any $z\in\mathbb{C}\backslash\mathbb{R}$, any $A\in\mathscr{H}$, $(z-A)^{-1}(\bar{z}-A)^{-1}$ has a spectral radius bounded independently of A. Diagonalizing the matrix A, it amounts to notice that for any $x\in\mathbb{R}$, $|z-x|^{-1}$ is bounded by the inverse of the absolute value of the imaginary part $\Im(z)$ of z.

For the second point, recall first that for any $(A_k,1\le k\le m)\in\mathscr{H}_N^m$, any $N\in\mathbb{N}$, if U is a $N\times N$ unitary matrix such that $F(A_k,1\le k\le m)=U^*DU$ for a diagonal matrix $D\in\mathscr{H}_N$, $f(F)(A_k,1\le k\le m)=U^*f(D)U$. Further, one can use Runge's Theorem 13.7 in [26] to approximate f uniformly by polynomial functions f_n on $[-||F||_{\infty,\infty}, ||F||_{\infty,\infty}]$. Since the elements of D are uniformly bounded by $||F||_{\infty,\infty}$, we conclude that

$$||f(F)(A_k, 1 \le k \le m) - U^* f_n(D) U||_\infty \le \sup_{x \in [-||F||_{\infty,\infty}, ||F||_{\infty,\infty}]} |f - f_n|(x) \equiv \varepsilon_n$$

and hence, taking the supremum over $(A_k, 1 \le k \le m) \in \mathscr{H}_N^m$ and $N \in \mathbb{N}$, $||f(F) - f_n(F)||_{\infty,\infty} \le \varepsilon_n$. Since for any $F \in \overline{\mathscr{C}\mathscr{C}}_{st}(\mathbb{R})$, any $n \in \mathbb{N}$, $(F)^n \in \overline{\mathscr{C}\mathscr{C}}_{st}(\mathbb{R})$, $f_n(F) \in \overline{\mathscr{C}\mathscr{C}}_{st}(\mathbb{R})$ for any $n \in \mathbb{N}$ and the claim follows. □

REMARK 4.3. As a consequence of Stone-Weierstrass theorem, if one considers the cylinder functions going to zero at infinity

$$\mathscr{C}\mathscr{C}_0(\mathbb{C}) = \left\{ \sum_{j=1}^{M} \prod_{1 \le i \le k_j}^{\rightarrow} f_i^j \left(\sum_{k=1}^{m} \alpha_i^j(k) X_k \right); M \in \mathbb{N}, (k_j)_{1 \le j \le M} \in \mathbb{N}^M, f_i^j \in \mathscr{C}_0(\mathbb{R}), \right.$$
$$\left. (\alpha_i^j(l), 1 \le l \le m) \in \mathbb{R}^m, 1 \le j \le M, 1 \le i \le k_j \right\},$$

the closure of $\mathscr{C}\mathscr{C}_0(\mathbb{C})$ by $|| \quad ||_{\infty,\infty}$ is $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})$.

We can now define the set of non-commutative probability measures; let $\overline{\mathscr{C}\mathscr{C}}_{st}(\mathbb{C})'$ be the algebraic dual of $\overline{\mathscr{C}\mathscr{C}}_{st}(\mathbb{C})$, that is the set of linear functionals on $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$. $\overline{\mathscr{M}}$ is the subset of $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})'$ of linear maps with real-valued restriction to the real vector space $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$. Any $\mu \in \overline{\mathscr{M}}$ can be decomposed as $\mu = \nu + i\nu$, $\nu \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})'$ since for any $F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{C})$,

$$\mu(F) = \mu\left(\frac{F + F^*}{2}\right) + i\mu\left(\frac{F - F^*}{2i}\right)$$

with $2^{-1}(F + F^*), (2i)^{-1}(F - F^*) \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})$. Hence $\overline{\mathscr{M}}$ is isomorphic to $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})'$. We furnish $\overline{\mathscr{M}}$ with the weak topology induced by $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$, denoted $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$-topology. It is the analogue of the topology on the space of measures on $\mathbb{R}$ inherited from the set of test functions going to zero at infinity according to the last remark.

We shall now introduce the analogue of the set of probability measures (i.e., the notions of boundedness, positiveness and mass 1).

For any positive real number a, we denote by $\overline{\mathscr{M}}_a$ the subset of $\overline{\mathscr{M}}$ of linear forms μ such that

$$\forall F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{C}), \qquad |\mu(F)| \le a \|F\|_{\infty,\infty}. \tag{4.16}$$

Remark that, for any $a > 0$, $\overline{\mathscr{M}}_a$ can be seen as a subset of the algebraic dual of $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})$ and that the $\overline{\mathscr{C}\mathscr{C}}_{st}(\mathbb{R})$-topology is equivalent to the $\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$-topology on $\overline{\mathscr{M}}_a$. Hereafter, we shall always consider $\overline{\mathscr{M}}_a$ as such.

We shall say that a linear form $\mu \in \overline{\mathscr{M}}_a$, $a > 0$, is positive iff

$$\forall F \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R}), \qquad F \ge 0 \Longrightarrow \mu(F) \ge 0$$

where the first inequality was defined in remark 4.1.

μ will be said to be tracial if

$$\forall F, G \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C}), \qquad \mu(GF) = \mu(FG).$$

REMARK 4.4. If a linear form μ is tracial and positive:
0. $\forall F, G \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})$,

$$\left[\mu\left(\frac{FG^* + GF^*}{2}\right)\right]^2 \leq \mu(FF^*)\mu(GG^*).$$

1. $\forall F \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R}), \quad G \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R}),\ F \geq 0$,

$$\mu(GF) \leq \|G\|_{\infty,\infty}\mu(F).$$

2. $\forall F \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$,

$$\mu(F) \leq \mu(|F|).$$

3. $\forall F \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R}),\ f \in \mathscr{C}^1(\mathbb{R}),\ f' \geq 0$, any $F, G \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R}),\ F \leq G$,

$$\mu(f(F)) \leq \mu(f(G)).$$

4. For any $F, G \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$, any Lipschitz function f on $\mathbb{R}$,

$$\left|\mu(f(F)) - \mu(f(G))\right| \leq |f|_{\mathscr{L}} \quad \mu(|F-G|)$$

with

$$|f|_{\mathscr{L}} = \sup_{x, y \in \mathbb{R}} \frac{|f(x) - f(y)|}{|x-y|}.$$

Point 0 can be easily demonstrated as a standard Cauchy-Schwarz formula. For point 1, notice that writing $F = F^{\frac{1}{2}}F^{\frac{1}{2}}$ [with $F^{\frac{1}{2}} \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$ according to Lemma 4.1] and noticing that $F^{\frac{1}{2}}(G - \|G\|_{\infty,\infty}1)F^{\frac{1}{2}} \leq 0$, one sees that

$$\mu(GF) - \|G\|_{\infty,\infty}\mu(F) = \mu\left(F^{\frac{1}{2}}(G - \|G\|_{\infty,\infty}1)F^{\frac{1}{2}}\right) \leq 0.$$

For the second point, we simply know that $|F| \geq F$ so that positiveness of μ gives the estimate. For point 3, note that

$$\mu(f(F)) - \mu(f(G)) = \mu\left([F-G]\int_0^1 f'(\alpha F + (1-\alpha)G)d\alpha\right) \tag{4.17}$$

and hence, if $F \leq G$ we can proceed as for point (1) to see that, if $f' \geq 0$,

$$\mu(f(F)) - \mu(f(G)) \leq 0.$$

For the last point, first assume that f is continuously differentiable and note that by (4.17) and point 2,

$$|\mu(f(F)) - \mu(f(G))| \leq \mu(|[F-G]\int_0^1 f'(\alpha F + (1-\alpha)G)d\alpha|). \tag{4.18}$$

Now,

$$[F-G]\left(\left(\int_0^1 f'(\alpha F+(1-\alpha)G)d\alpha\right)^2[F-G]\le\|f'\|_\infty^2[F-G]^2\right.$$

and applying (3) with $g(x)=\sqrt{x}$, we deduce

$$|\mu(f(F))-\mu(f(G))|\le\|f'\|_\infty\mu(|F-G|).$$

Since F and $G\in\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$ are uniformly bounded and $\mathscr{C}_b^1(\mathbb{R})$ is dense in the set of Lipschitz functions for $|\quad|_{\mathscr{L}}$ on compact sets, the result follows for Lipschitz functions.

Let $\overline{\mathscr{M}}_a^+$ be the subset of $\overline{\mathscr{M}}_a$ of positive tracial linear forms. We can define the notion of total mass for any linear form μ of $\overline{\mathscr{M}}_a^+$ by

$$m_\mu=\sup\{\mu(F),\ F\in\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R}),\ \|F\|_{\infty,\infty}\le 1\}=\inf\{\lambda\in\mathbb{R}^+,\ \mu\in\overline{\mathscr{M}}_\lambda^+\}$$

The analogue of the commutative set of probability measures will be the subset $\overline{\mathscr{M}}_1^=$ of $\overline{\mathscr{M}}_1^+$ of linear form with total mass m_μ exactly equal to one. Note that $m_\mu=\mu(1)$ since, as μ is positive, for any $F\in\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$,

$$\mu(F)=\mu(F.1)\le\|F\|_{\infty,\infty}\mu(1)$$

so that

$$m_\mu\le\mu(1)$$

and moreover that $1\in\mathscr{C}\mathscr{C}_{st}(\mathbb{R})$ implies $\mu(1)\le m_\mu$ and the desired equality.

By a standard diagonalization procedure, it is not hard to check as in the commutative setting that $\overline{\mathscr{M}}_1^=$ is compact for the $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$- topology since $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$ is separable. The $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$- topology is compatible on $\overline{\mathscr{M}}_1^=$ with the distance

$$\bar{d}(\mu,\nu)=|||\mu-\nu|||\equiv\sum_{p\in\mathbb{N}}\frac{1}{2^p}|\mu(F_p)-\nu(F_p)|$$

where $(F_p)_{p\in\mathbb{N}}$ is a basis of uniformly bounded functions of $ST(\mathbb{C})$ (or $ST(\mathbb{R})$) such as $\{\Phi_{\underline{\alpha},\underline{z}};(\underline{\alpha},\underline{z})\in(\mathbb{Q}^m\times(i+\mathbb{Q}))^n,n\in\mathbb{N}\}$. Hence, $\overline{\mathscr{M}}_1^=$ is a compact metric space, thus Polish.

Notice here that it is not clear whether the elements of $\overline{\mathscr{M}}_1^=$ satisfy a countably additive property (and how it should be stated), characterizing standard measures. However, the marginals of $\overline{\mathscr{M}}_1^=$ are standard probability measures, namely:

PROPERTY 4.2. *Let $F\in\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$. Then, the linear functional μ_F on $\mathscr{C}_b(\mathbb{R})$ given by*

$$\mu_F(f)=\mu(f\circ F)$$

is a compactly supported probability measure on $\mathbb{R}$. Further, the map $\mu\to\mu_F$ from $\overline{\mathscr{M}}_1^=$, furnished with the $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$-topology, into $\mathscr{P}(\mathbb{R})$, furnished with the weak topology, is continuous.

PROOF. First recall that, according to Lemma 4.1, $f\circ F\in\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$ so that μ_F is well-defined as a linear map on $\mathbb{C}_b(\mathbb{R})$. Further, μ_F is bounded since by (4.16), for any $f\in\mathscr{C}_b(\mathbb{R})$,

$$\mu_F(f)\le\|f\circ F\|_{\infty,\infty}\le\|f\|_\infty.$$

Moreover, F being uniformly bounded, μ_F can be seen as a linear map on $\mathscr{C}_c(\mathbb{R})$. Finally, μ_F is positive. Thus, by Riesz's representation Theorem 2.14 in [26], there exists a unique positive measure (hence countably additive), also denoted μ_F, so that for every $f\in\mathscr{C}_c(\mathbb{R})$,

$$\mu_F(f)=\int f(x)d\mu_F(x).$$

Moreover, μ_F is compactly supported [since $F\in\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$ is uniformly bounded] so that it has finite mass. This mass is necessarily smaller than 1 since $\mu\in\overline{\mathscr{M}_1^=}$, and can be seen to be exactly equal to one by taking the constant function equal to the unity. Hence, μ_F is a probability measure on $[-\|F\|_{\infty,\infty},\|F\|_{\infty,\infty}]$. Further, if we take a sequence $\mu^n\in\overline{\mathscr{M}_1^=}$ converging toward μ for the $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$-topology, for any $f\in\mathscr{C}_b(\mathbb{R})$, $\mu_F^n(f)$ converges toward $\mu_F(f)$ as n goes to infinity since $f\circ F\in\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$. Therefore, $\mu\to\mu_F$ is continuous. □

It can be useful to consider as well the marginal law of unbounded functions, to begin with the laws of the canonical coordinates. This requires an extra tightness property, which will be expressed as a second moment condition. Let $\mathscr{T}$ be the set of Stieljes functions $\{(z-x)^{-1}, z\in\mathbb{C}\backslash\mathbb{R}\}$.

Let us consider the marginals μ_{X_l}, $l\in\{1,\dots,m\}$, of $\mu\in\overline{\mathscr{M}_1^=}$ defined by

$$\forall f\in\mathscr{T}\qquad \mu_{X_l}(f)=\mu(f(X_l)),\quad 1\le l\le m.$$

Since the complex vector space generated by $\mathscr{T}$ is dense in $\mathscr{C}_c(\mathbb{R})$ by Runge's theorem, we can follow the arguments of the proof of Property 4.2, to see μ_{X_l}, $l\in\{1,\dots,m\}$, as positive linear maps on $\mathscr{C}_c(\mathbb{R})$ and therefore defined classically as probability measures on $\mathbb{R}$. In particular, they are countably additive and the monotone convergence theorem holds ((1.26), [26]). Hence, we can set $\mu(\sum_{l=1}^m X_l^2)=\sum_{l=1}^m\mu_{X_l}(x^2)$. Let, for $A\in\mathbb{R}^+$, $\mathscr{K}_1^=(A)$ be the closed subset of $\overline{\mathscr{M}_1^=}$,

$$\mathscr{K}_1^=(A)\equiv\left\{\mu\in\overline{\mathscr{M}_1^=},\ \max_{l\in\{1,\dots,m\}}\mu_{X_l}(x^2)\le A\right\}$$

and

$$\mathscr{K}_1^=(\infty)\equiv\bigcup_{A\in\mathbb{N}}\mathscr{K}_1^=(A)=\left\{\mu\in\overline{\mathscr{M}_1^=},\ \max_{l\in\{1,\dots,m\}}\mu_{X_l}(x^2)<\infty\right\}.$$

Further, consider

$$\overline{\overline{\mathscr{C}\mathscr{C}}}_{st}(\mathbb{R})\equiv\Bigg\{F\in\mathscr{E}(\mathbb{R});\exists(F_n)_{n\in\mathbb{N}}\in\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})^{\mathbb{N}},$$
$$|F-F_n|(X_k,1\le k\le m)\le\frac{1}{n}\left(\sum_{l=1}^m X_l^2+1\right)\Bigg\}.$$

$\overline{\overline{\mathscr{C}\mathscr{C}}}_{st}(\mathbb{R})$ contains the bounded continuous functions and the linear combination $\sum_{l=1}^m \alpha_l X_l$ of the canonical coordinates [approximate $\sum_{l=1}^m \alpha_l X_l$ by $\sum_{l=1}^m \alpha_l X_l (1+(1/n)X_l^2)^{-1} \in \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$ for some $z \in C/R$ as close as needed of one]. We can then extend, for any $A \in \mathbb{R}^+$, $\mathscr{K}_1^{=}(A)$ as a set isomorphic to a subset of $\overline{\overline{\mathscr{C}\mathscr{C}}}_{st}(\mathbb{R})'$ as follows:

LEMMA 4.2. *Let $F \in \overline{\overline{\mathscr{C}\mathscr{C}}}_{st}(\mathbb{R})$ and $\mu \in \mathscr{K}_1^{=}(A)$ for some $A \in \mathbb{R}^+$. Then, we can define*

$$\mu_F(f) = \lim_{n\to\infty} \mu_{F_n}(f) \qquad \forall f \in \mathscr{C}_b(\mathbb{R}),\ |f|_{\mathscr{L}} < \infty \tag{4.19}$$

and, more precisely, if d is the Wasserstein's distance (1.4),

$$\lim_{n\to\infty} \sup_{\mu \in \mathscr{K}_1^{=}(A)} d(\mu_F, \mu_{F_n}) \le \sup_{\mu \in \mathscr{K}_1^{=}(A)} \sup_{|f|_{\mathscr{L}} \le 1} |\mu_F(f) - \mu_{F_n}(f)| = 0. \tag{4.20}$$

μ_F is a probability measure on $\mathbb{R}$. Moreover, the map $\mu \to \mu_F$ is continuous from $\mathscr{K}_1^{=}(A)$ into $\mathscr{P}(\mathbb{R})$ for any $A \in \mathbb{R}^+$. Finally, $\mu \to \mu(F)$ is continuous from $\mathscr{K}_1^{=}(A)$ into $\mathbb{R}$ for any $A \in \mathbb{R}^+$.

PROOF. We first check that μ_F is well-defined, that is that (4.19) indeed converges. Indeed, for any $(n,p) \in \mathbb{N}$, $n \le p$, any Lipschitz function f, Remark 4.4 implies that

$$|\mu_{F_p}(f) - \mu_{F_n}(f)| \le |f|_{\mathscr{L}} \mu(|F_p - F_n|) \le \frac{1}{n} |f|_{\mathscr{L}} (2A+1).$$

Thus,

$$d(\mu_{F_p}, \mu_{F_n}) \le \sup_{\mu \in \mathscr{K}_1^{=}(A)} \sup_{|f|_{\mathscr{L}} \le 1} |\mu_F(f) - \mu_{F_n}(f)| \le \frac{1}{n}(2A+1)$$

so that $(\mu_{F_p})_{p\in\mathbb{N}}$ is Cauchy in the complete metric space $\mathscr{P}(\mathbb{R})$. Hence it converges as p goes to infinity and its limit $\mu_F \in \mathscr{P}(\mathbb{R})$ satisfies (4.20).

Further, if μ^n is a sequence in $\mathscr{K}_1^{=}(A)$ converging to μ, $\mu \in \mathscr{K}_1^{=}(A)$ as $\mathscr{K}_1^{=}(A)$ is closed for the $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$-topology. It is then not hard to verify that $d(\mu_F^n, \mu_F)$ goes to zero as n goes to infinity in view of the uniform approximation on $\mathscr{K}_1^{=}(A)$ obtained in (4.20) and Lemma 4.2. Hence, $\mu \to \mu_F$ is continuous. Finally, taking $f(x) = x$ in the right hand side of (4.20) shows that $\mu \to \mu(F)$ is also continuous, which achieves the proof of the lemma. □

REMARK. Note by the way that for any $A \in \mathbb{R}^+$, the $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$-topology is equivalent on $\mathscr{K}_1^{=}(A)$ to the topology inherited from the set of test functions given by

$$\mathscr{C}\mathscr{C}_b(\mathbb{C}) = \left\{ \sum_{j=1}^M \overrightarrow{\prod_{1\le i\le k_j}} f_i^j \left(\sum_{l=1}^m \alpha_i^j(l) X_l \right); M \in \mathbb{N}, (k_j)_{1\le j\le M} \in \mathbb{N}^M, f_i^j \in \mathscr{C}_b(\mathbb{R}), \right.$$
$$\left. (\alpha_i^j(l), 1 \le l \le m) \in \mathbb{R}^m, 1 \le j \le M, 1 \le i \le k_j \right\}.$$

Indeed, the functions of $\mathscr{C}\mathscr{C}_b(\mathbb{C})$ can be approximated, if for every $\eta > 0$, we set $h^{(\eta)}(x) = \frac{h(x)}{1+\eta x^2} \in \mathscr{C}_0(\mathbb{R})$, by replacing the bounded continuous functions f_i^j by $(f_i^j)^{(\eta)}$. We then obtain elements of $\overline{\mathscr{C}\mathscr{C}_0}(\mathbb{C}) = \overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})$. Any $\mu \in \mathscr{K}_1^{=}(A)$ can be extended to $\mathscr{C}\mathscr{C}_b(\mathbb{C})$ by controlling uniformly the difference of these functions and their approximations as in the last proof. Hence, $\mathscr{C}\mathscr{C}_b(\mathbb{C})$ (or its restriction to $\mathscr{E}(\mathbb{R})$) generates the same topology that $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$ on $\mathscr{K}_1^{=}(A)$.

REMARK. The choice of the function X^2 is only suitable for our purpose; we could in fact have taken any positive function going to infinity as X goes to infinity.

In the sequel, we shall restrict ourselves to $\mathscr{K}_1^{=}(\infty)$-valued non-commutative measures and furnish $\mathscr{K}_1^{=}(\infty)$ with the $\overline{\overline{\mathscr{C}\mathscr{C}}}_{st}(\mathbb{R})$-topology. Hence, since $\overline{\mathscr{M}_1^{=}}$ is compact for the $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$-topology, a subset $\mathscr{K}$ of $\mathscr{K}_1^{=}(\infty)$ is relatively compact if it is included in some $\mathscr{K}_1^{=}(A)$ for some finite $A > 0$. The $\overline{\overline{\mathscr{C}\mathscr{C}}}_{st}(\mathbb{R})$-topology will enable ourselves to get tightness for marginals of possibly unbounded functionals.

Let us finally introduce the topology on $\mathscr{K}_1^{=}(\infty)$-valued processes; it is generated by the uniform topology on the time variable and the previous $\overline{\overline{\mathscr{C}\mathscr{C}}}_{st}(\mathbb{R})$-topology on the marginals. It is compatible on $\mathscr{C}([0,1], \mathscr{K}_1^{=}(A))$, $A > 0$, with the distance

$$\bar{\mathscr{D}}(\mu, \nu) = \sup_{t \in [0,1]} \bar{d}(\mu_t, \nu_t).$$

4.4. *Proof of the large deviations upper bound.* To obtain the large deviation upper bound of Theorem 4.1 is easy once we have defined the right topology and notice that for every $N \in \mathbb{N}$,

$$\mathbb{P}(\hat{\mu}_t^{(N)} \in \mathscr{C}([0,1], \mathscr{K}_1^{=}(\infty))) = 1$$

so that we can consider $\hat{\mu}^{(N)}$ as a $\mathscr{K}_1^{=}(\infty)$-valued process.

The proof of the large deviation upper bound for the process of the time marginals of the non-commutative law of m Hermitian Brownian motions follows now the usual scheme; first we shall check that $\bar{S}$ is a good rate function, then that the law of our $\mathscr{K}_1^{=}(\infty)$- valued processes are exponentially tight and then use Itô's calculus to get a weak large deviation upper bound.

4.4.1. *$\bar{S}$ is a good rate function.* It is straightforward to see that $\bar{S}$ is non negative as in (1.4) [note that for $F \in \mathscr{C}\mathscr{C}_{st}([0,1], \mathbb{R})$, $S^{0,1}(F, \mu)$ and $\langle\langle\langle F, F\rangle\rangle\rangle_\mu^{0,1}$ are real valued]. Further, for any $F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})$, $S^{0,1}(., F)$ is continuous for the $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$-topology since we noticed that, for any $l \in \{1, \ldots, m\}$, $L_{X_l}(F) \in \mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ and $|\mathscr{D}_{X_l} F|^2 \in \mathscr{C}\mathscr{C}_{st}(\mathbb{C})$. Hence, $S^{0,1}$, as a supremum of continuous functions, is lower semi-continuous for the $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$-topology. To prove that the level sets E_M are in fact compact in $\mathscr{C}([0,1], \mathscr{K}_1^{=}(\infty))$, note that the relatively compact subsets of $\mathscr{C}([0,1], \mathscr{K}_1^{=}(\infty))$ can be included, following [13], Lemma 5.4, in subsets of the form

$$\mathscr{K} = \mathscr{K}'_M \bigcap \left(\bigcap_{n\in\mathbb{N}} \mathscr{K}'_n \right)$$

with

$$\mathscr{K}'_M = \{\nu \in \mathscr{C}([0,1], \mathscr{K}_1^{=}(\infty))/\nu_u \in K'_M \ \forall u \in [0,1]\},$$

$$\mathscr{K}'_n = \{\nu \in \mathscr{C}([0,1], \mathscr{K}_1^{=}(\infty))/ \text{ the function } (u \to \nu_u(F_n)) \text{ belongs to } K'_n\}$$

with K'_M a compact subset of $\mathscr{K}_1^{=}(\infty)$, $(K'_n)_{n\in\mathbb{N}}$ a sequence of compact subsets of $\mathscr{C}([0,1],\mathbb{R})$ and $(F_n)_{n\in\mathbb{N}}$ a basis of $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$ (recall that on K'_M, the $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{R})$-topology is equivalent to the weak topology). In view of the description we gave of the relatively compact subsets of $\mathscr{K}_1^{=}(\infty)$ and Arzéla-Ascoli theorem, we can follow the proof given in Subsection 2.2 to see that we need to show that:

1. There exists $L^M > 0$ so that

$$\forall \nu \in E_M \qquad \sup_{u\in[0,1]} \nu_u \left(\sum_{l=1}^{m} X_l^2 \right) \le L^M.$$

2. For every $F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})$, and every $m > 0$, there exists $\delta_m^M(F)$ so that

$$\forall \nu \in E_M \qquad \sup_{|t-s|\le \delta_m^M(F)} |\nu_t(F) - \nu_s(F)| \le \frac{1}{m}.$$

PROOF. For the first point, note that for any $M \in \mathbb{N}$, there exists a finite integer number A so that

$$E_M \equiv \{\bar{S} \le M\} \subset \left\{ \nu \in \mathscr{C}([0,1], \mathscr{K}_1^{=}(\infty)), \max_{1\le l\le m} \sup_{t\in[0,1]} \nu_t(X_l^2)_t \le A, \right\} \\ \subset \left\{ \nu \in \mathscr{C}([0,1], \mathscr{K}_1^{=}(\infty)), \nu_t \in \mathscr{K}_1^{=}(A), \forall t \in [0,1] \right\}. \tag{4.21}$$

In fact, this is clear since, if $\nu_0 = \delta_0^{\star m}$, taking

$$F_\delta^l(X_l, 1 \le k \le m) = f_\delta(X_l) = \frac{X_l^2}{1+\delta^2 X_l^2} = \frac{X_l}{i+\delta X_l} \frac{X_l}{-i+\delta X_l} \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})$$

for some δ's as small as we wish, we find as in (2.10) that $\sup_{t\in[0,1]} \nu_t(F_\delta^l(X_k, 1 \le k \le m)) \le (8+M)e^8$ and hence by monotone convergence theorem

$$\max_{1\le l\le m} \sup_{t\in[0,1]} \nu_t(X_l^2) \le (8+M)e^8$$

so that the proof of the first point is complete.

For the second point, we follow the same strategy as the one developed in Lemma 2.4. Since by definition we have $\forall F \in \mathscr{C}\mathscr{C}_{st}(\mathbb{R})$, $\forall \nu \in E_M$,

$$(\bar{S}^{s,t}(\nu, F))^2 \le 2\bar{S}^{0,1}(\nu) \langle\langle\langle F, F \rangle\rangle\rangle_\nu^{s,t} \le 2M \langle\langle\langle F, F \rangle\rangle\rangle_\nu^{s,t}$$

we deduce

$$|\nu_t(F)-\nu_s(F)| \leq \tfrac{1}{2}\left|\int_s^t \nu_u\otimes\nu_u\left(\sum_{l=1}^m D_{X_l}\circ\mathcal{D}_{X_l}F\right)du\right| + \sqrt{2M\int_s^t \nu_u\left(\sum_{l=1}^m |\mathcal{D}_{X_l}F|^2\right)du}$$

By definition of $\mathcal{C}\mathcal{C}_{st}(\mathbb{R})$, all the functions appearing in the above right hand side are uniformly bounded for $\|\quad\|_{\infty,\infty}$ so that we conclude that there exists a finite constant $\Delta^M(F)$ so that

$$|\nu_t(F)-\nu_s(F)|\leq\Delta^M(F)\sqrt{|t-s|}. \qquad \square$$

4.4.2. *Exponential tightness.* We have the following lemma.

LEMMA 4.3. *There exist relatively compact subsets $(\mathcal{K}_L)_{L\in\mathbb{N}}$ of $\mathcal{C}([0,1], \mathcal{K}_1^{=}(\infty))$ so that*

$$\limsup_{N\to\infty}\frac{1}{N^2}\ln\mathbb{P}(\hat{\mu}^{(N)}\in\mathcal{K}_L)\leq -L.$$

We shall not go into the details of the proof since, in view of the previous description of the relatively compact subsets of $\mathcal{C}([0,1],\mathcal{K}_1^{=}(\infty))$ and by our Itô's formula of Theorem 4.2, we can proceed exactly as in Subsection 2.3.

4.4.3. *Weak large deviation upper bound.* We here state the following bound:

THEOREM 4.3. *For every process ν in $\mathcal{C}([0,1],\mathcal{K}_1^{=}(\infty))$, if $B_\delta(\nu)$ denotes the open ball with center ν and radius δ for the distance $\overline{\mathcal{D}}$, then*

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\ln\mathbb{P}\big(\hat{\mu}^{(N)}\in B_\delta(\nu)\big)\leq -\overline{S}(\nu)$$

PROOF. Thanks to Theorem 4.2, we can proceed exactly as in subsection 4.4. The only two points to notice is first that $\mathcal{C}\mathcal{C}_{st}(\mathbb{R},[0,1])$ was chosen so that for any $F\in\mathcal{C}\mathcal{C}_{st}(\mathbb{R},[0,1])$, $F(H_N^l(t), 1\leq k\leq m, t)$ is an Hermitian matrices, and therefore has real valued eigenvalues and second that all the functions appearing in Itô's formula belong to $\mathcal{C}\mathcal{C}_{st}(\mathbb{R})$. Hence, the analogue exponential super-martingales of $\zeta_f^{(N)}$ are real valued with continuous exponents for the $\overline{\mathcal{C}\mathcal{C}_{st}}(\mathbb{R})$-topology so that we can apply our strategy. $\square$

4.5. *Proof of Corollary* 4.2. To prove Corollary 4.2, note first that the functions F_n belong to $(\overline{\overline{\mathscr{C}\mathscr{C}_{st}}}(\mathbb{R})+i\overline{\overline{\mathscr{C}\mathscr{C}_{st}}}(\mathbb{R}))^m$ for any $n\in\mathbb{N}$. Indeed, it can first be approximated by

$$F_n^\varepsilon(X_l,1\le l\le 2m)=F_n\left(\frac{X_l}{1+\varepsilon X_l^2},1\le l\le 2m\right)$$

up to an error

$$\max_{1\le k\le m}|F_n^\varepsilon(X_l,1\le l\le 2m)_k-F_n(X_l,1\le l\le 2m)_k|\le\sqrt{n\varepsilon}\max_{1\le k\le 2m}X_k^2.$$

Then, by Runge's theorem, for any $\varepsilon>0$, for any $k\in\{1,\dots,m\}$, $F_n^\varepsilon(X_l,1\le l\le 2m)_k$ can be approximated uniformly by polynomial functions of

$$\left(\frac{X_{2k-1}}{1+\varepsilon X_{2k-1}^2},\frac{X_{2k}}{1+\varepsilon X_{2k}^2}\right)$$

and therefore belongs to $\overline{\mathscr{C}\mathscr{C}_{st}}(\mathbb{C})$. Hence, we can proceed as in the proof of 4.1 to see that $\mu\in\overline{\mathscr{M}_1^=}\to\mu\circ F_n^{-1}\in\mathscr{P}(U_l,1\le l\le m)^*$ is continuous on $\mathscr{K}_1^=(A)$, $A>0$ and that Theorem 4.1 implies that $\bar\mu_n^{(N)}=\hat\mu_1^{(N)}\circ F_n^{-1}$ satisfies a large deviation upper bound with good rate function

$$\bar{\mathscr{U}}_n(\mu)=\inf\{S(\nu),\nu_1\circ F_n^{-1}=\mu\}.$$

To get Corollary 4.2, we need to verify that $\bar\mu_n^{(N)}$ is an exponentially good approximation of the non-commutative law $\bar\mu^{(N)}$ of $(U_N^1,\dots,U_N^m)$ according to Theorem 4.2.23 of [14]. According to (4.3), $\bar\mu^{(N)}$ and $\hat\mu_1^{(N)}\circ F_\infty^{-1}$ have the same law. Hence, constructing $\bar\mu^{(N)}=\hat\mu_1^{(N)}\circ F_\infty^{-1}$ and $\bar\mu_n^{(N)}=\hat\mu_1^{(N)}\circ F_n^{-1}$ on the space generated by the same $2m$ independent Hermitian Brownian motions $(H_N^k,1\le k\le 2m)$ with non-commutative law $(\hat\mu_t^{(N)},t\in[0,1])$, we find

$$\begin{aligned}\hat{\mathscr{D}}(\bar\mu_n^{(N)},\bar\mu^{(N)})&\le 2\max_{1\le k\le m}\mathrm{tr}_N\left(\frac{n^{-1}}{H_N^{2k-1}(1)^2+H_N^{2k}(1)^2+n^{-1}}\right)\\&\le 2\max_{1\le k\le m}\mathrm{tr}_N\left(\frac{2n^{-1}}{(H_N^{2k-1}(1)+H_N^{2k}(1))^2+2n^{-1}}\right)\end{aligned}\tag{4.22}$$

so that $\bar\mu_n^{(N)}$ is an exponentially good approximation of $\bar\mu^{(N)}$ as soon as we can prove that for any $\varepsilon>0$,

$$\begin{aligned}&\limsup_{n\to\infty}\limsup_{N\to\infty}\mathbb{P}\left(\max_{1\le k\le m}\mathrm{tr}_N\left(\frac{2n^{-1}}{(H_N^{2k-1}(1)+H_N^{2k}(1))^2+2n^{-1}}\right)\ge\varepsilon\right)\\&\quad=-\infty.\end{aligned}\tag{4.23}$$

m being finite, we need only to prove this result for $m=1$, up to replace ε by (ε/m). Since $H_N^1(1)+H_N^2(1)$ is a standard Wigner's matrix, and for any $n\in\mathbb{N}$,

$A_n^\varepsilon = \{\mu \in \mathscr{P}(\mathbb{R}); \int \frac{2n^{-1}}{x^2+2n^{-1}} d\mu(x) \geq \varepsilon\}$ is a closed set for the weak topology, we can use the full large deviation principle obtained in [2] to get for any $n \in \mathbb{N}$,

$$\begin{aligned}\limsup_{N\to\infty} \mathbb{P}\left(\mathrm{tr}_N\left(\frac{2n^{-1}}{(H_N^1(1)+H_N^2(1))^2+2n^{-1}}\right) \geq \varepsilon\right) \\ \leq -\inf\{I_2(\mu), \mu \in A_n^\varepsilon\}\end{aligned} \tag{4.24}$$

with

$$I_2(\mu) = \frac{1}{4}\int x^2 d\mu(x) - \int \log|x-y| d\mu(x) d\mu(y) - \frac{3}{4} + \frac{1}{2}\log(2).$$

It was noticed in [2] that $g(x,y) = (1/8)(x^2+y^2) - \log|x-y|$ is bounded from below so that there exists a constant $C > -\infty$ so that for any $\delta \in [0,1]$,

$$\begin{aligned} I_2(\mu) &\geq -\int_{|x|\leq\delta, |y|\leq\delta} \log|x-y| d\mu(x) d\mu(y) + C \\ &\geq \log(2\delta)^{-1} \mu(|x|\leq\delta)^2 + C. \end{aligned} \tag{4.25}$$

Writing

$$\int \frac{2n^{-1}}{x^2+2n^{-1}} d\mu(x) = \int_0^\infty \frac{2n^{-1}}{(y+2n^{-1})^2} \mu(|x| \leq \sqrt{y}) dy$$

gives with (4.25)

$$\begin{aligned} \int \frac{2n^{-1}}{x^2+2n^{-1}} d\mu(x) &\leq \int_0^1 \frac{2n^{-1}}{(y+2n^{-1})^2} \left(\frac{I_2(\mu)-C}{\log(2\sqrt{y})^{-1}}\right)^{\frac{1}{2}} dy + 2n^{-1} \\ &\simeq_{n\to\infty} C' \left(\frac{I_2(\mu)-C}{\log(\sqrt{n})}\right)^{\frac{1}{2}} \int_0^\infty (1+y)^{-2} dy \end{aligned}$$

for a finite constant $C' > 0$. Thus, for n large, we have found two finite constants $c > 0$ and c' so that for $\mu \in A_n^\varepsilon$,

$$I_2(\mu) \geq c((\log n)\varepsilon^2 + c'). \tag{4.26}$$

Since I_2 is a good rate function and A_n^ε is closed, I_2 achieves its minimum value on A_n^ε and (4.26) results with

$$\inf\{I_2(\mu), \mu \in A_n^\varepsilon\} \geq c((\log n)\varepsilon^2 + c'). \tag{4.27}$$

Inequalities (4.24) and (4.27) give (4.23). As a conclusion, $\bar{\mu}^{(N)}$ satisfies a large deviation upper bound with the rate function

$$\bar{\mathscr{U}}(\mu) = \liminf_{n\to\infty} \bar{\mathscr{U}}_n(\mu) = \liminf_{n\to\infty} \inf\left\{\overline{S}(\nu), \nu_1 \circ F_n^{-1} = \mu\right\}.$$

REMARK. In view of Theorem 4.2.23 of [14],

$$\bar{\mathscr{U}}(\mu) = \inf\left\{\overline{S}(\nu), \nu_1 \circ F_\infty^{-1} = \mu\right\}$$

if we can prove that for any $M\in\mathbb{N}$,

$$\lim_{n\to\infty}\sup_{\mu\in\{\bar{S}\le M\}}\hat{\mathscr{D}}(\mu_1\circ F^{-1},\mu_1\circ F_n^{-1})=0.$$

Following the above strategy, we see that it is enough to prove, if $\phi(X_l,1\le l\le 2m)=X_1-X_2$, that

$$\lim_{n\to\infty}\sup_{\mu\in\{\bar{S}\le M\}}\int\frac{2n^{-1}}{x^2+2n^{-1}}d\mu\circ\phi^{-1}(x)=0$$

which by the above computation is verified if we can show that $\inf\{\bar{S}(\nu),\nu_1\circ\phi^{-1}=\mu\}\ge I_2(\mu)$. We believe it to be true, and refer the reader to [8] for similar computations. However, such identification will be the subject of a separate paper.

4.6. *Study of the minimizers of* $\bar{S}$. The minimizers of $\bar{S}$ are elements of $\mathscr{C}([0,1],\mathscr{K}_1^{=}(\infty))$ with initial data $\delta_0^{\star m}$ so that

$$\bar{S}^{s,t}(\mu,F)=0 \tag{4.28}$$

for any $0\le s\le t\le 1$ and any $F\in\mathscr{C}\mathscr{C}_{st}(\mathbb{R}\times[0,1])$. We shall prove:

LEMMA 4.4. *There exists a unique* $\mu\in\mathscr{C}([0,1],\mathscr{K}_1^{=}(\infty))$ *with initial data* $\delta_0^{\star m}$ *satisfying* (4.28).

PROOF. Existence of the solutions of (4.28) is trivial since it is obtained as the limit of the non-commutative law of $(H_N^k(t),1\le k\le m)_{t\in[0,1]}$. Let μ be any solution. First notice that, by linearity, (4.28) can be extended to functions F with values in $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$. Second, note that we can show that the marginals $(\mu_{X_l},1\le l\le m)$ are the semicircular processes defined in Section 2 (see the proof in [25] of the uniqueness of the solutions of (1.2)) and, since they are compactly supported, deduce that there exists a finite constant C so that for any $n\in\mathbb{N}$,

$$\max_{1\le l\le m}\sup_{t\in[0,1]}\mu_t(X_l^{2n})\le C^{2n}. \tag{4.29}$$

As a consequence, we can approximate the non-commutative polynomial functions

$$P_{\underline{n}}(X_l,1\le l\le m)=\overrightarrow{\prod_{1\le i\le M}}X_{l_i}^{n_i},\qquad n_i,M\in\mathbb{N},l_i\in\{1,\dots,m\}$$

by functions of $\mathscr{C}\mathscr{C}_{st}(\mathbb{C})$ and see that (4.28) is still valid when F is such a polynomial function. Note then that L_{X_l}, $l\in\{1,\dots,m\}$, map $\mathbb{C}_k(X_l,1\le l\le m)$, the set of non-commutative monomial functions $P_{\underline{n}}$ of degree $\sum_{i=1}^M n_i$ less or equal to k into $\mathbb{C}_{k-1}(X_l,1\le l\le m)\otimes\mathbb{C}_k(X_l,1\le l\le m)$. In particular, if μ and ν are two solutions of (4.28) and if we denote

$$\Delta_k(t)=\sup_{P_{\underline{n}}\in\mathbb{C}_k(X_l,1\le l\le m)}|\mu_t(P_{\underline{n}})-\nu_t(P_{\underline{n}})|,$$

then we find a finite constant c_k for any $k \in \mathbb{N}$ so that

$$\Delta_k(t) \leq c_k \int_0^t \Delta_k(s) ds.$$

Gronwall's lemma then implies that $\Delta_k(t) \equiv 0$ for all $k \in \mathbb{N}$. With (4.29), this is enough to characterize uniquely $\mu \in \mathscr{C}([0,1], \mathscr{K}_1^{=}(\infty))$. Indeed $\mu(\Phi_{\underline{\alpha},\underline{z}})$ is uniquely determined for any $(\underline{\alpha}, \underline{z}) \in \bigcup_{n\in\mathbb{N}} (\mathbb{R}^m \times \mathbb{C}\backslash\mathbb{R})^n$ so that for all $i \in \mathbb{N}$, $|z_i| > (\sum_{k=1}^m |\alpha_i^k|)C$ (just expend the ratio). Since $\underline{z} \to \mu(\Phi_{\underline{\alpha},\underline{z}})$ is analytic in $(\mathbb{C}\backslash\mathbb{R})^n$, this is enough to characterize $\mu(\Phi_{\underline{\alpha},\underline{z}})$ in $(\mathbb{R}^m \times \mathbb{C}\backslash\mathbb{R})^n$ for all $n \in \mathbb{N}$. □

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