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ON THE EXISTENCE OF A QUASISTATIONARY MEASURE FOR A MARKOV CHAIN

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We consider a Markov chain on a locally compact metric space with an absorbing set. Necessary and sufficient conditions are provided for the existence of a quasistationary probability distribution.

1. Introduction. Consider a Markov chain Φ on a locally compact separable metric space X with an absorbing set $S \subset X$; that is, once in S the chain Φ remains in S with probability 1.

As absorption can take a long time, one is often interested in the evolution of the distribution of Φ conditional on absorption not yet having taken place.

This issue has been investigated in the pioneering papers of Seneta and Vere-Jones [13] and others for countable state spaces. For a review, see [11]. It has been shown that under various conditions this conditional probability has a limit distribution, which is called a *quasistationary* distribution or QSD for short. For instance, it was shown that the existence of the Yaglom limit for some initial state x implies the existence of a QSD. For accounts of limiting conditional distributions, the reader is referred to [8] and [14]–[16].

More recently, still in the countable case and in continuous time, Ferrari, Kesten, Martinez and Picco [4] have also proved the existence of a QSD using renewal arguments and under assumptions on the distribution of the absorption time. Finally, in a recent paper, Hognas [7] considered a parametrized single-species population model of the Ricker type and proved the existence of a QSD under easily checked assumptions on the model. He then analyzed the asymptotic behavior of the QSD as the parameter γ vanishes.

Interestingly enough, two assumptions in [7] and [4] are quite opposite. Hognas [7] assumes that the one-step probability of absorption goes to unity as the distance to the absorbing set becomes large, whereas the discrete-time version of one condition in Ferrari, Kesten, Martinez and Picco [4] implies that this one-step probability vanishes as the state becomes large! The former hypothesis is particular to population growth models.

In all of the previously cited works, the state space was countable and the arguments for proving the existence of a QSD dependent on the discrete nature of X. Seneta and Vere-Jones [13] and Hognas [7] used the Perron–Frobenius theory of nonnegative matrices. Ferrari, Kesten, Martinez and Picco [4] made elegant use of renewal arguments and fixed-point techniques and it is possible that their approach can be extended to the present context. However, as acknowledged by the authors, the conditions are annoying for they restrict

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significantly the applicability of the results and therefore weakening those conditions is highly desirable.

The contribution of this paper is twofold. First, we consider discrete-time Markov chains on Borel spaces which are not necessarily countable. Second, we assume only that the substochastic kernel maps $C_0(X)$, the space of bounded functions that vanish at ∞, into itself. We prove that this condition is implied by those of Hognas [7] or by the discrete-time version of those of Ferrari, Kesten, Martinez and Picco [4] and indeed is much weaker than the latter. In addition, and in the spirit of Lasserre [9], we also obtain a Foster–Liapounov-type necessary and sufficient condition for existence.

Finally, we consider two particular classes of Markov chains with quite opposite hypotheses on the distribution of the absorption time. In the first, we assume that the one-step probability of absorption increases to unity as $|x| \to \infty$, as with the population growth models of Hognas [7]. In the second, we assume instead that this one-step probability tends to 0 as $|x| \to \infty$. In both cases, the requisite existence conditions for a QSD simplify.

2. Preliminaries. Let $(X, \mathscr{B})$ be a measurable space, with X a locally compact separable metric space and $\mathscr{B}$ its usual Borel σ-field. Suppose $C(X)$ is the Banach space of real-valued bounded continuous functions on X and $C_0(X) \subset C(X)$ the Banach space of continuous functions that vanish at ∞, both endowed with the sup norm.

Let $\mathscr{M}$ be the Banach space of finite signed Borel measures on $\mathscr{B}$, endowed with that total variation norm, and $\mathscr{P}$ the subspace of probability measures on $\mathscr{B}$.

Let $\Phi := \{\Phi_0, \Phi_1, \dots\}$ be a discrete-time Markov process with values in X and stochastic transition kernel P; that is, $P(x, \cdot)$ is a probability measure on $\mathscr{B}$ for every $x \in X$ and $P(\cdot, B)$ is a real-valued measurable function on X for every Borel set $B \in \mathscr{B}$.

The chain Φ is said to be *weak Feller* if its kernel P maps $C(X)$ into itself; that is, the function

$$x \mapsto g(x) := \int f(y) P(x, dy) \in C(X)$$

whenever $f \in C(X)$.

We assume that $S \in \mathscr{B}$ is an absorbing set; that is, if $\Phi_t \in S$, then with probability 1, $\Phi_{t+n} \in S$ for all $n = 1, 2, \dots$. For instance, if μ is an invariant probability measure for Φ, then there is an ergodic class decomposition, in which one may find at least one absorbing set $S \in \mathscr{B}$ such that $P(x, S) = 1$ for all $x \in S$ (see, e.g., Hernandez-Lerma and Lasserre [6]). If the chain Φ is strong Feller, then S is closed. We are interested in the case when there exists an S which is a proper subset of X.

Conditional on absorption not yet having taken place, the chain Φ evolves on the set $X \setminus S$ with probability 1. As $X \setminus S$ is also a metric space with a Borel σ-field $\mathscr{B}'$, we may replace $(X \setminus S, \mathscr{B}')$ by $(X, \mathscr{B})$ and consider a Markov chain Φ on $(X, \mathscr{B})$ with now a *substochastic* rather than a stochastic kernel P.

Denote by $s(x)$ the nonnegative function

$$x \mapsto s(x) = \int_X P(x, dy) = P(x, X). \tag{2.1}$$

Without loss of generality we may assume that $s > 0$, since the Borel set $\{x \in X \mid s(x) = 0\}$ can be taken as part of the absorbing set.

We can regard P as an operator acting on $\mathscr{M}$ via

$$\nu \mapsto \nu P(B) := \int P(x, B)\,\nu(dx), \qquad B \in \mathscr{B}, \tag{2.2}$$

and an operator acting on $C(X)$ via

$$f \mapsto Pf(x) := \int f(y)P(x, dy), \qquad x \in X. \tag{2.3}$$

A probability measure $\nu \in \mathscr{P}$ is said to be a *quasistationary distribution* or QSD if and only if

$$\frac{\int P(x, B)\,\nu(dx)}{\int P(x, X)\,\nu(dx)} = \nu(B), \qquad B \in \mathscr{B}. \tag{2.4}$$

Equivalently, $\nu P = [\int s\nu(dx)]\nu$; that is, ν is a left eigenvector of P with eigenvalue $\alpha := \int s\nu(dx)$.

Finally, given a probability measure $\nu \in \mathscr{P}$, define the family of probability measures

$$\mu_n^\nu(B) := \frac{\int P^n(x, B)\,\nu(dx)}{\int P^n(x, X)\,\nu(dx)}, \qquad B \in \mathscr{B},\ n = 1, 2, \ldots. \tag{2.5}$$

In the case where μ_n^ν converges weakly to some $\mu^\nu \in \mathscr{P}$ (denoted as usual by $\mu_n^\nu \Rightarrow \mu^\nu$), we say that μ^ν is the Yaglom limit for the initial distribution ν, by analogy with the situation for finite and countable spaces. It was proved by Seneta and Vere-Jones that in that context, if the Yaglom limit exists for some $x \in X$, then this implies the existence of a QSD.

The usual *vague* convergence of probability measures is denoted by $\overset{*}{\to}$; that is,

$$\mu_n \overset{*}{\to} \mu \Leftrightarrow \int f\mu_n(dx) \to \int f\mu(dx) \qquad \forall f \in C_0(X).$$

3. Main result. In this section, we discuss the issue of the existence of a QSD. First, we make the following Feller-like assumption on the kernel P.

ASSUMPTION 3.1. The map P takes $C_0(X)$ into itself; that is,

$$x \mapsto g(x) := \int f(y)P(x, dy) \in C_0(X) \quad \text{whenever } f \in C_0(X). \tag{3.1}$$

Further, $P1 \in C(X)$.

3.1. *Necessary and sufficient conditions.* We provide next various necessary and sufficient conditions for the existence of a QSD. For a discussion on limiting conditional distributions in an even more general framework, the interested reader is referred to [15] and [16].

THEOREM 3.2. *Under Assumption* 3.1, *the following three propositions are equivalent*:

(a) *P has a QSD*;

(b) $\mu_n^\nu \Rightarrow \mu^\nu$ *for some initial probability measure* $\nu \in \mathscr{P}$, *in which case the eigenvalue* α *is given by*

$$\alpha = \lim_{n\to\infty} \frac{\int P^{n+1}(x,\ X)\nu(dx)}{\int P^n(x,\ X)\nu(dx)}; \tag{3.2}$$

(c) *for some probability measure* $\nu \in \mathscr{P}$, *the limit*

$$\alpha = \lim_{n\to\infty} \frac{\int P^{n+1}(x,\ X)\nu(dx)}{\int P^n(x,\ X)\nu(dx)} \tag{3.3}$$

exists, *is positive and*

$$0 < \liminf_{n\to\infty} \frac{\int P^{n}(x,\ K)\nu(dx)}{\int P^n(x,\ X)\nu(dx)} \tag{3.4}$$

for some compact set $K \in \mathscr{B}$.

The proof is omitted, but the interested reader may find details in [10].

REMARK 3.3. (i) The equivalence of (a) and (b) in Theorem 3.2 is true in an arbitrary metric space with a Feller kernel P. However, to prove that (b) $\Rightarrow$ (c) as well as (c) $\Rightarrow$ (a) requires repeated use of Assumption 3.1 and the fact that X is locally compact separable.

(ii) From Theorem 3.2, one may see that the existence of a Yaglom limit for some $x \in X$ is a sufficient condition for the existence of a QSD. In this case the Yaglom limit is itself a QSD. What we have shown in Theorem 3.2 is that the existence of a nontrivial weak* limit point in the Yaglom sequence is also a sufficient condition for the existence of a QSD. However, the limit point itself is not in general a QSD.

3.2. *Foster–Liapounov-type conditions.* In this section we provide another necessary and sufficient existence condition based on a totally different approach. This follows from the observation that a QSD μ exists if and only if the system

$$\begin{aligned} \mu P &= \alpha\mu, \\ \mu(X) &\le 1, \\ \mu(K) &\ge \beta \end{aligned} \tag{3.5}$$

has a solution for some compact set K and some positive scalar β. We recall that a kernel P is *weak Feller* if Pf is continuous whenever f is bounded continuous.

THEOREM 3.4. *Assume that P is weak Feller. Then P has a QSD if and only if there exist a compact set K, positive scalars β and $0 < \alpha < 1$ such that whenever $f \in C_0(X)$ and $\gamma \in R^+$ satisfy*

$$Pf - \alpha f + \gamma - I_K \geq 0, \tag{3.6}$$

we have $\gamma \geq \beta$.

The proof is postponed to the Appendix. It is based on a generalized Farkas lemma for infinite-dimensional linear systems. An excellent exposition of the use of the Farkas lemma in proving the existence of invariant measures for finite Markov chains is given by Franklin [5].

4. Two particular classes. In this section, we particularize our results to two classes of Markov chains. The first is the one for which the function $s \in C_0(X)$, as in population growth models in the countable case investigated by Hognas [7]. This condition simply says that the one-step probability of absorption tends to unity as the initial state $|x| \to \infty$.

The second class is quite different in that the function $1 - s$, not s, is in $C_0(X)$; that is, the one-step probability of absorption vanishes for large initial state x. This condition is just the discrete-time version of the condition

$$P_x(R < 2) \to 0 \quad \text{as } |x| \to \infty \tag{4.1}$$

of Ferrari, Kesten, Martinez and Picco [4] (with R the absorption time).

4.1. *The case $s \in C_0(X)$.* In this section, we consider the case where the function $s \in C_0(X)$, which is a typical situation for population growth models. For a recent paper in this vein, the reader is referred to Hognas [7]. If $s \in C_0(X)$ and P is weak Feller, then P maps $C_0(X)$ into $C_0(X)$. To see this, observe that

$$Pf \leq \|f\| P1 = \|f\| s \quad \text{whenever } f \in C(X),$$

so that $Pf \in C_0(X)$.

In this case, condition (3.4) in part (c) of Theorem 3.2 can be dropped. Therefore, as an immediate consequence, a sufficient condition for the existence of a QSD is that the scalar

$$\alpha := \lim_{n\to\infty} \frac{P^{n+1}(x, X)}{P^n(x, X)}$$

exists and is positive for some initial state $x \in X$.

In addition, the Foster–Liapounov-type condition in Theorem 3.4 also simplifies as follows.

THEOREM 4.1. *Suppose that P is weak Feller and $s \in C_0(X)$. Then P has a QSD if and only if there is a positive scalar $0 < \alpha < 1$ such that whenever $0 \le f \in C_0(X)$ and $\gamma \in R^+$ satisfy*

$$Pf - \alpha f + \gamma - s \ge 0, \tag{4.2}$$

we have $\gamma \ge \alpha$.

PROOF. P has a QSD if and only if the system

$$\begin{aligned} \mu P &\le \alpha\mu, \\ \mu(X) &\le 1, \\ \int s\mu(dx) &\ge \alpha \end{aligned} \tag{4.3}$$

has a nonnegative solution in $\mu \in \mathscr{M}$. Indeed, if $\mu \in \mathscr{P}$ and $\mu P = \alpha\mu$, then, evidently, $\mu P(X) = \int s\mu(dx) = \alpha$. Conversely, assume that $\mu \in \mathscr{M}$ satisfies (4.3); that is, equivalently, $\mu P + \nu = \alpha\mu$ for some $\nu \in \mathscr{M}$. Integrating the constant 1 in both sides yields $\int s\mu(dx) + \nu(X) = \alpha\mu(X)$, so that $\int s\mu(dx) \le \alpha$. Therefore, we must have $\nu(X) = 0$ and $fs\mu(dx) = \alpha$, so that $\mu \in \mathscr{P}$ and $\mu P = \alpha\mu$; that is, μ is a QSD.

The rest of the proof is similar to that of Theorem 3.4 in the Appendix. □

One may note that function f in Theorem 3.4 is not restricted to be nonnegative as in Theorem 4.1. This is because in (3.5) we have an equality $\mu P = \alpha\mu$ which can be relaxed into an inequality in (4.3). The Foster–Liapounov-type conditions are also simpler in Theorem 4.1 for we do not need to introduce a compact set K to ensure that μ is nontrivial. This is now implied by the constraint $\int s\mu(dx) \ge \alpha$. However, we can do that because $s \in C_0(X)$, so that the conditions to apply the generalized Farkas lemma are fulfilled.

4.2. *The case* $1 - s \in C_0(X)$. In this section, we consider the case where in place of s the function $1 - s$ is now in $C_0(X)$. It is immediate that with R the absorption time, this condition is exactly the condition $P_x(R < 2) \to 0$ as $|x| \to \infty$ of Ferrari, Kesten, Martinez and Picco [4], where, in fact, one also has $P_x(R < n) \to 0$ as $|x| \to \infty$ for every $n = 2, 3, \ldots$. In this case, Theorem 3.2(c) simplifies since we do not need the condition (3.4).

It is worth noting that from Theorem 3.2 a simple sufficient condition for the existence of a QSD is just that

$$\lim_{n\to\infty} \frac{P^{n+1}(x, X)}{P^n(x, X)} = \alpha > 0,$$

since condition (3.4) is useless.

We have also an adapted version of Theorem 3.4 which reads as follows.

THEOREM 4.2. *Let Assumption* 3.1 *hold and suppose that* $1 - s \in C_0(X)$. *Then* P *has a QSD if and only if there is a positive scalar* $0 < \alpha < 1$ *such that whenever* $f \in C_0(X)$ *and* $\gamma \in R^+$ *satisfy*

$$Pf - \alpha f + \gamma + s \geq 1, \tag{4.4}$$

one must have $\gamma \geq 1 - \alpha$.

PROOF. The existence of a QSD is equivalent to the existence of a nonnegative solution to the linear system

$$\begin{gathered} \mu P = \alpha\mu, \\ \mu(X) \leq 1, \\ \int (1 - s)\mu(dx) \geq 1 - \alpha, \end{gathered} \tag{4.5}$$

where now the constraint $\int(1 - s)\mu(dx) \geq 1 - \alpha$ ensures that the solution is not trivial. The rest of the proof is similar to that of Theorem 3.4 in the Appendix. □

4.3. *Discussion of the conditions of Ferrari, Kesten, Martinez and Picco* [4]. We have already seen in the previous section that the discrete-time version of the condition (4.1) of Ferrari, Kesten, Martinez and Picco [4] implies that $1 - s \in C_0(X)$ provided P maps $C(X)$ into itself.

In this section, we prove that their other conditions

$$P_x(R < t) \to 0 \quad \text{as } |x| \to \infty, \ \forall t = 3, 4, \ldots, \tag{4.6}$$

and $E_x(R) < \infty$ for all $x \in X$ imply that P maps $C_0(X)$ into itself if P is weak Feller. For this we need only show that for every compact $K \in \mathscr{B}$, $P(x, K) \to 0$ whenever $|x| \to \infty$.

Let K be an arbitrary compact set in $\mathscr{B}$ and define

$$K_0 := \{x \in K | s(x) = 1\}, \qquad K_i := \{x \in K_{i-1} | P^i s(x) = 1\}, \qquad i \geq 1.$$

Since P is weak Feller and s is continuous, then so is $P^n s$ for each positive integer n. Thus (K_i) forms a nonincreasing sequence of compact sets. Let $K^* := \cap_0^\infty K_i$ and define

$$B := \{x \in X | P^n(x, X) = 1, \ \forall n = 1, 2, \ldots\}.$$

Then

$$K_n = \{x \in K : P_x(R > n + 1) = 1\}$$

so that

$$K^* = \{x \in K : P_x(R = \infty) = 1\},$$

where R is the absorption time. Likewise

$$B = \{x \in X : P_x(R = \infty) = 1\}.$$

The hypothesis $E_x(R) < \infty$ implies that B is empty and so a fortiori that K^* is empty. By compactness, K_n is empty for all sufficiently large n. Fix such an n. Then $P^n s(x) < 1$ for all $x \in K$. Consequently, if

$$B_k^n := \{x \in X : P^n s(x) \le 1 - k^{-1}\},$$

then

$$\bigcup_k B_k^n = \{x \in X : P^n s(x) < 1\} \supset K.$$

It follows that $B_k^n \supset K$ for all k sufficiently large. But $1_{B_k^n} \le k \cdot (1 - P^n s)$, so

$$P(x, B_k^n) \le k \cdot \left[P(1 - P^n)s(x)\right] \le k \cdot \left[1 - P^{n+1}s(x)\right] = k \cdot P_x(R \le n+1).$$

As, from (4.6), the probability on the extreme right goes to 0 as $|x| \to \infty$, we must have $\lim_{|x|\to\infty} P(x, B_k^n) = 0$. Thus $\lim_{|x|\to\infty} P(x, K) = 0$, as desired.

APPENDIX

PROOF OF THEOREM 3.4. Consider the linear system

$$\begin{aligned} \mu P &= \alpha\mu, \\ \mu(X) &\le 1, \\ \mu(K) &\ge \beta \end{aligned} \tag{A.1}$$

and define the operator T: $\mathscr{M} \times R^2 \to \mathscr{M} \times R^2$ by

$$T(\mu, y, w) = \begin{bmatrix} \mu P - \alpha\mu \\ \mu(X) + y \\ \mu(K) - w \end{bmatrix}.$$

We have seen that the existence of a QSD is equivalent to that of a nonnegative solution $\mu \in \mathscr{M}$ for the system (A.1), which in turn is equivalent to the existence of a nonnegative solution $(\mu, y, w) \in \mathscr{M} \times R^2$, that is, a solution $(\mu, y, w) \in \Omega$, where Ω is the positive cone in $\mathscr{M} \times R^2$.

Consider the dual pairs of vector spaces $(\mathscr{X}, \mathscr{Y})$ and $(\mathscr{Z}, \mathscr{W})$ with $\mathscr{X} := \mathscr{M} \times R^2$, $\mathscr{Y} := B(X) \times R^2$, $\mathscr{Z} := \mathscr{M} \times R^2$ and $\mathscr{W} := C_0(X) \times R^2$, with $B(X)$ the Banach space of bounded measurable functions on X. The preceding spaces are equipped with the weak topologies $\sigma(\mathscr{X}, \mathscr{Y})$, $\sigma(\mathscr{Y}, \mathscr{X})$, $\sigma(\mathscr{Z}, \mathscr{W})$ and $\sigma(\mathscr{W}, \mathscr{Z})$, respectively. We view T as a map from $\mathscr{X}$ to $\mathscr{Z}$.

The adjoint linear mapping T^*: $\mathscr{W} \to \mathscr{Y}$ is defined by

$$T^*(f, s, t) := \begin{bmatrix} Pf - \alpha f + s + t1_K \\ s \\ -t \end{bmatrix}$$

and, as obviously $T^*(\mathscr{W}) \subset \mathscr{Y}$, T is continuous with respect to the previously defined topologies. First, we need the following result.

PROPOSITION A.1. *Suppose that P maps $C_0(X)$ into itself. Then $T(\Omega)$ is closed.*

PROOF. Note first that the $\sigma(\mathscr{Z},\mathscr{W})$ topology is the weak* topology on $\mathscr{Z}$, so closures of convex sets can be characterized by converging sequences (cf. Dunford and Schwartz [3], Volume 1, page 437).

Therefore, as Ω is obviously convex, consider a sequence $\{(\mu_n, y_n, w_n)\} \in \Omega$ such that the sequence $T(\mu_n, y_n, w_n) \in \mathscr{Z}$ converges to some $(a, b, c) \in \mathscr{Z}$, that is,

$$\text{(A.2)} \qquad \mu_n P - \alpha\mu_n \overset{*}{\to} a, \qquad \mu_n(X) + y_n \to b, \qquad \mu_n(K) - w_n \to c.$$

As $y_n \geq 0$, from (A.2), $\mu_n(X) \leq 2b$ for n sufficiently large, so that by weak* sequential compactness of the unit ball in $\mathscr{M}$, there is a nonnegative measure $\mu \in \mathscr{M}$ and a subsequence $\{n_k\}$ such that $\mu_{n_k} \overset{*}{\to} \mu$.

Suppose, if possible, that $\mu(X) > b$. By tightness there is a compact set K with $\mu(K) > b$. Choose $f \in C_0(X)$ such that $1_K \leq f \leq 1$. Then

$$b < \mu(K) \leq \int f\, d\mu = \lim_k \int f\, d\mu_{n_k} \leq \limsup_n \mu_n(X) \leq b,$$

a contradiction. Hence we must have $\mu(X) \leq b$.

In addition, as K is compact and $\mu_{n_k} \overset{*}{\to} \mu$ we have $c \leq \limsup_k \mu_{n_k}(K) \leq \mu(K)$ (see, e.g., Doob [2]).

Finally, as P maps $C_0(X)$ into itself, we have $\mu_{n_k}(P - \alpha I) \overset{*}{\to} \mu(P - \alpha I)$. Therefore, with $y := b - \mu(X) \geq 0$ and $w := \mu(K) - c \geq 0$, we have $T(\mu, y, w) = (a, b, c)$, so that $T(\Omega)$ is closed. □

As $T(\Omega)$ is closed, we can apply the generalized Farkas lemma of Craven and Koliha ([1], Theorem 2, page 987) so that (A.1) has a nonnegative solution if and only if

$$\text{(A.3)} \qquad T^*(f, \gamma, t) \in \Omega^* \Rightarrow \langle (0, 1, \beta), (f, \gamma, t)\rangle \geq 0,$$

with Ω^* the "dual" cone in $\mathscr{Y}$ of the convex cone Ω and $\langle \cdot, \cdot \rangle$ the duality bracket between $\mathscr{Z}$ and $\mathscr{W}$.

Relation (A.3) simply translates into

$$\left.\begin{aligned} Pf - \alpha f + \gamma + t1_K \geq 0 \\ \gamma \geq 0 \\ -t \geq 0 \end{aligned}\right\} \Rightarrow \gamma + t\beta \geq 0.$$

The only interesting case is when $t \neq 0$. Dividing by $-t > 0$ and relabeling provides

$$Pf - \alpha f + \gamma - 1_K \geq 0 \Rightarrow \gamma \geq \beta,$$

which is the condition in Theorem 3.4.

The proofs of Theorem 4.1 and 4.2 are in the same vein.

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