THE BEHAVIOR OF THE SPECIFIC ENTROPY IN THE HYDRODYNAMIC SCALING LIMIT

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The paper studies the behavior of the specific entropy for one-dimensional simple exclusion processes under the hydrodynamic scaling of time and space. It is shown that if the initial configurations possess a macroscopic profile then for each positive macroscopic time the specific microscopic entropy converges to the macroscopic entropy. The latter is defined in terms of the solution of the corresponding hydrodynamic equation.

1. Introduction. The paper concerns simple exclusion models on the periodic integer lattice. The symmetric simple exclusion process is reversible and we study it under diffusive (parabolic) scaling of time and space. The corresponding hydrodynamic equation describing the behavior of the limiting macroscopic density is the heat equation. It has smooth solutions uniquely defined by the initial data. Asymmetric simple exclusion models are non-reversible and require Euler (hyperbolic) scaling which leads to inviscid Burgers-type equations. Solutions of such equations are not unique and may develop shocks. Relevant solutions are known to be the entropic solutions of these equations.

We study the behavior of the microscopic entropy relative to uniform measures and to local Gibbs measures. In words, a version of the main result can be stated as follows: let the initial configurations be deterministic and possess a macroscopic profile. Then under the appropriate scaling of time and space for any positive macroscopic time, the specific microscopic entropy converges to the macroscopic entropy calculated from the relevant solution of the hydrodynamic equation.

The main emphasis is placed on the asymmetric case when the solution of the hydrodynamic equation develops shocks and the macroscopic entropy is lost at each shock. Our results show that in the scaling limit the particle system closely follows the hydrodynamic equation and loses the same amount of the specific entropy at the same macroscopic time.

The key idea of this paper is presented in Lemma A.1. It is a rather general statement, which could be adopted for applications to other models. One of the important ingredients of the proof of Lemma A.1 is the logarithmic Sobolev inequality (Theorem A.1). Let us stress that we do not need an estimate on the constant in the logarithmic Sobolev inequality. The main difficulty in applying

Lemma A.1 is the condition (A3), which is often called the two-block estimate. The two-block estimate is well known for the symmetric case [see, e.g., Jensen and Yau (1999), Lecture 4]. The two-block estimate for the asymmetric case was obtained by Rezakhanlou [(1991), Lemma 6.2] as a part of the derivation of the scaling limit. The coupling technique employed there required restrictive assumptions on the initial distributions. In the work by Venkatsubramani (1995) the results on existence of the scaling limit were extended to deterministic initial data. A recent paper by Seppäläinen (1999) contains another coupling approach to the derivation of the hydrodynamic limit for more general totally asymmetric K-exclusion processes. In the present paper we use the existence of the scaling limit to prove the two-block estimate for asymmetric simple exclusion processes starting from deterministic initial data.

Section 2 gives a brief description of basic facts about simple exclusion processes and sets up the notation. The precise formulation of the main result (Theorem 3.1) is given in Section 3. There we also introduce local Gibbs measures and study the specific entropy relative to those measures (Theorem 3.2). Further, we prove Theorem 3.1 assuming the two-block estimate, which is derived later in Sections 4 and 5. The Appendix contains Lemma A.1 and two technical results that we use in Sections 3 and 4.

2. Preliminaries.

2.1. *Description of simple exclusion processes and notation.* The simple exclusion process on the periodic integer lattice $\Lambda_N = \mathbb{Z}/N\mathbb{Z}$ is the process with pairwise interactions between nearest neighbor particles. At most one particle per site is allowed. Particles are indistinguishable and move from site to site of Λ_N according to the following rules. Each particle waits a random time distributed exponentially with mean 1 then chooses one of the two neighboring sites of Λ_N with probabilities p and q, $p \in [0, 1]$, $q = 1 - p$ and jumps to the chosen site if that site is not occupied. Otherwise it suppresses the jump and stays at the same place. Throughout the paper p denotes the probability to jump to the right.

The state of the system at any time τ is described by a random vector $\eta(\tau) \in \mathbb{X}_N = \{0, 1\}^N$. Each component of η is equal to the occupation number of the corresponding site: $\eta_x = 1$ if site x is occupied, and $\eta_x = 0$ if it is vacant, $x \in \Lambda_N$. Vector $\eta(\tau)$ is called the configuration of particles at time τ. The infinitesimal generator of this process is given by

$$(1) \quad \mathscr{L}_p f(\eta) = \sum_{x \in \Lambda_N} \{ p\eta_x(1 - \eta_{x+1}) + q\eta_{x+1}(1 - \eta_x) \}(f(\eta^{x, x+1}) - f(\eta)),$$

where the configuration $\eta^{x, x+1}$ is obtained from η by exchanging the xth and the $(x + 1)$th coordinates,

$$\eta_y^{x, x+1} = \begin{cases} \eta_y, & \text{if } y \neq x, \ x+1, \\ \eta_{x+1}, & \text{if } y = x, \\ \eta_x, & \text{if } y = x + 1. \end{cases}$$

Since the notions of a "particle" (occupied site) and a "hole" (empty site) are interchangeable, we assume that $p \in [\frac{1}{2}, 1]$.

The number of particles is preserved in time. This implies a natural decomposition of the configuration space $\mathbb{X}_N = \{0, 1\}^N$ into "hyperplanes,"

$$\mathbb{X}_{N, n} = \left\{ \eta \in \mathbb{X}_N \,\Big|\, \sum_{x \in \Lambda_N} \eta_x = n \right\}, \qquad n = 0, 1, \ldots, N,$$

each of which consists of all configurations with a fixed number of particles. If the process starts from $\eta(0) \in \mathbb{X}_{N, n}$ then for all τ it stays in $\mathbb{X}_{N, n}$.

Let μ_N be the uniform measure on $\mathbb{X}_N$: $\mu_N(\eta) = 2^{-N}$ for each $\eta \in \mathbb{X}_N$. It is an invariant measure for simple exclusion processes. On $L^2(\mathbb{X}_N, \mu_N)$ the generator $\mathscr{L}_q$ is adjoint to $\mathscr{L}_p$; that is, $\mathscr{L}_p^* = \mathscr{L}_q$, $q = 1 - p$. The Dirichlét form for the simple exclusion process with the generator $\mathscr{L}_p$ is given by

$$(2) \qquad D_N(f) \stackrel{\text{def}}{=} -\langle \mathscr{L}_p f, f \rangle_{\mu_N} = \tfrac{1}{4} \sum_{\eta \in \mathbb{X}} \sum_{x \in \Lambda_N} \left(f(\eta^{x, x+1}) - f(\eta) \right)^2 \mu_N(\eta).$$

$D_N(f)$ does not depend on p.

For arbitrary probability measures μ and ν on $\mathbb{X}_N$, define the entropy of measure ν and the relative entropy of measure ν with respect to μ by

$$H(\nu) = \sum_{\eta \in \mathbb{X}} \nu(\eta) \log \nu(\eta) \quad \text{and} \quad H(\nu \,|\, \mu) = \sum_{\eta \in \mathbb{X}} \nu(\eta) \log \frac{\nu(\eta)}{\mu(\eta)},$$

respectively. We agree to set $0 \log 0$ to 0. Most often we shall use the uniform measure μ_N as a reference measure. If ν is absolutely continuous with respect to μ_N then $\nu(\eta) = f(\eta)\mu_N(\eta)$ for some f and we denote by $H_N(f)$ the relative entropy $H(\nu \,|\, \mu_N)$. The following lemma provides an estimate for the time derivative of the entropy for a simple exclusion process.

LEMMA 2.1. *Let f^τ be the solution of the forward equation*

$$(3) \qquad \frac{\partial f^\tau}{\partial \tau} = \mathscr{L}_p^* f^\tau, \qquad (\tau, \eta) \in [0, +\infty) \times \mathbb{X}_N,$$

with initial data f^0 such that $f^0 \geq 0$ and $E_{\mu_N} f^0 = 1$. Then for any $\tau \geq 0$,

$$(4) \qquad \frac{dH_N(f^\tau)}{d\tau} \leq -2D_N(\sqrt{f^\tau}).$$

The proof of this lemma can be found, for example, in Kipnis and Landim (1999), Theorem 9.2, page 342.

Throughout the paper we use the following notation for averages over the block of size $(2l + 1)$ centered at $x \in \Lambda_N$:

$$(5) \qquad \bar{\eta}_{x, l} = \frac{1}{2l + 1} \sum_{|y - x| \leq l} \eta_y.$$

For any $x \in \Lambda_N$ define the translation operator τ_x by

(6)
$$(\tau_x \eta)_y \stackrel{\text{def}}{=} \eta_{x+y}, \qquad y \in \Lambda_N,$$

$$\tau_x f(\eta) \stackrel{\text{def}}{=} f(\tau_x \eta) \quad \text{for any } f \colon \mathbb{X}_N \to \mathbb{R}.$$

2.2. *Space–time scaling, macroscopic profiles and hydrodynamic equations.*
The space scaling is given by the mapping $\Lambda_N \to S = \mathbb{R}/\mathbb{Z}$, which takes each
point x of the lattice Λ_N to the point $\theta = x/N$ of the circle S.

DEFINITION 2.1.   For each $N \in \mathbb{N}$ let ν_N be a probability measure on $\mathbb{X}_N$.
We shall say that the set $\{\nu_N\}$ possesses an asymptotic macroscopic profile
$\rho \in L^1(S)$ as $N \to \infty$ and denote this by $\nu_N \sim \rho$ if for every function $J \in C(S)$
and each $\delta > 0$,

(7)
$$\lim_{N \to \infty} \nu_N \left\{ \eta \in \mathbb{X}_N \colon \left| \frac{1}{N} \sum_{x \in \Lambda_N} J\left(\frac{x}{N}\right) \eta_x - \int_S J(\theta)\rho(\theta) \, d\theta \right| > \delta \right\} = 0.$$

REMARK.   It is clear that if $\nu_N \sim \rho$ then $0 \leq \rho(\theta) \leq 1$ a.e.

We use the standard notation $\delta_\eta(\cdot)$ for the measure concentrated on a single
configuration $\eta \in \mathbb{X}_N$; for any $A \subset \mathbb{X}_N$,

$$\delta_\eta(A) = \begin{cases} 1, & \text{if } \eta \in A, \\ 0, & \text{if } \eta \notin A. \end{cases}$$

Thus the relation $\delta_{\eta^{(N)}} \sim \rho$ simply means that for every $J \in C(S)$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{x \in \Lambda_N} J\left(\frac{x}{N}\right) \eta_x^{(N)} = \int_S J(\theta)\rho(\theta) \, d\theta.$$

To introduce the time scaling we have to distinguish between the following
two cases.

1. *Symmetric case* ($p = 1/2$). The time is speeded up by N^2. If τ denotes
   the microscopic time then the macroscopic time t is defined by the relation
   $t = \tau/N^2$. In terms of the macroscopic time the expected waiting time
   between jumps becomes $1/N^2$.
2. *Asymmetric case* ($p \neq 1/2$). The time is scaled by the same factor as the
   space, that is, $t = \tau/N$.

For an arbitrary $p \in [\frac{1}{2}, 1]$ consider the appropriately scaled simple exclu-
sion process with a deterministic initial condition $\eta^{(N)}(0)$. Let ν_N^t be the proba-
bility measure on the configuration space $\mathbb{X}_N$, which describes the distribution
of $\eta^{(N)}(t)$. Since the number of particles is conserved, ν_N^t is supported on $\mathbb{X}_{N, n_N}$,
where

(8)
$$n_N = \sum_{x \in \Lambda_N} \eta_x^{(N)}(0).$$

The restriction of ν_N^t to $\mathbb{X}_{N, n_N}$ will be denoted by ν_{N, n_N}^t.

We formulate a version of well-known results about the existence of the hydrodynamic scaling limit for simple exclusion processes [see, e.g., Spohn (1991) and Varadhan (1993) for $p = 1/2$; Rezakhanlou (1991), Venkatsubramani (1995), and Seppäläinen (1999) for $p \neq 1/2$].

THEOREM 2.1. *Assume that $\delta_{\eta^{(N)}(0)} \sim \rho_0$ for some $\rho_0 \in L^1(S)$. Let ν_N^t be the distribution of configurations at time $t > 0$ for the simple exclusion process with generator $\mathscr{L}_p$, $p \in [0, 1]$ and the initial data $\eta^{(N)}(0)$.*

(i) *If $p = 1/2$ then for any $t > 0$ we have the relation $\nu_N^t \sim \rho(t, \cdot)$, where ρ is the solution of the heat equation*

$$(9) \qquad \frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho}{\partial \theta^2}, \qquad (t, \theta) \in (0, +\infty) \times S,$$

with the initial condition ρ_0.

(ii) *If $p \neq 1/2$ then for any $t > 0$ we have the relation $\nu_N^t \sim \rho(t, \cdot)$, where ρ is the entropic solution of the inviscid Burgers equation*

$$(10) \qquad \frac{\partial \rho}{\partial t} + (2p - 1)\frac{\partial \rho(1 - \rho)}{\partial \theta} = 0, \qquad (t, \theta) \in (0, +\infty) \times S,$$

with the initial condition ρ_0.

REMARK. The same statements hold if we consider simple exclusion processes on $\mathbb{Z}$, replace $\mathbb{X}_N$ by $\{0, 1\}^{\mathbb{Z}}$ and S by $\mathbb{R}$ in Definition 2.1 and Theorem 2.1, and require that test functions J in (7) have compact support.

Equations (9) and (10) are called the *hydrodynamic equations*. Whenever we refer to a solution of the hydrodynamic equation we shall always mean either the solution of (9) or the entropic solution of (10) with the initial condition ρ_0, where ρ_0 is a measurable function and $0 \leq \rho_0 \leq 1$ a.e. In both cases the solution of the hydrodynamic equation is uniquely defined by the initial data.

3. Main results.

THEOREM 3.1. *Assume that $\delta_{\eta^{(N)}(0)} \sim \rho_0$ for some $\rho_0 \in L^1(S)$. Let ν_N^t be the distribution of configurations at time $t > 0$ for the simple exclusion process with generator $\mathscr{L}_p$, $p \in [0, 1]$, and initial data $\eta^{(N)}(0)$. Then for each $t > 0$,*

$$\lim_{N \to \infty} \frac{1}{N} H(\nu_N^t) = \int_S h(\rho(t, \theta)) \, d\theta,$$

where ρ is the solution of the hydrodynamic equation with the initial condition ρ_0 and h is defined by the following formula:

$$(11) \qquad h(y) = y \log y + (1 - y) \log(1 - y), \qquad y \in [0, 1], \ 0 \log 0 = 0.$$

Our second result is a consequence of Theorem 3.1. It concerns the behavior of the specific entropy relative to local Gibbs measures. Let $\rho(t, \cdot)$ be a solution of the hydrodynamic equation. For each $t \geq 0$ construct a sequence of step functions $\rho_N(t, \cdot)$ by averaging $\rho(t, \cdot)$ over the interval of size $1/N$,

$$\rho_N(t, \theta) = N \int_{(x-1/2)/N}^{(x+1/2)/N} \rho(t, \alpha) \, d\alpha$$

for all $\theta \in [(x - 1/2)/N, (x + 1/2)/N)$ and $x \in \Lambda_N$. Then, clearly, $\rho_N(t, \cdot) \to \rho(t, \cdot)$ as $N \to \infty$ in $L^1(S)$. Define the *local Gibbs measures* $\gamma_N^{\rho_t}$ on $\mathbb{X}_N$ by setting

$$(12) \quad \gamma_N^{\rho_t}(\eta) = \prod_{x \in \Lambda_N} \left(\rho_N\left(t, \frac{x}{N}\right) \eta_x + \left(1 - \rho_N\left(t, \frac{x}{N}\right)\right)(1 - \eta_x) \right) \quad \text{for each } \eta \in \mathbb{X}_N.$$

In other words, $\gamma_N^{\rho_t}$ is a product measure on $\{0, 1\}^N$ whose xth marginal is a Bernoulli measure with the probability of "success" given by $\rho_N(t, \frac{x}{N})$. It is not difficult to check that not only

$$(13) \qquad\qquad\qquad \gamma_N^{\rho_t} \sim \rho(t, \cdot)$$

but also for every $J \in C(S)$ and each $\delta > 0$,

$$(14) \quad \varlimsup_{N \to \infty} \frac{1}{N} \log \gamma_N^{\rho_t}\left\{ \eta \in \mathbb{X}_N : \left| \frac{1}{N} \sum_{x \in \Lambda_N} J\left(\frac{x}{N}\right) \eta_x \right. \right.$$
$$\left. \left. - \int_S J(\theta) \rho(t, \theta) \, d\theta \right| > \delta \right\} < 0$$

[see, e.g., Jensen and Yau (1999), Example 1.3]. A standard large deviation argument shows that the same relations remain true if we assume that

$$\frac{n_N}{N} \to \int_S \rho(t, \theta) \, d\theta \quad \text{as } N \to \infty,$$

and replace $\gamma_N^{\rho_t}$ by conditional measures $\gamma_{N, n_N}^{\rho_t}(\cdot) = \gamma_N^{\rho_t}(\cdot \mid \sum_{x \in \Lambda_N} \eta_x = n_N)$ and $\mathbb{X}_N$ by $\mathbb{X}_{N, n_N}$.

Unfortunately, if we start a simple exclusion process from $\gamma_N^{\rho_0}$, then at any $t > 0$ the measure ψ_N^t, which describes the distribution of this process at time t, will diverge from $\gamma_N^{\rho_t}$. It is known [see, e.g., Jensen and Yau (1999), Proof of Theorem 5.5] that if $\rho(t, \cdot)$ is continuously differentiable, then

$$\frac{1}{N} H(\psi_N^t | \gamma_N^{\rho_t}) \to 0 \qquad \text{as } N \to \infty.$$

The next theorem states that even if we start very far from the equilibrium, almost instantly a similar statement becomes true: the specific entropy relative to local Gibbs measures converges to zero.

THEOREM 3.2.   *Let the condition of Theorem* 3.1 *be satisfied. If* $p \neq 1/2$ *assume also that* $\rho_0 \in [\delta_0, 1 - \delta_0]$ *a.e. for some* $\delta_0 \in (0, 1/2)$. *Define* n_N *by* (8). *Then for any* $t > 0$,

$$\lim_{N \to \infty} \frac{1}{N} H\big(\nu^t_{N, n_N} \,|\, \gamma^{\rho_t}_{N, n_N}\big) = 0.$$

PROOF.   The relative entropy is nonnegative. Therefore Theorem 3.2 will follow from Theorem 3.1 if we prove that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{\eta \in \mathbb{X}_{N, n_N}} \nu^t_{N, n_N}(\eta) \log \gamma^{\rho_t}_{N, n_N}(\eta) \geq \int_S h(\rho(t, \theta)) \, d\theta.$$

We have

$$\log \gamma^{\rho_t}_{N, n_N}(\eta) = \log \gamma^{\rho_t}_N(\eta) - \log \sum_{\eta \in \mathbb{X}_{N, n_N}} \gamma^{\rho_t}_N(\eta).$$

Since $\log \sum_{\eta \in \mathbb{X}_{N, n_N}} \gamma^{\rho_t}_N(\eta) \leq 0$, it is enough to show that

(15) $$\lim_{N \to \infty} \frac{1}{N} E_{\nu^t_{N, n_N}} \log \gamma^{\rho_t}_N(\eta) = \int_S h(\rho(t, \theta)) \, d\theta.$$

This equality is a consequence of the existence of the scaling limit. We rewrite $\log \gamma^{\rho_t}_N(\eta)$ as

$$\log \prod_{x \in \Lambda_N} \Big(\rho_N\Big(t, \frac{x}{N}\Big)\Big)^{\eta_x} \Big(1 - \rho_N\Big(t, \frac{x}{N}\Big)\Big)^{1 - \eta_x}$$

$$= \sum_{x \in \Lambda_N} F\Big(\rho_N\Big(t, \frac{x}{N}\Big)\Big) \eta_x + \sum_{x \in \Lambda_N} \log\Big(1 - \rho_N\Big(t, \frac{x}{N}\Big)\Big),$$

where $F(y) = \log y - \log(1 - y)$. At first, notice that, under our assumptions on the initial data, the function $\rho(t, \cdot)$ is bounded away from 0 and 1 for each $t > 0$. [In the symmetric case this follows from the inequalities $0 \leq \rho_0(\theta) \leq 1$ a.e. and the strong maximum principle for solutions of the heat equation.] By Lemma A.5,

(16) $$\lim_{N \to \infty} \frac{1}{N} \sum_{x \in \Lambda_N} F\Big(\rho_N\Big(t, \frac{x}{N}\Big)\Big) \eta_x = \int_S F\Big(\rho(t, \theta)\Big) \rho(t, \theta) \, d\theta.$$

Since $\rho_N(t, \cdot) \to \rho(t, \cdot)$ as $N \to \infty$ in $L^1(S)$, we also have that

(17) $$\frac{1}{N} \sum_{x \in \Lambda_N} \log\Big(1 - \rho_N\Big(t, \frac{x}{N}\Big)\Big) = \int_S \log(1 - \rho_N(t, \theta)) \, d\theta$$

$$\to \int_S \log(1 - \rho(t, \theta)) \, d\theta \qquad \text{as } N \to \infty.$$

Combining (16) and (17) we obtain (15). The proof of Theorem 3.2 is complete. □

PROOF OF THEOREM 3.1. Lemma A.1 (see the Appendix) provides a general framework for the proof. This lemma is a dynamics-free statement. Below we formulate a slightly different version, which involves time averages.

LEMMA 3.1. *Let ρ be the solution of the hydrodynamic equation. Suppose that probability measures ν_N^t satisfy the following conditions*:

(A1$'$) *For every $t \in [0, T]$, $\delta > 0$ and $J \in C(S)$,*

$$\nu_N^t \left\{ \eta \in \mathbb{X}_N \colon \left| \frac{1}{N} \sum_{x \in \Lambda_N} J\left(\frac{x}{N}\right) \eta_x - \int_S J(\theta) \rho(t, \theta) \, d\theta \right| > \delta \right\} \to 0 \qquad as\ N \to \infty.$$

(A2$'$) $\int_0^T D_N\left(\sqrt{f_N^t}\right) dt = o(N)$ *as* $N \to \infty$, *where* $f_N^t(\eta) = \frac{\nu_N^t(\eta)}{\mu_N(\eta)}$.

(A3$'$)

$$\lim_{\varepsilon \to 0} \overline{\lim_{l \to \infty}} \, \overline{\lim_{N \to \infty}} \int_0^T E_{\nu_N^t} \frac{1}{N} \sum_{x \in \Lambda_N} \left| \bar{\eta}_{x, l} - \bar{\eta}_{x, \varepsilon N} \right| dt = 0.$$

Then

$$\lim_{N \to \infty} \frac{1}{N} H(\nu_N^t) = \int_S h(\rho(t, \theta)) \, d\theta \qquad for\ every\ t \in (0, T].$$

PROOF. The proof of the lower bound is identical to the one of Lemma A.1. For the upper bound we notice that the entropy is a decreasing function of time [see (4)]. Repeating the proof of Lemma A.1 we obtain for every $t \in (0, T]$ and all sufficiently small $\varepsilon > 0$,

$$\lim_{N \to \infty} \frac{1}{N} H(\nu_N^t) \le \lim_{N \to \infty} \frac{1}{N\varepsilon} \int_{t-\varepsilon}^t H(\nu_N^s) \, ds \le \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_S h(\rho(s, \theta)) \, d\theta \, ds.$$

The statement of the lemma follows if we let ε go to zero, since the function $G(t) =^{\mathrm{def}} \int_S h(\rho(t, \theta)) \, d\theta$ is continuous. The continuity of G is obvious if ρ is the solution of the heat equation. If ρ is the entropic solution of (10) then

$$\int_S |\rho(t, \theta) - \rho(s, \theta)| \, d\theta \to 0 \quad as\ |t - s| \to 0, \ t, s \in [0, T]$$

[see, e.g., DiBenedetto (1995), Chapter 7, relations (10.4) and (20.1)]. This implies the continuity of G. $\square$

We have to check that all the conditions of Lemma 3.1 are satisfied. Theorem 2.1 provides us with (A1$'$). The estimate (A2$'$) on the Dirichlét form follows from the lemma.

LEMMA 3.2. *Let f_N^t be the solution of the problem*

(18)
$$\frac{\partial f_N^t(\eta)}{\partial t} = N^c \mathcal{L}_p^* f_N^t(\eta), \qquad (t, \eta) \in [0, T] \times \mathbb{X}_N,$$

$$f_N^t(\eta)\big|_{t=0} = f_N^0(\eta) \ge 0,$$

where $E_{\mu_N} f_N^0 = 1$, $c = 2$ if $p = 1/2$ and $c = 1$ if $p \neq 1/2$. Then

$$\int_0^T D_N\left(\sqrt{f_N^t}\right) dt \leq \frac{\log 2}{2N^{c-1}}.$$

PROOF. Since $E_{\mu_N} f_N^t = 1$ for any $t \in [0, T]$, by Jensen's inequality $H_N(f_N^t) = E_{\mu_N} f_N^t \log f_N^t \geq 0$. From (4) with $\tau = tN^c$ we obtain

$$(19) \qquad \int_0^T D_N\left(\sqrt{f_N^t}\right) dt \leq \frac{1}{2N^c}\left(H_N(f_N^0) - H_N(f_N^T)\right) \leq \frac{1}{2N^c} H_N(f_N^0).$$

To estimate the right-hand side we use the following obvious proposition.

PROPOSITION 3.3. *Let $\mathscr{P}$ be the space of probability measures on $\Omega = \{1, 2, \ldots, M\}$. Then for any $P, Q \in \mathscr{P}$ with $P(i) = p_i$, $Q(i) = q_i$, $i = 1, \ldots, M$,*

$$H(P|Q) = \sum_{i=1}^M p_i \log \frac{p_i}{q_i} \leq \max_{1 \leq i \leq M} \log \frac{1}{q_i}.$$

In our case $q_i \equiv q = 2^{-N}$ and

$$(20) \qquad\qquad\qquad E_{\mu_N} f_N^0 \log f_N^0 \leq N \log 2.$$

Inequalities (19) and (20) imply the statement of the lemma. □

To finish the proof of Theorem 3.1 we have to show that the two-block estimate (A3′) holds. It is a well-known result for the symmetric case [see, e.g., Jensen and Yau (1999)]. The two-block estimate for the asymmetric case is the content of the next two sections.

4. The two-block estimate. Denote by P_N the measure on the Skorokhod space $D([0, +\infty); \mathbb{X}_N)$, which corresponds to the process $\eta^{(N)}(t)$, $t \geq 0$, with the initial condition $\eta^{(N)}(0)$ and by E_N the expectation relative to the measure P_N.

THEOREM 4.1. *Under the assumptions of Theorem 3.1 for any $T > 0$,*

$$\lim_{\varepsilon \to 0} \overline{\lim_{l \to \infty}} \, \overline{\lim_{N \to \infty}} \, E_N \int_0^T \frac{1}{N} \sum_{x \in \Lambda_N} |\bar{\eta}_{x, l} - \bar{\eta}_{x, \varepsilon N}| \, dt = 0.$$

PROOF. We prove this theorem for $p \neq 1/2$. The proof is based on a number of lemmas. Let $j^{(N)}(t)$ be the difference between the total number of jumps to the right and the total number of jumps to the left up to time t. The first lemma gives us a "microscopic" description of the behavior of $j^{(N)}(t)$ as $N \to \infty$ for a fixed t.

LEMMA 4.1. *For any $t \geq 0$ and any $\delta > 0$*

$$(21) \quad \lim_{N \to \infty} P_N \left(\left| \frac{j^{(N)}(t)}{N^2} - \frac{(2p-1)}{N} \int_0^t \sum_{x \in \Lambda_N} \eta_x(s)(1 - \eta_{x+1}(s)) \, ds \right| > \delta \right) = 0.$$

PROOF. Consider the process $(\eta^{(N)}(t), j^{(N)}(t))$ with the state space $\mathbb{X}_N \times (\mathbb{N} \cup \{0\})$ and the generator

$$\widetilde{\mathcal{L}}_p f(\eta, j) = N \sum_{x \in \Lambda_N} \left[p \eta_x (1 - \eta_{x+1})(f(\eta^{x,\,x+1}, j+1) - f(\eta, j)) \right.$$
$$\left. + q \eta_{x+1}(1 - \eta_x)(f(\eta^{x,\,x+1}, j-1) - f(\eta, j)) \right].$$

Assume that the process starts from $(\eta^{(N)}(0), 0)$. Choose $f(\eta, j) = j$ then

$$(22) \qquad \widetilde{\mathcal{L}}_p j^{(N)} = (2p-1)N \sum_{x \in \Lambda_N} \eta_x^{(N)}(1 - \eta_{x+1}^{(N)}),$$

$$(23) \qquad dj^{(N)}(t) = \widetilde{\mathcal{L}}_p j^{(N)}(t) \, dt + dM_1^{(N)}(t)$$

and

$$(24) \quad d\big(M_1^{(N)}(t)\big)^2 = \big[\widetilde{\mathcal{L}}_p (j^{(N)}(t))^2 - 2 j^{(N)}(t) \widetilde{\mathcal{L}}_p j^{(N)}(t) \big] \, dt + dM_2^{(N)}(t),$$

where $M_1^{(N)}(t)$ and $M_2^{(N)}(t)$ are martingales. Since

$$\widetilde{\mathcal{L}}_p (j^{(N)}(t))^2 - 2 j^{(N)}(t) \widetilde{\mathcal{L}}_p j^{(N)}(t) = N \sum_{x \in \Lambda_N} \eta_x^{(N)}(t)(1 - \eta_{x+1}^{(N)}),$$

we obtain from (24) that for any $t \geq 0$,

$$(25) \qquad\qquad E_N \big(M_1^{(N)}(t) \big)^2 = O(N^2).$$

Therefore, by (23), (22) and (25) we conclude that

$$P_N \left(\left| \frac{j^{(N)}(t)}{N^2} - \frac{(2p-1)}{N} \int_0^t \sum_{x \in \Lambda_N} \eta_x(s)(1 - \eta_{x+1}(s)) \, ds \right| > \delta \right)$$

$$= P_N \big(|M_1^{(N)}(t) - M_1^{(N)}(0)| > \delta N^2 \big) \leq \frac{1}{\delta^2 N^4} E_N |M_1^{(N)}(t) - M_1^{(N)}(0)|^2$$

$$\leq \frac{C(t)}{\delta^2 N^2} \to 0 \quad \text{as } N \to \infty. \qquad\qquad \square$$

We use the following one-block estimate to calculate $j^{(N)}(t)$ in terms of large microscopic blocks.

LEMMA 4.2. *Let G be a local function on the configuration space. Then for every test function $J \in C(S)$ and each $\delta > 0$,*

$$\lim_{l \to \infty} \overline{\lim_{N \to \infty}} \, P_N \left(\frac{1}{N} \int_0^T \left| \sum_{x \in \Lambda_N} J \left(\frac{x}{N} \right) \left(\tau_x G(\eta(t)) - \overline{G}(\bar{\eta}_{x,\,l}(t)) \right) \right| dt > \delta \right) = 0,$$

where (*assuming that G depends on n coordinates*)

$$\overline{G}(u) = E_{\mu_n^u} G \overset{\text{def}}{=} \sum_{(\alpha_1,\ldots,\alpha_n)\in\{0,1\}^n} G(\alpha_1,\ldots,\alpha_n) u^{\sum \alpha_i}(1-u)^{n-\sum \alpha_i}$$

and τ_x is defined by (6).

For the proof see, for example, Jensen and Yau (1999).
Apply this lemma with $G(\eta) = \eta_0(1-\eta_1)$. Then $\overline{G}(u) = u(1-u)$ and

(26)
$$\lim_{l\to\infty} \overline{\lim_{N\to\infty}} P_N\left(\frac{1}{N}\int_0^T \left| \sum_{x\in\Lambda_N} \eta_x(1-\eta_{x+1}) \right.\right.$$
$$\left.\left. - \sum_{x\in\Lambda_N} \bar{\eta}_{x,l}(1-\bar{\eta}_{x,l}) \right| dt > \delta \right) = 0.$$

From (21) and (26) we obtain for any $t \in [0,T]$ and $\delta > 0$,

(27)
$$\lim_{l\to\infty} \overline{\lim_{N\to\infty}} P_N\left(\left| \frac{j^{(N)}(t)}{N^2} - \frac{(2p-1)}{N} \right.\right.$$
$$\left.\left. \times \int_0^t \sum_{x\in\Lambda_N} \bar{\eta}_{x,l}(s)(1-\bar{\eta}_{x,l}(s))\,ds \right| > \delta \right) = 0.$$

The next lemma is the key step of the proof. Combined with the existence of the scaling limit it will allow us to substitute averages over small macroscopic blocks for the averages over large microscopic blocks in (27) and will essentially imply the two-block estimate.

LEMMA 4.3. *Let ρ be the entropic solution of the hydrodynamic equation with the initial condition m_0. Then for any $t \geq 0$ and $\delta > 0$,*

$$\lim_{N\to\infty} P_N\left(\left| \frac{j^{(N)}(t)}{N^2} - (2p-1)\int_0^t \int_S \rho(s,\theta)(1-\rho(s,\theta))\,d\theta\,ds \right| > \delta \right) = 0.$$

We postpone the proof of this lemma until the next section and finish the proof of Theorem 4.1. From (27), Lemma 4.3 and Lemma A.4 it follows that

(28) $$\lim_{\varepsilon\to 0} \overline{\lim_{l\to\infty}} \overline{\lim_{N\to\infty}} P_N\left(\left| \frac{1}{N}\int_0^T \left(\sum_{x\in\Lambda_N} \bar{\eta}_{x,l}^2(t) - \sum_{x\in\Lambda_N} \bar{\eta}_{x,\varepsilon N}^2(t) \right) dt \right| > \delta \right) = 0.$$

By Lemma A.6 we can strengthen (28) and conclude that

$$\lim_{\varepsilon\to 0} \overline{\lim_{l\to\infty}} \overline{\lim_{N\to\infty}} P_N\left(\frac{1}{N}\int_0^T \sum_{x\in\Lambda_N} \left(\bar{\eta}_{x,l}(t) - \bar{\eta}_{x,\varepsilon N}(t) \right)^2 dt > \delta \right) = 0.$$

This immediately implies the statement of Theorem 4.1. □

5. Proof of Lemma 4.3. The additivity property of $j^{(N)}(t)$ and of the Lebesgue integral allows us to assume without loss of generality that t is small. Fix an arbitrary $t < 1/2$ and $\varepsilon \in (0, 1 - 2t)$. Consider an arc of length ε on S. At first, we study the dynamics of particles within this arc. We "straighten" this arc into an interval of the real line, surround it at $t = 0$ with the same environment that it had on the circle and start a new process on $\mathbb{R}$. Using a coupling argument, we show that for a given small t and all sufficiently large N the dynamics of particles, present in that interval at any time less than t, is practically the same as the dynamics of particles on the circle. The problem on the line allows a straightforward computation. Then we divide the circle into small arcs and apply the above procedure to each arc separately. Having done this, we rebuild the circle, putting all arcs back together, and obtain the statement of the lemma.

STEP 1 (The problem on $\mathbb{R}$). Fix an arbitrary $\tilde{\theta}_0 \in [0, 1)$. Consider the asymmetric simple exclusion process on $\mathbb{R}$ with the following initial data: for $x \in \mathbb{Z}$ we define

$$\tilde{\eta}_x^{(N)}(0) = \begin{cases} \eta_{x \,(\mathrm{mod}\, N)}^{(N)}(0), & \text{if } x/N \in [\tilde{\theta}_0 - \tfrac{1}{2}, \tilde{\theta}_0 + \tfrac{1}{2}), \\ 0, & \text{otherwise.} \end{cases}$$

Then $\delta_{\tilde{\eta}^{(N)}(0)} \sim \tilde{\rho}_0$, where

$$\tilde{\rho}_0(\theta) = \begin{cases} \rho_0(\theta \,(\mathrm{mod}\, 1)), & \text{if } \theta \in [\tilde{\theta}_0 - \tfrac{1}{2}, \tilde{\theta}_0 - \tfrac{1}{2}), \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $\widetilde{P}_N$ the probability measure corresponding to this process and by $\widetilde{E}_N$ the expectation relative to this measure. For the rest of the proof all quantities related to the problem on $\mathbb{R}$ will be marked with a "tilde." By the existence of the scaling limit (see Theorem 2.1),

$$\tilde{\nu}_N^t \sim \tilde{\rho}(t, \cdot),$$

where $\tilde{\rho}$ is the entropic solution of the problem

(29)
$$\tilde{\rho}_t + (2p - 1)(\tilde{\rho}(1 - \tilde{\rho}))_\theta = 0, \qquad (t, \theta) \in (0, \infty) \times \mathbb{R},$$
$$\tilde{\rho}\big|_{t=0} = \tilde{\rho}_0.$$

Observe that for any $t \geq 0$ the function $\tilde{\rho}(\cdot, t)$ has a compact support.

Fix an arbitrary interval $[\tilde{\alpha}, \tilde{\beta}) \subset \mathbb{R}$. Let $\tilde{j}_{\tilde{\alpha}, \tilde{\beta}}^{(N)}(t)$ be the difference of the total number of jumps to the right and the total number of jumps to the left made during time t by the particles in the interval $[\tilde{\alpha}, \tilde{\beta})$.

LEMMA 5.1. *For any interval $[\tilde{\alpha}, \tilde{\beta}) \subset \mathbb{R}$,*

$$\lim_{N \to \infty} \widetilde{E}_N \left(\left| \frac{\tilde{j}_{\tilde{\alpha}, \tilde{\beta}}^{(N)}(t)}{N^2} - (2p - 1) \int_0^t \int_{\tilde{\alpha}}^{\tilde{\beta}} \tilde{\rho}(s, \theta)(1 - \tilde{\rho}(s, \theta)) \, d\theta \, ds \right| \right) = 0,$$

where $\tilde{\rho}$ is the entropic solution of problem (29).

PROOF.   For any $x \in \mathbb{Z}$ denote by $\tilde{j}^{(N)}_{x,\,x+1}(t)$ the difference between the number of particle crossings from x/N to $(x+1)/N$ and the number of particle crossings from $(x+1)/N$ to x/N up to time t. Then, evidently,

$$
\text{(30)} \qquad \tilde{j}^{(N)}_{x,\,x+1}(t) = \sum_{y=-\infty}^{x} \tilde{\eta}^{(N)}_{y}(0) - \sum_{y=-\infty}^{x} \tilde{\eta}^{(N)}_{y}(t)
$$

and

$$
\text{(31)} \qquad \tilde{j}^{(N)}_{\tilde{\alpha},\,\tilde{\beta}}(t) = \sum_{\substack{x \in \mathbb{Z}: \\ x/N \in [\tilde{\alpha}, \tilde{\beta})}} \tilde{j}^{(N)}_{x,\,x+1}(t).
$$

For any $c \in \mathbb{R}$ and each N choose $x \in \mathbb{Z}$ such that $x/N \leq c < (x+1)/N$. Then by (30) and the existence of the scaling limit

$$
\text{(32)} \qquad \lim_{N \to \infty} \widetilde{E}_N \left| \frac{\tilde{j}^{(N)}_{x,\,x+1}(t)}{N} - \int_{-\infty}^{c} \big(\tilde{\rho}_0(\theta) - \tilde{\rho}(t, \theta) \big) \, d\theta \right| = 0.
$$

From (31) and (32) we obtain

$$
\widetilde{E}_N \left| \frac{\tilde{j}^{(N)}_{\tilde{\alpha},\,\tilde{\beta}}(t)}{N^2} - \int_{\tilde{\alpha}}^{\tilde{\beta}} \int_{-\infty}^{c} \big(\tilde{\rho}_0(\theta) - \tilde{\rho}(t, \theta) \big) \, d\theta \, dc \right|
$$

$$
\leq \widetilde{E}_N \; \frac{1}{N} \sum_{\substack{x \in \mathbb{Z}: \\ x/N \in [\tilde{\alpha}, \tilde{\beta})}} \left| \frac{\tilde{j}^{(N)}_{x,\,x+1}(t)}{N} - \int_{-\infty}^{x/N} \big(\tilde{\rho}_0(\theta) - \tilde{\rho}(t, \theta) \big) \, d\theta \right| \to 0
$$

as $N \to \infty$. Since $\tilde{\rho}$ is a weak solution of problem (29), we also have that

$$
\int_{\tilde{\alpha}}^{\tilde{\beta}} \int_{-\infty}^{c} \big(\tilde{\rho}_0(\theta) - \tilde{\rho}(t, \theta) \big) \, d\theta \, dc = (2p - 1) \int_{0}^{t} \int_{\tilde{\alpha}}^{\tilde{\beta}} \tilde{\rho}(s, \theta)(1 - \tilde{\rho}(s, \theta)) \, d\theta \, ds.
$$

The proof of Lemma 5.1 is complete. □

STEP 2 (Back to the circle).   For a given $\tilde{\theta}_0 \in [0, 1)$ define the bijection ϕ between the interval $[\tilde{\theta}_0 - 1/2, \tilde{\theta}_0 + 1/2)$ and the circle $S = \mathbb{R}/\mathbb{Z}$ by

$$
\phi(\tilde{\theta}) = \{\tilde{\theta}\} \quad \text{for every } \tilde{\theta} \in [\tilde{\theta}_0 - 1/2, \tilde{\theta}_0 + 1/2),
$$

where $\{y\}$ denotes the fractional part of y. Thus each site x/N, $x \in \Lambda_N$, of the circle has a corresponding site on the line.

Choose $\tilde{\alpha} = \tilde{\theta}_0 - \varepsilon/2$ and $\tilde{\beta} = \tilde{\theta}_0 + \varepsilon/2$. Let θ_0, α and β be the images of $\tilde{\theta}_0$, $\tilde{\alpha}$ and $\tilde{\beta}$ under ϕ. On the macroscopic level, for our choice of t and ε we have the equality

$$
\int_{0}^{t} \int_{\tilde{\alpha}}^{\tilde{\beta}} \tilde{\rho}(s, \theta)(1 - \tilde{\rho}(s, \theta)) \, d\theta \, ds = \int_{0}^{t} \int_{\alpha}^{\beta} \rho(s, \theta)(1 - \rho(s, \theta)) \, d\theta \, ds,
$$

since the solutions of (29) have a bounded domain of dependence on the initial data.

On the particle level, we couple the process on the line, constructed in Step 1, with the process on the circle. For convenience, we turn to the following informal description of a simple exclusion process on S. Every site of the circle has an associated Poisson "clock," which "ticks" at rate N. All these "clocks" are independent. When the "clock" at a given site "ticks" and there is a particle at that site, the particle attempts to jump according to the rules given in Section 2. The coupling is performed by synchronizing at $t = 0$ the clock at each site on the circle with the clock at the corresponding site on the line, so that the particles at these sites always try to jump together. All other sites on the line carry their independent Poisson "clocks."

The rightmost (respectively, leftmost) particle on the line can always accomplish a jump to the right (left). This is not always the case for its counterpart on the circle. If a particle on the circle cannot jump while its pair on the line can, the particle on the circle becomes "infected" and turns red. Every time a "healthy" particle attempts to jump to a site occupied by a red particle it also becomes red. Nothing happens if a red particle attempts a jump to the site occupied by a "healthy" particle. As t increases, the "infection" spreads. Let τ_N be the first time a red particle reaches the arc $[\alpha, \beta) \in S$.

LEMMA 5.2. $P_N(\tau_N > t) \to 1$ as $N \to \infty$.

PROOF. Let us mark with the letter "r" (respectively, "l") the particle on the circle attached to the rightmost (respectively, leftmost) particle on the line. All particles initially are considered to be "healthy." To obtain an estimate on τ_N we simplify the problem by keeping only the two marked particles on the circle. Moreover, we assume that these particles are independent, particle l can only jump to the left, particle r can only jump to the right, and the number of jumps made by each particle during time t has the Poisson distribution with parameter Nt. It is clear that τ_N is greater or equal to the minimum of the time when r reaches $[\alpha, \beta)$ and the time when l reaches $[\alpha, \beta)$. Therefore,

$$P_N(\tau_N > t) \geq \left(\mathrm{Prob}\left(X < \frac{N(1 - \varepsilon)}{2} \right) \right)^2,$$

where X is a Poisson random variable with parameter Nt. Since X can be thought of as the sum of N independent Poisson random variables with parameter t, by the law of large numbers,

$$\mathrm{Prob}\left(X < \frac{N(1 - \varepsilon)}{2} \right) = \mathrm{Prob}\left(\frac{X}{N} < \frac{(1 - \varepsilon)}{2} \right) \to 1$$

as $N \to \infty$ if $t < (1 - \varepsilon)/2$. □

Let A_N be the event that

$$\left| \frac{j_{\alpha, \beta}^{(N)}(t)}{N^2} - (2p - 1) \int_0^t \int_\alpha^\beta \rho(s, \theta)(1 - \rho(s, \theta)) \, d\theta \, ds \right| > \delta.$$

Then by Lemmas 5.1 and 5.2,

(33)
$$P_N(A_N) = P_N(A_N \cap \{\tau_N \leq t\}) + P_N(A_N \cap \{\tau_N > t\})$$
$$\leq P_N(\tau_N \leq t) + \widetilde{P}_N(\widetilde{A}_N) \to 0 \quad \text{as } N \to \infty,$$

where $\widetilde{A}_N$ is the corresponding event for the process on the line.

STEP 3. Divide the circle into arcs of length $\varepsilon < (1 - 2t)$. We obtain $k = 1/\varepsilon$ arcs $[\alpha_1, \alpha_2)$, $[\alpha_2, \alpha_3)$, ..., $[\alpha_k, \alpha_{k+1})$, where $\alpha_i = (i - 1)\varepsilon \pmod{1}$, $i = 1, 2, \ldots, k + 1$. Applying (33) with A_N constructed for $\alpha = \alpha_i$ and $\beta = \alpha_{i+1}$, $i = 1, 2, \ldots, k$, we conclude that

$$P_N\left(\left| \frac{j^{(N)}(t)}{N^2} - (2p - 1) \int_0^t \int_0^1 \rho(s, \theta)(1 - \rho(s, \theta)) \, d\theta \, ds \right| > \delta \right)$$

$$\leq \sum_{i=1}^k P_N\left(\left| \frac{j_{\alpha_i, \alpha_{i+1}}^{(N)}(t)}{N^2} - (2p - 1) \int_0^t \int_{\alpha_i}^{\alpha_{i+1}} \rho(s, \theta)(1 - \rho(s, \theta)) \, d\theta \, ds \right| > \frac{\delta}{k} \right),$$

which goes to zero as $N \to \infty$ by Lemma 5.1. $\square$

APPENDIX

Basic lemma. We use the notation introduced in Section 2.

LEMMA A.1. *Let $\{\nu_N\}$ be a sequence of probability measures which satisfies the following conditions*:

(A1) *Each ν_N is a measure on $\mathbb{X}_N$ and $\nu_N \sim \rho$ for some Lebesgue measurable function ρ on S, $\rho(\theta) \in [0, 1]$ a.e.*
(A2) $D_N(\sqrt{f_N}) = o(N)$ *as $N \to \infty$, where $f_N(\eta) = \frac{\nu_N(\eta)}{\mu_N(\eta)}$.*
(A3) $\lim_{\varepsilon \to 0} \overline{\lim}_{l \to \infty} \overline{\lim}_{N \to \infty} E_{\nu_N} \frac{1}{N} \sum_{x \in \Lambda_N} |\bar{\eta}_{x, l} - \bar{\eta}_{x, \varepsilon N}| = 0.$

Then

$$\lim_{N \to \infty} \frac{1}{N} H(\nu_N) = \int_S h(\rho(\theta)) \, d\theta,$$

where h is given by (11).

PROOF. At first notice that

(34)
$$\lim_{N \to \infty} \frac{1}{N} H(\nu_N) = \lim_{N \to \infty} \frac{1}{N} H(\nu_N \mid \mu_N) - \log 2.$$

Below we estimate the specific relative entropy $\frac{1}{N} H(\nu_N \mid \mu_N)$.

Lower bound. For this part we only need to assume (A1). Then we claim that

(35) $$\lim_{N \to \infty} \frac{1}{N} H(\nu_N \,|\, \mu_N) \geq \int_S h(\rho(\theta)) \, d\theta + \log 2.$$

By the entropy inequality for any $J \in C(S)$ we have that

$$\frac{1}{N} H(\nu_N \,|\, \mu_N) \geq \frac{1}{N} E_{\nu_N} \sum_{x \in \Lambda_N} J\left(\frac{x}{N}\right) \eta_x - \frac{1}{N} \log E_{\mu_N} \exp\left(\sum_{x \in \Lambda_N} J\left(\frac{x}{N}\right) \eta_x\right).$$

Using the convergence (A1) we obtain

$$\lim_{N \to \infty} \frac{1}{N} H(\nu_N \,|\, \mu_N) \geq \sup_{J \in C(S)} \left(\int_S J(\theta)\rho(\theta) - \log\left(1 + e^{J(\theta)}\right) d\theta\right) + \log 2.$$

Let $\mathcal{F}(J) \stackrel{\text{def}}{=} \int_S J(\theta)\rho(\theta) - \log\left(1 + e^{J(\theta)}\right) d\theta$. We show that

(36) $$\sup_{J \in C(S)} \mathcal{F}(J) = \int_S h(\rho(\theta)) \, d\theta.$$

Clearly,

$$\mathcal{F}(J) \leq \int_S \sup_{y \in \mathbb{R}} \left(y\rho(\theta) - \log(1 + e^y)\right) d\theta = \int_S h(\rho(\theta)) \, d\theta,$$

where the supremum is attained at

$$y(\theta) = \begin{cases} \log \dfrac{\rho(\theta)}{1 - \rho(\theta)}, & \text{if } \rho(\theta) \neq 0, 1, \\ -\infty, & \text{if } \rho(\theta) = 0, \\ +\infty, & \text{if } \rho(\theta) = 1. \end{cases}$$

Therefore, if ρ is continuous and does not take values 0 and 1, the functional $\mathcal{F}$ attains its maximum at $J_*(\theta) = y(\theta)$, and (36) holds. For a general ρ we need an additional argument to complete the proof.

Let ρ_n be a sequence of continuous functions such that $\rho_n \to \rho$ as $n \to \infty$ a.e. on S and $\rho_n \in [0, 1]$ for all n. By Egorov's theorem for any $\varepsilon > 0$ there is a set E_ε such that:

  (i) The Lebesgue measure of E_ε is at least $1 - \varepsilon$.
  (ii) $\rho_n \to \rho$ uniformly on E_ε as $n \to \infty$.

For any $\varepsilon > 0$ define

$$\rho_{n,\,\varepsilon}(\theta) = \begin{cases} \rho_n(\theta), & \text{if } \varepsilon < \rho_n(\theta) < 1 - \varepsilon, \\ \varepsilon, & \text{if } \rho_n(\theta) \leq \varepsilon, \\ 1 - \varepsilon, & \text{if } \rho_n(\theta) \geq 1 - \varepsilon. \end{cases}$$

Functions $\rho_{n,\,\varepsilon}$ are continuous on S. Let

$$J_{n,\,\varepsilon}(\theta) = \log \frac{\rho_{n,\,\varepsilon}(\theta)}{1 - \rho_{n,\,\varepsilon}(\theta)}.$$

Functions $J_{n,\varepsilon}$ are continuous and $|J_{n,\varepsilon}| < 2|\log \varepsilon|$ uniformly in n. We claim that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathscr{F}(J_{n,\varepsilon}) = \int_S h(\rho(\theta))\,d\theta.$$

The proof is straightforward. For an arbitrary $\varepsilon > 0$, all $n \geq n_0(\varepsilon)$ and all $\theta \in E_\varepsilon$ we have that

$$|\rho_{n,\varepsilon}(\theta) - \rho(\theta)| \leq |\rho_{n,\varepsilon}(\theta) - \rho_n(\theta)| + |\rho_n(\theta) - \rho(\theta)| < 2\varepsilon.$$

We write

$$\left| \mathscr{F}(J_{n,\varepsilon}) - \int_S h(\rho(\theta))\,d\theta \right| \leq \left| \mathscr{F}(J_{n,\varepsilon}) - \int_S h(\rho_{n,\varepsilon}(\theta))\,d\theta \right|$$

$$+ \left| \int_S h(\rho_{n,\varepsilon}(\theta))\,d\theta - \int_S h(\rho(\theta))\,d\theta \right| = I_1 + I_2.$$

Estimating I_1 and I_2 we find that

$$I_1 = \left| \int_S J_{n,\varepsilon}(\theta)(\rho(\theta) - \rho_{n,\varepsilon}(\theta))\,d\theta \right|$$

$$\leq \left(\int_{E_\varepsilon} + \int_{S \setminus E_\varepsilon} \right) |J_{n,\varepsilon}(\theta)||\rho_{n,\varepsilon}(\theta) - \rho|(\theta)\,d\theta$$

$$\leq 2|\log \varepsilon|(2\varepsilon + \varepsilon) = 6\varepsilon|\log \varepsilon|$$

and

$$I_2 \leq \left(\int_{E_\varepsilon} + \int_{S \setminus E_\varepsilon} \right) |h(\rho_{n,\varepsilon}(\theta)) - h(\rho(\theta))|\,d\theta \leq |h(2\varepsilon)| + \varepsilon \log 2.$$

By letting ε go to zero we obtain (36) and, therefore, (35).

Upper bound. We have to show that

$$(37) \qquad \overline{\lim_{N \to \infty}} \frac{1}{N} H(\nu_N \,|\, \mu_N) \leq \int_S h(\rho(\theta))\,d\theta + \log 2.$$

We start with the following simple fact.

LEMMA A.2 (Martingale decomposition). *Let $(\Omega, \mathscr{G}_0, P)$ be a probability space and $\mathscr{G}_0 \supset \mathscr{G}_1 \supset \cdots \supset \mathscr{G}_k$ be a decreasing sequence of σ-algebras. For a nonnegative random variable X_0, such that $EX_0|\log X_0| < \infty$, define $X_i = E(X_0|\mathscr{G}_i)$, $i = 0, 1, \ldots, k$. Then*

$$E X_0 \log X_0 = \sum_{i=1}^{k} E E\left(X_{i-1} \log \frac{X_{i-1}}{X_i} \,\Big|\, \mathscr{G}_i \right) + E X_k \log X_k.$$

PROOF. We have

$$E X_0 \log X_0 = \sum_{i=1}^{k} E\left(X_{i-1} \log X_{i-1} - X_i \log X_i \right) + E X_k \log X_k.$$

We can rewrite the first sum in the right-hand side of the above equality as

$$\sum_{i=1}^{k} E\, E\Big(X_{i-1} \log X_{i-1} - X_i \log X_i \mid \mathscr{G}_i \Big)$$

$$= \sum_{i=1}^{k} E\, E\Big(X_{i-1} \log X_{i-1} - X_{i-1} \log X_i \mid \mathscr{G}_i \Big)$$

$$= \sum_{i=1}^{k} E\, E\Big(X_{i-1} \log \frac{X_{i-1}}{X_i} \,\Big|\, \mathscr{G}_i \Big). \qquad \square$$

We apply the above lemma in the following context. Consider f_N defined in (A2) as a random variable on the probability space $(\mathbb{X}_N, \Sigma_{0,N}, \mu_N)$, where $\Sigma_{0,N}$ is the natural σ-algebra for μ_N. Fix a large number $l \in \mathbb{N}$. Divide a circle into $k_N = \left[\frac{N}{2l}\right]$ equal parts, which we call "boxes." Each box contains $2l$ or $(2l+1)$ sites. Let $B_i \subset \Lambda_N$ be the set of indexes corresponding to the ith box. For $i = 1, \ldots, k_N$ define

$$\bar{\eta}_i = \frac{1}{|B_i|} \sum_{x \in B_i} \eta_x,$$

where $|B_i|$ is the number of elements in B_i, and

$$\Sigma_i = \Big\{ \sigma\text{-algebra generated by the averages } \bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_i \text{ and}$$

$$\text{by } \eta_x, \ x \in \bigcup_{j=i+1}^{k_N} B_j \Big\}.$$

Let $f_i = E_{\mu_N}(f_N \mid \Sigma_i)$. The dependence of Σ_i and f_i on N will not be reflected in the notation. By Lemma A.2,

(38)
$$\frac{1}{N} H(\nu_N \mid \mu_N) = \frac{1}{N} E_{\mu_N} f_N \log f_N$$

$$= \frac{1}{N} \sum_{i=1}^{k_N} E_{\mu_N} E_{\mu_N} \Big(f_i \log \frac{f_{i-1}}{f_i} \,\Big|\, \Sigma_i \Big) + \frac{1}{N} E_{\mu_N} f_{k_N} \log f_{k_N}.$$

We are going to show that the sum of the first k_N terms on the right-hand side of (38) can be made arbitrarily small as $N \to \infty$. Then we use the two-block estimate (A3) to obtain a bound on the last term of (38).

STEP 1. For the first part of our plan we use a version of the logarithmic Sobolev inequality.

THEOREM A.1. Let $\mathbb{X}_{l,n} = \{\eta \in \{0,1\}^l : \sum_{x=1}^{l} \eta_x = n\}$ and $\mu_{l,n}$ be the uniform measure on $\mathbb{X}_{l,n}$. Then there is a constant $C(l)$ such that for any

nonnegative function f on $\mathbb{X}_{l,n}$, which satisfies the relation $E_{\mu_{l,n}} f = 1$, the following inequality holds uniformly in $n \in \{0, 1, \ldots, l\}$:

$$E_{\mu_{l,n}} f \log f \leq C(l) E_{\mu_{l,n}} \sum_{x=1}^{l-1} \left(\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)} \right)^2.$$

This is an obvious statement, since we do not specify the order of $C(l)$. In fact, much stronger results are known. Namely, it was shown in Lee and Yau (1998) [see also Yau (1997)] that $C(l) = O(l^2)$. But for our purposes this information is not needed.

Let $\mu_i(\cdot) \overset{\text{def}}{=} \mu_N(\cdot \mid \Sigma_i)$. For any fixed $\bar{\eta}_1, \ldots, \bar{\eta}_i$, and $\eta_x, x \in \cup_{j=i+1}^{k_N} B_j$, we regard μ_i as a measure on the first i boxes. Denote by $\widetilde{\mu}_i$ the marginal of μ_i on the ith box. It is easy to see that $\widetilde{\mu}_i$ is the uniform measure. Namely, $\widetilde{\mu}_i$ can be identified with $\mu_{l,n}$, where $l = |B_i|$ and $n = \bar{\eta}_i |B_i|$ if we use the notation of Theorem A.1. The function f_i can be written as $E_{\mu_i} f_N$ or as $E_{\widetilde{\mu}_i} f_{i-1}$. Therefore the ratio $\widetilde{f}_i = f_{i-1}/f_i$ satisfies the condition $E_{\widetilde{\mu}_i} \widetilde{f}_i = 1$. By Theorem A.1,

(39)
$$E_{\widetilde{\mu}_i} (\widetilde{f}_i \log \widetilde{f}_i) \leq C(l) E_{\widetilde{\mu}_i} \sum_{x, x+1 \in B_i} \left(\sqrt{\widetilde{f}_i(\eta^{x,x+1})} - \sqrt{\widetilde{f}_i(\eta)} \right)^2$$

$$\leq \frac{C(l)}{f_i} E_{\widetilde{\mu}_i} \sum_{x, x+1 \in B_i} \left(\sqrt{f_{i-1}(\eta^{x,x+1})} - \sqrt{f_{i-1}(\eta)} \right)^2.$$

For any $x, x+1 \in B_i$ we have that

(40)
$$\left(\sqrt{f_{i-1}(\eta^{x,x+1})} - \sqrt{f_{i-1}(\eta)} \right)^2 = \left(\sqrt{E_{\mu_{i-1}} f_N(\eta^{x,x+1})} - \sqrt{E_{\mu_{i-1}} f_N(\eta)} \right)^2$$

$$\leq E_{\mu_{i-1}} \left(\sqrt{f_N(\eta^{x,x+1})} - \sqrt{f_N(\eta)} \right)^2.$$

Here we used the fact that μ_{i-1} is invariant under the transformation which takes η to $\eta^{x,x+1}$ for $x, x+1 \in B_i$. Substituting (40) into (39), taking the summation over i and averaging with respect to μ_N we obtain

$$\sum_{i=1}^{k_N} E_{\mu_N} E_{\mu_N} \left(f_{i-1} \log \frac{f_{i-1}}{f_i} \Big| \Sigma_i \right) \leq C(l) E_{\mu_N} \sum_{x=1}^{N-1} \left(\sqrt{f_N(\eta^{x,x+1})} - \sqrt{f_N(\eta)} \right)^2$$

$$\leq 4C(l) D_N(\sqrt{f_N}).$$

The assumption (A2) allows us to conclude that for any fixed l,

(41)
$$\overline{\lim_{N \to \infty}} \frac{1}{N} \sum_{i=1}^{k_N} E_{\mu_N} E_{\mu_N} \left(f_{i-1} \log \frac{f_{i-1}}{f_i} \Big| \Sigma_i \right) = 0.$$

STEP 2. We turn now to the last term of (38). Let $\bar{\mu}_{k_N}$ be the joint distribution of averages $\bar{\eta}_1, \bar{\eta}_2 \ldots, \bar{\eta}_{k_N}$ under μ_N; that is,

$$(42) \qquad \bar{\mu}_{k_N}(\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_{k_N}) = \frac{1}{2^N} \prod_{i=1}^{k_N} \binom{|B_i|}{\bar{\eta}_i |B_i|}.$$

Then $E_{\bar{\mu}_{k_N}} f_{k_N} = 1$ and

$$
\begin{aligned}
E_{\mu_N} f_{k_N} \log f_{k_N} &= E_{\bar{\mu}_{k_N}} f_{k_N} \log f_{k_N} \\
(43) \qquad &= E_{\bar{\mu}_{k_N}} f_{k_N} \log(f_{k_N} \bar{\mu}_{k_N}) - E_{\bar{\mu}_{k_N}} f_{k_N} \log \bar{\mu}_{k_N} \\
&\leq -E_{\bar{\mu}_{k_N}} f_{k_N} \log \bar{\mu}_{k_N} = E_{\nu_N} \log \bar{\mu}_{k_N}.
\end{aligned}
$$

The following lemma is a consequence of Stirling's formula.

LEMMA A.3. *For any sequence* $\{k_n\}_{n=1}^{\infty}$, $k_n \in \{0, 1, \ldots, n\}$,

$$\lim_{n \to \infty} \left(\frac{1}{n} \log \binom{n}{k_n} + h\left(\frac{k_n}{n}\right) \right) = 0.$$

The proof of this statement can be found, for example, in Ellis [(1985), Lemma 1.3.2].

The relations (42), (43), and the above lemma imply that

$$(44) \qquad \overline{\lim_{N \to \infty}} \frac{1}{N} E_{\mu_N} f_{k_N} \log f_{k_N} \leq \overline{\lim_{N \to \infty}} E_{\nu_N} \frac{2l}{N} \sum_{i=1}^{k_N} h(\bar{\eta}_i) + \log 2,$$

since $|B_i| \geq 2l$ and h is nonpositive.

CONCLUSION. We notice that the left-hand side of (38) does not depend on the way we divide the circle into boxes. Therefore from (38), (41) and (44) we obtain

$$
\begin{aligned}
\overline{\lim_{N \to \infty}} \frac{1}{N} H(\nu_N \mid \mu_N) &\leq \lim_{l \to \infty} \overline{\lim_{N \to \infty}} \frac{1}{N} E_{\nu_N} \sum_{x=0}^{2l-1} \sum_{i=1}^{k_N} \tau_x h(\bar{\eta}_i) + \log 2 \\
&= \lim_{l \to \infty} \overline{\lim_{N \to \infty}} \frac{1}{N} E_{\nu_N} \sum_{x \in \Lambda_N} h(\bar{\eta}_{x,l}) + \log 2,
\end{aligned}
$$

where τ_x is defined in (6). By the two-block estimate (A3) we have that

$$(45) \qquad \overline{\lim_{N \to \infty}} \frac{1}{N} H(\nu_N | \mu_N) \leq \lim_{\varepsilon \to 0} \overline{\lim_{N \to \infty}} \frac{1}{N} E_{\nu_N} \sum_{x \in \Lambda_N} h(\bar{\eta}_{x, \varepsilon N}) + \log 2.$$

Next we use the following simple consequence of condition (A1).

LEMMA A.4. *Let* (A1) *hold. Then for any* $F \in C(S)$,

$$\lim_{\varepsilon \to 0} \overline{\lim_{N \to \infty}} E_{\nu_N} \left| \frac{1}{N} \sum_{x \in \Lambda_N} F(\bar{\eta}_{x, \varepsilon N}) - \int_S F(\rho(\theta)) \, d\theta \right| = 0.$$

PROOF.   Let

$$\rho_\varepsilon(u) = \frac{1}{2\varepsilon} \int_{u-\varepsilon}^{u+\varepsilon} \rho(\theta)\, d\theta.$$

Then ρ_ε is continuous and

(46) $\rho_\varepsilon \overset{L^1(S)}{\longrightarrow} \rho$ as $\varepsilon \to 0$.

For any $\varepsilon > 0$ and N we have

$$E_{\nu_N}\left| \frac{1}{N} \sum_{x\in\Lambda_N} F(\bar\eta_{x,\varepsilon N}) - \int_S F(\rho(\theta))\,d\theta \right|$$

$$\leq E_{\nu_N}\left| \frac{1}{N} \sum_{x\in\Lambda_N} F(\bar\eta_{x,\varepsilon N}) - \frac{1}{N} \sum_{x\in\Lambda_N} F\left(\rho_\varepsilon\left(\frac{x}{N}\right)\right) \right|$$

$$+ \left| \frac{1}{N} \sum_{x\in\Lambda_N} F\left(\rho_\varepsilon\left(\frac{x}{N}\right)\right) - \int_S F(\rho_\varepsilon(\theta))\,d\theta \right| + \left| \int_S F(\rho_\varepsilon(\theta))\,d\theta - \int_S F(\rho(\theta))\,d\theta \right|$$

$$= I_{N,\varepsilon} + II_{N,\varepsilon} + III_\varepsilon.$$

Let N go to infinity. Then $I_{N,\varepsilon} \to 0$ by (A1) and the bounded convergence theorem and $II_{N,\varepsilon} \to 0$ by the continuity of ρ_ε. Finally let ε go to zero. Relation (46) implies that $III_\varepsilon \to 0$.   □

Applying Lemma A.4 with $F = h$ we obtain

(47) $\displaystyle \lim_{\varepsilon\to 0} \overline{\lim_{N\to\infty}} \frac{1}{N} E_{\nu_N} \sum_{x\in\Lambda_N} h(\bar\eta_{x,\varepsilon N}) = \int_S h(\rho(\theta))\,d\theta.$

The upper bound (37) follows from (45) and (47). This finishes the proof of Lemma A.1.   □

Two technical lemmas.   The first lemma shows that if the hydrodynamic scaling limit exists (see Definition 2.1) then the analog of (7) holds for any bounded measurable function J. More precisely, we have the following.

LEMMA A.5.   *Let $\nu_N \sim \rho$ for some measurable $\rho \in L^1$. For any bounded measurable function J on S and for each $N \in \mathbb{N}$ define the step function J_N by*

$$J_N(\theta) = N \int_{(x-1/2)/N}^{(x+1/2)/N} J(\alpha)\, d\alpha,$$

for all $\theta \in [(x-1/2)/N, (x+1/2)/N)$ and $x \in \Lambda_N$.
   Then for any continuous function F,

$$\lim_{N\to\infty} E_{\nu_N}\left| \frac{1}{N} \sum_{x\in\Lambda_N} F\left(J_N\left(\frac{x}{N}\right)\right)\eta_x - \int_S F(J(\theta))\rho(\theta)\,d\theta \right| = 0.$$

PROOF. Since $\nu_N \sim \rho$ and $J_N \to J$ as $N \to \infty$ in $L^1(S)$, we have that

$$\lim_{n\to\infty}\lim_{N\to\infty} E_{\nu_N}\left|\frac{1}{N}\sum_{x\in\Lambda_N} F\left(J_n\left(\frac{x}{N}\right)\right)\eta_x - \int_S F(J(\theta))\rho(\theta)\,d\theta\right| = 0.$$

Therefore it suffices to show that

$$(48)\qquad \lim_{n\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{x\in\Lambda_N}\left|F\left(J_n\left(\frac{x}{N}\right)\right) - F\left(J_N\left(\frac{x}{N}\right)\right)\right| = 0.$$

We have

$$\frac{1}{N}\sum_{x\in\Lambda_N}\left|F\left(J_n\left(\frac{x}{N}\right)\right) - F\left(J_N\left(\frac{x}{N}\right)\right)\right|$$

$$= \sum_{y\in\Lambda_n}\int_{(y-1/2)/n}^{(y+1/2)/n}\left|F(J_n(y/n)) - F(J_N(\theta))\right|d\theta + O(n/N)$$

$$\leq \sum_{y\in\Lambda_n}\int_{(y-1/2)/n}^{(y+1/2)/n}\left|F(J_n(y/n)) - F(J(\theta))\right|d\theta$$

$$+ \sum_{y\in\Lambda_n}\int_{(y-1/2)/n}^{(y+1/2)/n}\left|F(J_N(\theta)) - F(J(\theta))\right|d\theta + O(n/N)$$

$$= \int_S\left|F(J_n(\theta)) - F(J(\theta))\right|d\theta + \int_S\left|F(J_N(\theta)) - F(J(\theta))\right|d\theta + O(n/N).$$

Passing to the limit as $N \to \infty$ and then as $n \to \infty$ we obtain (48). □

The next lemma is a purely algebraic fact, which we use in the proof of the two-block estimate.

LEMMA A.6. *For any $l \in \mathbb{N}$, sufficiently small $\varepsilon > 0$ and all N such that $\varepsilon N \geq l$,*

$$(49)\qquad \sum_{x\in\Lambda_N}\left(\bar{\eta}_{x,l} - \bar{\eta}_{x,\varepsilon N}\right)^2 \leq 4\sum_{x\in\Lambda_N}\left(\bar{\eta}_{x,l}^2 - \bar{\eta}_{x,\varepsilon N}^2\right) + C,$$

where the constant C depends only on l and ε.

PROOF. For $l \in \mathbb{N}$ define the operator $M_l: [0,1]^N \to [0,1]^N$ by $(M_l\eta)_x = \bar{\eta}_{x,l}$. We prove that

$$(50)\qquad \|M_l\eta - M_{\varepsilon N}\circ M_l\eta\|^2 \leq 2\Big(\|M_l\eta\|^2 - \|M_{\varepsilon N}\circ M_l\eta\|^2\Big).$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^N$. Since

$$\|M_{\varepsilon N}\circ M_l\eta - M_{\varepsilon N}\eta\|^2 \leq \frac{C(\varepsilon,l)}{N},$$

the inequality (50) implies the statement of the lemma.

To prove (50) we use the Fourier transform defined by

$$\widehat{\eta}(\lambda) = \sum_{x \in \Lambda_N} e^{2\pi i x \lambda} \eta_x, \qquad \lambda \in S.$$

The function $\widehat{\eta}$ belongs to $L^2(S)$ and satisfies the identity

$$\|\widehat{\eta}\|_2^2 = \int_0^1 |\widehat{\eta}(\lambda)|^2 \, d\lambda = \|\eta\|^2.$$

Let $\chi_l \colon \Lambda_N \to \{0, 1\}$ be the indicator function of the set $\{x \in \Lambda_N \colon |x| \le l\}$,

$$\chi_l(x) = \begin{cases} 1, & \text{if } |x| \le l, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$M_l \eta = \frac{1}{2l+1}(\chi_l * \eta),$$

$$\widehat{M_l \eta} = \frac{1}{2l+1} \widehat{\eta} \, \widehat{\chi_l},$$

where

$$(\chi_l * \eta)_x = \sum_{y \in \Lambda_N} \chi_l(x - y)\eta_y = \sum_{y \in \Lambda_N} \chi_l(y)\eta_{x+y}.$$

Using this notation we can write the left-hand side of (50) as

$$\|M_l \eta - M_{\varepsilon N} \circ M_l \eta\|^2 = \|\widehat{M_l \eta} - \widehat{M_{\varepsilon N} \circ M_l \eta}\|^2$$

$$(51) \qquad = \frac{1}{(2l+1)^2} \left\| \widehat{\eta} \, \widehat{\chi_l} - \frac{1}{2\varepsilon N + 1} \widehat{\eta} \, \widehat{\chi_l} \widehat{\chi_{\varepsilon N}} \right\|^2$$

$$= \frac{1}{(2l+1)^2} \int_0^1 |\widehat{\eta} \, \widehat{\chi_l}(\lambda)|^2 \left(1 - \frac{1}{2\varepsilon N + 1} \widehat{\chi_{\varepsilon N}}(\lambda)\right)^2 d\lambda$$

Observe that $\widehat{\chi_{\varepsilon N}}$ is real-valued. For the right-hand side of (50) we get

$$\|M_l \eta\|^2 - \|M_{\varepsilon N} \circ M_l \eta\|^2$$

$$(52) \qquad = \frac{1}{(2k+1)^2} \int_0^1 |\widehat{\eta} \, \widehat{\chi_l}(\lambda)|^2 \left(1 - \frac{1}{(2\varepsilon N + 1)^2} \left(\widehat{\chi_{\varepsilon N}}(\lambda)\right)^2\right) d\lambda$$

Therefore to obtain a bound on (51) in terms of (52) it is enough to show that

$$f(\lambda) \stackrel{\text{def}}{=} \frac{1}{2\varepsilon N + 1} \widehat{\chi_{\varepsilon N}}(\lambda)$$

is bounded away from (-1). Indeed, the inequality $f(\lambda) \geq -1 + \delta$ implies that

$$1 - f(\lambda) \leq 2 - \delta \leq \frac{2 - \delta}{\delta}(1 + f(\lambda))$$

and

$$(1 - f(\lambda))^2 \leq \frac{2 - \delta}{\delta}(1 - f^2(\lambda)).$$

This proves (50) with the constant $\frac{2-\delta}{\delta}$. We show that $f(\lambda) \geq -\frac{1}{3}$ which gives us $\delta = \frac{2}{3}$ and the constant equal to 2.

It is not difficult to compute that

$$f(\lambda) = \frac{1}{2\varepsilon N + 1} \frac{\sin \lambda(2\varepsilon N + 1)/2}{\sin \lambda/2}.$$

Since $f(\lambda) = f(2\pi - \lambda), \ \lambda \in [0, \pi]$,

$$\inf_{\lambda \in [0, 2\pi]} f(\lambda) = \inf_{\lambda \in [0, \pi]} f(\lambda)$$

$$= \inf_{\lambda \in [0, \pi]} \frac{\sin \lambda(2\varepsilon N + 1)/2}{\lambda(2\varepsilon N + 1)/2} \cdot \frac{\lambda/2}{\sin \lambda/2}$$

$$\geq \inf_{\lambda \in [0, \pi]} \frac{\sin \lambda(2\varepsilon N + 1)/2}{\lambda(2\varepsilon N + 1)/2} \cdot \sup_{\lambda \in [0, \pi]} \frac{\lambda/2}{\sin \lambda/2}$$

$$= -\frac{2}{3\pi} \cdot \frac{\pi}{2} = -\frac{1}{3}. \qquad \qquad \square$$

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