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SUPERPROCESSES OF STOCHASTIC FLOWS[1]

BY ZHI-MING MA AND KAI-NAN XIANG

Academia Sinica, Peking University and University of Lisbon

We construct a continuous superprocess X on $M(R^d)$ which is the unique weak Feller extension of the empirical process of consistent k-point motions generated by a family of differential operators. The process X differs from known Dawson–Watanabe type, Fleming–Viot type and Ornstein–Uhlenbeck type superprocesses. This new type of superprocess provides a connection between stochastic flows and measure-valued processes, and determines a stochastic coalescence which is similar to those of Smoluchowski. Moreover, the support of X describes how an initial measure on R^d is transported under the flow. As an example, the process realizes a viewpoint of Darling about the isotropic stochastic flows under certain conditions.

1. Introduction. Both measure-valued processes and stochastic flows have been studied by many authors. Motivated by [9], [11] and [22], in this paper stochastic flows arising from stochastic differential equations with a finite number of driving Brownian motions are used to construct measure-valued processes over R^d.

Let $C_b^2((R^d)^k)$ be the set of all functions on $(R^d)^k$ with bounded continuous derivatives of orders up to and including 2. Define an operator A_k for each natural number k as follows:

$$A_k f(z_1, \dots, z_k) = \frac{1}{2} \sum_{i,\, j=1}^{k} \sum_{p,\, q=1}^{d} a^{pq}(z_i, z_j) \frac{\partial^2 f}{\partial z_i^p \partial z_j^q}(z_1, \dots, z_k) + \sum_{i=1}^{k} \sum_{p=1}^{d} b^p(z_i) \frac{\partial f}{\partial z_i^p}(z_1, \dots, z_k),$$

$$\forall\, f \in C_b^2((R^d)^k), \qquad (z_1, \dots, z_k) \in (R^d)^k,\ z_i = (z_i^1, \dots, z_i^d) \in R^d.$$

We assume that:

(1.1) $b^p(\cdot)$ is a bounded Borel measurable function on R^d, $1 \le p \le d$.

(1.2) $(a^{pq}(\cdot, \cdot))_{1 \le p,\, q \le d}$ is a nonnegative definite $d \times d$ symmetric matrix such that

$$a^{pq}(z_i, z_j) = a^{pq}(z_j, z_i) \qquad \forall (z_i, z_j) \in (R^d)^2$$

and $a^{pq}(\cdot, \cdot)$ is a bounded Borel measurable function on $(R^d)^2$, $\forall p, q$.

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(1.3) The martingale problem for $(A_k, C_b^2((R^d)^k))$ is well posed and generates a unique continuous strong Markov process on $(R^d)^k$ with a weak Feller semigroup $(V_t^k)_{t\geq 0}$ on the space of all bounded continuous functions on $(R^d)^k$.

Denote the process corresponding to $(V_t^k)_{t\geq 0}$ starting from $(z_1, \dots, z_k) \in (R^d)^k$ at time s by $Y^k = (Y_{st}(z_1), \dots, Y_{st}(z_k))_{t\geq s}$. Then one can check that $\{A_k\}_{k\geq 1}$ satisfies the following consistency property:

(1.4) Each $r(r \leq k)$ components of Y^k evolves according to the $(A_r, C_b^2((R^d)^r))$ diffusion process.

Furthermore, $Y^k = (Y_{st}(z_1), \dots, Y_{st}(z_k))_{t\geq s}$ can be viewed as the k-point motion under the A_1 diffusion stochastic flow with interaction term $(a^{pq}(\cdot, \cdot))$ (see [22]).

It is easy to check that

$$A_1 F(z) = A_2 f(z, z)$$

for any $f \in C_b^2((R^d)^2)$ and $F(z) = f(z, z)$, $z \in R^d$. In addition, if $f(z_1, z_2) = |z_1 - z_2|^2$ for $(z_1, z_2) \in O_r \times O_r$, where O_r is the open ball centered at 0 with radius $r(0 < r < \infty)$ in R^d, then

$$A_2 f(z_1, z_1) = 0 \qquad \text{for } z_1 \in O_r.$$

Particularly, if $f(z_1, z_2) = |z_1 - z_2|^2$, $\forall (z_1, z_2) \in O_r \times O_r$, then

$$F(z_1) = 0, \quad A_2 f(z_1, z_1) = A_1 F(z_1) = 0 \qquad \forall\, z_1 \in O_r.$$

Therefore $(Y_{st}(x), Y_{st}(x))_{t\geq s}$ is a martingale solution to $(A_2, C_b^2((R^d)^2))$ starting from (x, x) at time s. Combining with the fact that $(A_2, C_b^2((R^d)^2))$ generates a unique strong Markov process and the consistency of $\{A_k\}_{k\geq 1}$, we see that A_k-diffusion has the property that two particles have to stay together whenever they meet. So the path space of Y^k should be

$$\Omega_k = \{\omega = (\omega^1, \dots, \omega^k) \in C([0, \infty); (R^d)^k) | \omega^i(t) = \omega^j(t)$$
$$\text{for some } t \text{ and } i \neq j \Longrightarrow \omega^i(s) = \omega^j(s) \text{ for } s > t\}.$$

Isotropic stochastic flows constructed by the reproducing kernel Hilbert space method and generally perturbed isotropic flows satisfy the above conditions (1.1)–(1.3) (see [9] or Section 8 in the present paper). See [22] for more examples which meet our conditions.

Let $M(R^d)$ be the set of all finite measures on R^d endowed with the topology of weak convergence. Write $M_r(R^d) = \{\mu \in M(R^d) | \mu(R^d) = r\}$ for $r > 0$, and write $M_{1,k}(R^d) = \{\mu \in M_1(R^d) | \mu = (1/k) \sum_{i=1}^k \delta_{z_i}; (z_1, \dots, z_k) \in (R^d)^k\}$ for $k \geq 1$.

Let $(Y_t^k) = (Y_{0t}(y_1), \dots, Y_{0t}(y_k))$ be the process corresponding to $(V_t^k)_{t \ge 0}$ starting from $(y_1, \dots, y_k) \in (R^d)^k$; define

$$X_t^k = \frac{1}{k} \sum_{i=1}^{k} \delta_{Y_{0t}(y_i)}. \tag{1.5}$$

Then it follows from Proposition 2.3.3 in [11] that $X^k = (X_t^k)$ is a continuous strong Markov process on $M_{1,k}(R^d)$ (cf. Lemma 2.1 below).

A main result of this paper is to show that there exists a unique weak Feller extension X of the above X^k to a continious Markov process on the whole space of $M(R^d)$. Moreover, we show that X is a new type of measure-valued process differing from the known types of superprocesses. More precisely, we obtain the following results.

THEOREM 1.1 (cf. Theorem 4.5 and Proposition 3.8 below). *There exists a unique continuous strong Markov process $X = (X_t)$ in $M(R^d)$ associated with a weak Feller semigroup $\{T_t^X\}_{t \ge 0}$ such that for each r and k, if the initial value of the process is an atomic measure in $M_r(R^d)$ distributing its mass equally among k points $y_1, \dots, y_k$, then the distribution of X coincides with rX^k, where X^k is specified by* (1.5).

REMARK 1.2. (i) The weak Feller property of $\{T_t^X\}$ implies the uniqueness of X; see Proposition 3.8(ii). Otherwise, one may have a different extension. For example, one may construct a trivial extension by letting all nonatomic measures remain fixed.

(ii) The analysis of the generator of X (cf. Sections 5 and 6) shows that X is neither a type of Dawson–Watanabe process nor a type of Fleming–Viot process. Also X is not an Ornstein–Uhlenbeck superprocess. Indeed Theorem 1.1 suggests a new type of superprocess. This new type of superprocess is closely related to stochastic flows, and hence is of interest to be studied further. The present paper is just a beginning.

(iii) For each k, if the initial value of X is an atomic measure distributing its mass equally among k points, then the support of X evolves according to the diffusion with the generator A_k with the rule that paths in R^d which meet must coalesce. Thus the process X obtained here determines a stochastic coalescence which is similar to those of [32] (see [1]) in some sense. Moreover, it realizes a view point of Darling [9] that the real object of interest in stochastic flows is not the family of mappings $\{Y_{st}\}$, but the way that an initial measure on R^d is transported under the flow. In particular, the framework in this paper is applicable to certain isotropic stochastic flows constructed by the reproducing kernel Hilbert space method. See Sections 7 and 8 for details.

REMARK 1.3. We are grateful to the referee who pointed out that the framework adopted here includes the case where the k-point motions are associated with a stochastic differential equation on R^d. An interesting extreme example is the case of independent Brownian motion. In this case atoms will a.s. not collide provided $d > 1$. The referee conjectured that for independent Brownian motion, the $M(R^d)$-valued process will evolve by the atoms moving randomly and the rest under the semigroup. We shall discuss this subject in a forthcoming paper.

This paper is organized as follows. After some preliminaries in Section 2, we construct the weak Feller semigroup on $M(R^d)$ and discuss some consequences of the associated process in Section 3. Section 4 is devoted to showing the continuity of the sample paths. Then we calculate the generator of $\overline{X}$ in Section 5 and compare it to other known superprocesses in Section 6. The relation of our process to the stochastic coalescence is briefly discussed in Section 7. Finally, in Section 8 we discuss some examples coming from isotropic flows.

2. Preliminaries. Throughout this paper, we shall use the following notation:

$M(R^d)$	all finite measures on R^d endowed with the weak topology
$M_1(R^d)$	$= \{\mu \in M(R^d) \mid \mu(R^d) = 1\}$
$M_1(M_1(R^d))$	all probability measures on $M_1(R^d)$ endowed with the weak topology
$C_b(R^d)$	all bounded continuous functions on R^d.
$C_0(R^d)$	all continuous functions on R^d vanishing at infinity
$C_b^\infty(R^d)$	all smooth functions on R^d with bounded derivatives of any order
$\mathrm{Per}(N)$	all permutations of $\{1, \dots, N\}$
$C_{\mathrm{sym}}((R^d)^n)$	all continuous symmetric functions from $R^d \times \cdots \times R^d$ (n folds) to R
$\tilde{\pi}$	$(R^d)^n \to (R^d)^n$ is defined by $(\tilde{\pi}x)_i = x_{\pi i}$ for $x = (x_1, \dots, x_n) \in (R^d)^n$, $\pi \in \mathrm{Per}(n)$
$\langle \mu, f \rangle := \int f\, d\mu$	the integral of measurable function f with respect to measure μ on R^d
Ξ_n	$(R^d)^n \to M_1(R^d)$, $\Xi_n(z_1, \dots, z_n) = \dfrac{1}{n}\sum_{i=1}^n \delta_{z_i}$, where δ_{z_i} is the unit measure centered at z_i
$M_{1,n}(R^d)$	$= \{\Xi_n(z_1, \dots, z_n) \mid (z_1, \dots, z_n) \in (R^d)^n\}$
μ^n	$:= \mu \times \cdots \times \mu$ (n folds), $\forall\, \mu \in M(R^d)$
$F_{f,n}(\mu) := \langle \mu^n, f \rangle$	$\forall\, \mu \in M(R^d)$, bounded Borel measurable function f on $(R^d)^n$.

LEMMA 2.1. *Let* $(Y_t^k) = (Y_{0t}(y_1), \dots, Y_{0t}(y_k))$ *be the process corresponding to* $(V_t^k)_{t\geq 0}$ *starting from* $(y_1, \dots, y_k) \in (R^d)^k$; *define*

$$(X_t^k) = (\Xi_k(Y_t^k)).$$

Then $X^k = (X_t^k)$ *is a continuous strong Markov process on* $M_{1,k}(R^d)$.

PROOF. The proof is similar to Proposition 2.3.3 in [11]. For the reader's convenience, we provide an outline.

(i) We have the following claim:

$$\sigma\{C_{\mathrm{sym}}((R^d)^k)\} = \sigma\{\Xi_k\}.$$

In particular, if $f \in C_{\mathrm{sym}}((R^d)^k)$, then f is $\sigma\{\Xi_k\}$-measurable.

(ii) For any $\pi \in \mathrm{Per}(k)$ and any $f \in C_b^2((R^d)^k)$, we have

$$\begin{aligned}
&(A_k(\tilde\pi f))(\tilde\pi^{-1}(z_1, \dots, z_k))\\
&\quad = (A_k(\tilde\pi f))(z_{\pi^{-1}1}, \dots, z_{\pi^{-1}k})\\
&\quad = \frac{1}{2}\sum_{i,j=1}^{k}\sum_{p,q=1}^{d} a^{pq}(z_{\pi^{-1}j}, z_{\pi^{-1}i})\frac{\partial^2 \tilde\pi f}{\partial z^p_{\pi^{-1}i}\partial z^q_{\pi^{-1}j}}(z_{\pi^{-1}1}, \dots, z_{\pi^{-1}k})\\
&\qquad + \sum_{i=1}^{k}\sum_{p=1}^{d} b^p(z_{\pi^{-1}i})\frac{\partial \tilde\pi f}{\partial z^p_{\pi^{-1}i}}(z_{\pi^{-1}1}, \dots, z_{\pi^{-1}k})\\
&\quad = \frac{1}{2}\sum_{i,j=1}^{k}\sum_{p,q=1}^{d} a^{pq}(z_{\pi^{-1}j}, z_{\pi^{-1}i})\frac{\partial^2 f}{\partial z^p_{\pi^{-1}i}\partial z^q_{\pi^{-1}j}}(z_1, \dots, z_k)\\
&\qquad + \sum_{i=1}^{k}\sum_{p=1}^{d} b^p(z_{\pi^{-1}i})\frac{\partial f}{\partial z^p_{\pi^{-1}i}}(z_1, \dots, z_k)\\
&\quad = (A_k f)(z_1, \dots, z_k),
\end{aligned}$$

where $(\tilde\pi f)(z_1, \dots, z_k) = f(z_{\pi 1}, \dots, z_{\pi k})$ for any $(z_1, \dots, z_k) \in (R^d)^k$. Namely, A_k satisfies the exchangeable condition.

(iii) We now prove that (X_t^k) is a continuous strong Markov process. In fact, the continuity of (X_t^k) is obvious. Let

$$\mathscr{F}_t(Y^k) = \sigma\{Y_s^k;\, s \leq t\},\ \mathscr{F}_t(X^k) = \sigma\{X_s^k;\, s \leq t\}.$$

Then for any finite $(\mathscr{F}_t(X^k))$-stopping time τ and any positive number s, we have

$$\begin{aligned}
E\big[X^k_{\tau+s} \in \cdot \,|\mathscr{F}_\tau(X^k)\big] &= E\big[E[X^k_{\tau+s} \in \cdot\,|\mathscr{F}_\tau(Y^k)]|\mathscr{F}_\tau(X^k)\big]\\
&= E\big[E[X^k_{\tau+s} \in \cdot\,|\sigma(Y^k_\tau)]|\mathscr{F}_\tau(X^k)\big].
\end{aligned}$$

Noticing that A_k satisfies the exchangeable condition, we get

$$\begin{aligned} E\big[X^k_{\tau+s} \in \cdot | \sigma(Y^k_\tau)\big] &= E\big[\Xi_k(\tilde{\pi} Y^k_{\tau+s}) \in \cdot | \sigma(\tilde{\pi} Y^k_\tau)\big] \\ &= E\big[\Xi_k(Y^k_{\tau+s}) \in \cdot | \sigma(\tilde{\pi} Y^k_\tau)\big]. \end{aligned}$$

Thus

$$E\big[X^k_{\tau+s} \in \cdot | \mathscr{F}_\tau(X^k)\big] = E\big[X^k_{\tau+s} \in \cdot | \sigma(X^k_\tau)\big]. \qquad \square$$

REMARK 2.2. Let $R^d_\triangle$ be the one-point compactification of R^d, $M_1(R^d_\triangle)$ the set of all probabilities on $R^d_\triangle$, and $C_b(R^d_\triangle)$ the set of all (bounded) continuous functions on $R^d_\triangle$. Then $M_1(R^d_\triangle)$ is compact in the topology of weak convergence and

$$C_b(R^d_\triangle) = \{f + c \mid f \in C_0(R^d),\ c \text{ is a constant}\}.$$

Here and at other similar places, we view any $f \in C_0(R^d)$ as a function on $R^d_\triangle$ by letting $f(\triangle) = 0$.

Notice that there exists a countable subset $\{g_j\}_{j\geq 1}$ of $C_0(R^d) \cap C_b^\infty(R^d)$ which is dense in $C_0(R^d)$ with respect to the uniform norm. Let $\{r_k\}_{k\geq 1}$ be the set of all rational numbers, then $\{f_n\}_{n\geq 1} := \{g_j + r_k;\ j \geq 1,\ k \geq 1\}$ is dense in $C_b(R^d_\triangle)$ with respect to the uniform norm. We may always assume $f_1 = 1$. We define for $\mu, \nu \in M_1(R^d_\triangle)$,

$$\rho(\mu, \nu) = \sum_{n=1}^{\infty} \frac{1}{2^n}\big(1 \wedge |F_{f_n, 1}(\mu) - F_{f_n, 1}(\nu)|\big).$$

Then $(M_1(R^d_\triangle), \rho)$ becomes a compact metric space and the ρ-topology is compatible with the weak topology on $M_1(R^d_\triangle)$. If we consider $M_1(R^d)$ as a Borel subset of $M_1(R^d_\triangle)$ by identifying

$$M_1(R^d) = \{\mu \in M_1(R^d_\triangle) \,|\, \mu(\{\triangle\}) = 0\}.$$

Then $M_1(R^d)$ equipped with the metric ρ is a separable metric space and the topology induced by ρ coincides with the vague topology. But since any element μ in $M_1(R^d)$ must have total mass 1, hence any vague limit in $M_1(R^d)$ [not in $M(R^d_\triangle)$!] is also a weak limit. Therefore the ρ-topology coincides with the weak topology on $M_1(R^d)$. Moreover, one can check that the closure of $M_1(R^d)$ in $(M_1(R^d_\triangle), \rho)$ is just $M_1(R^d_\triangle)$.

Let $C_b(M_1(R^d))$ be the space of all bounded continuous functions on $M_1(R^d)$. For the further use we introduce its several subspaces:

$$C_p(M_1(R^d)) := \{F_{f,n} \mid f \in C_b((R^d)^n),\ n \geq 1\}, \tag{2.1}$$

$$C_p^\infty(M_1(R^d)) := \{F_{f,n} \mid f \in C_b^\infty((R^d)^n),\ n \geq 1\} \tag{2.2}$$

and

$$C_u(M_1(R^d)) := \{F \in C_b(M_1(R^d);\ F \text{ is uniformly continuous}\}. \tag{2.3}$$

LEMMA 2.3. *$C_p^\infty(M_1(R^d))$ is an algebra and is convergence determining in $M_1(M_1(R^d))$. Moreover, $C_p^\infty(M_1(R^d)) \cap C_u(M_1(R^d))$ is dense in $C_u(M_1(R^d))$ with respect to the uniform norm.*

PROOF. We divide the proof into four parts.

(i) Noticing $F_{f,n}(\cdot)F_{g,m}(\cdot) = F_{h,n+m}(\cdot)$ for

$$h(x_1, \dots, x_{n+m}) = f(x_1, \dots, x_n)g(x_{n+1}, \dots, x_{n+m})$$

and

$$F_{f,n}(\cdot) = F_{r,n+m}(\cdot) \quad \text{for } r(x_1, \dots, x_{n+m}) = f(x_1, \dots, x_n),$$

we see that $C_p^\infty(M_1(R^d))$ is an algebra.

(ii) Note $(M_1(R^d_\triangle), \rho)$ is a compact metric space and is the ρ-closure of $M_1(R^d)$. Hence any $F \in C_u(M_1(R^d))$ is uniquely extended to an element $\overline{F} \in C(M_1(R^d_\triangle))$ [$C(M_1(R^d_\triangle))$: all the continuous functions on $M_1(R^d_\triangle)$]. Moreover, $F \to \overline{F}$ is a one-to-one correspondence between $C_u(M_1(R^d)$ and $C(M_1(R^d_\triangle))$.

(iii) Let $\overline{C}_{up} = \{\overline{F} \in C(M_1(R^d_\triangle)) \,|\, F \in C_u(M_1(R^d)) \cap C_p^\infty(M_1(R^d))\}$. Then $\overline{C}_{up}$ is an algebra and contains the constant function 1. Moreover $\{F_{f_n,1}(\cdot);\, n \geq 1\}$ used in the definition of ρ is a subset of $\overline{C}_{up}$ and $\{F_{f_n,1}(\cdot);\, n \geq 1\}$ separates the points of $M_1(R^d_\triangle)$. Therefore, by the Stone–Weierstrass theorem, $\overline{C}_{up}$ is dense in $C(M_1(R^d_\triangle))$. Combining with (ii), we see that $C_p^\infty(M_1(R^d)) \cap C_u(M_1(R^d))$ is dense in $C_u(M_1(R^d))$.

(iv) Let $P, P_n \in M_1(M_1(R^d))$, $n \geq 1$. Suppose that

$$\int F\, dP_n \to \int F\, dP, \qquad n \to \infty, \tag{2.4}$$

for all $F \in C_p^\infty(M_1(R^d))$, then it follows from (iii) that (2.4) holds for all $F \in C_u(M_1(R^d))$. Thus $C_p^\infty(M_1(R^d))$ is convergence determining in $M_1(M_1(R^d))$. □

3. Weak Feller semigroup on $M(R^d)$. For $F = F_{f,n} \in C_p(M_1(R^d))$ [cf. (2.1)] and $\mu \in M_1(R^d)$, we define

$$T_t F_{f,n}(\mu) = \int V_t^n f(x_1, \dots, x_n)\mu^n(dx_1 \dots dx_n). \tag{3.1}$$

Note that by the consistency property [cf. (1.4)] and the fact that μ is a probability measure, (3.1) is independent of the expression $F = F_{f,n}$, and hence $\{T_t;\, t \geq 0\}$ is well defined on $C_p(M_1(R^d))$.

LEMMA 3.1. *$\{T_t;\, t \geq 0\}$ satisfies the following properties:*

(i) $T_t T_s F_{f,n}(\mu) = T_{t+s} F_{f,n}(\mu)$.
(ii) $|T_t F_{f,n}(\mu) - F_{f,n}(\mu)| \to 0,\ t \to 0$.
(iii) $\|T_t F_{f,n}\| \leq \|F_{f,n}\|$,

where $\|\cdot\|$ *denotes the uniform norm. If in addition each* $(V_t^k)_{t\geq 0}$ *is strongly Fellerian, then* (ii) *is strengthened by*

(ii)′ $\|T_t F_{f,n}(\cdot) - F_{f,n}(\cdot)\| \to 0, \qquad t \to 0.$

PROOF. By (3.1), we have $T_t F_{f,n}(\mu) = F_{V_t^n f,n}(\mu)$, hence by the consistency property (1.4),

$$T_t T_s F_{f,n}(\mu) = T_t F_{V_s^n f,n}(\mu) = F_{V_{t+s}^n f,n}(\mu) = T_{t+s} F_{f,n}(\mu),$$

which verifies (i). Also by (3.1),

$$|T_t F_{f,n}(\mu) - F_{f,n}(\mu)| \leq \int |V_t^n f - f| \mu^n(dx) \leq \sup_{x\in(R^d)^n} |V_t^n f(x) - f(x)|,$$

thus (ii) [resp. (ii)′] follows from the weak (resp. strong) Feller property of $\{V_t^n\}_{t\geq 0}$. We now prove (iii). For a given $\mu \in M_1(R^d)$, let $\mu_N = (1/N)\sum_{j=1}^N \delta_{z_j}$, where $\{z_j\}$ are independent random variables with the same distribution μ. Then $\{\mu_N\}$ converges weakly to μ a.s. by the large number law. Denote by $\{T_t^k\}$ the semigroup of X^k, with the consistency property (1.4), for $n \leq N$, we have

$$T_t^N F_{f,n}(\mu_N) = \frac{1}{N^N} \sum_{j_1,\dots,j_N=1}^{N} V_t^N \bar{f}(z_{j_1},\dots,z_{j_N})$$

$$[\text{where } \bar{f}(y_1,\dots,y_N) = f(y_1,\dots,y_n)]$$

$$\begin{aligned} &= \frac{1}{N^N} \sum_{j_1,\dots,j_N=1}^{N} V_t^n f(z_{j_1},\dots,z_{j_n}) \qquad (3.2)\\ &= \frac{1}{N^n} \sum_{j_1,\dots,j_n=1}^{N} V_t^n f(z_{j_1},\dots,z_{j_n})\\ &= T_t F_{f,n}(\mu_N) = F_{V_t^n f,n}(\mu_N). \end{aligned}$$

Let $N \to \infty$, since $V_t^n f \in C_b((R^d)^n)$, we have

$$T_t^N F_{f,n}(\mu_N) \to T_t F_{f,n}(\mu) = F_{V_t^n f,n}(\mu) \quad \text{a.s.} \tag{3.3}$$

Thus by the contraction property of $\{T_t^N\}$,

$$|T_t F_{f,n}(\mu)| \leq \limsup_{N\to\infty} |T_t^N F_{f,n}(\mu_N)| \leq \limsup_{N\to\infty} \|F_{f,n}\| \quad \text{a.s.} \tag{3.4}$$

Hence

$$\|T_t F_{f,n}\| \leq \|F_{f,n}\|,$$

proving (iii). □

LEMMA 3.2. *There exists a unique kernel probability measure* $P_t(\mu, d\nu)$ *from* $M_1(R^d)$ *to* $M_1(R^d)$ *such that*

$$T_t F(\mu) = \int F(\nu) P_t(\mu, d\nu) \quad \text{for all } F \in C_p(M_1(R^d)). \tag{3.5}$$

PROOF. Given $\mu \in M_1(R^d)$, let $\mu_N = (1/N)\sum_{j=1}^N \delta_{z_j}$, where $\{z_j\}$ are independent random variables with the same distribution μ. Denote the law of X_t^N $(X_0^N = \mu_N)$ on $M_{1,N}(R^d) \subset M_1(R^d)$ by $P_t^N(\mu, d\nu)$. From (3.3), we can check that

$$\int F(\nu) P_t^N(\mu, d\nu) \to T_t F(\mu) \qquad \forall\, F \in C_p(M_1(R^d)). \tag{3.6}$$

We claim that $\{P_t^N(\mu, d\nu)\}_{N \ge 1}$ is tight in $M_1(M_1(R^d))$.

In fact, for any $r > 0$, let $B_r = \{|x| \le r\} \subset R^d$. Then with the consistency property of A_N, we have

$$\begin{aligned}\int_{M_1(R^d)} P_t^N(\mu, d\nu)\nu(B_r^c) &= P\big[\langle X_t^N, B_r^c\rangle | X_0^N = \mu_N\big] \\ &= P\Bigg[\frac{1}{N}\sum_{i=1}^N I_{\{Y_{0t}(z_i) \notin B_r\}}\Bigg] \\ &= P[Y_{0t}(z_1) \notin B_r].\end{aligned}$$

Since $Y_{0t}(z_1)$ is an R^d-valued random variable, we must have

$$\limsup_{r\to\infty} \sup_N \int_{M_1(R^d)} P_t^N(\mu, d\nu)\nu(B_r^c) = \limsup_{r\to\infty} P[Y_{0t}(z_1) \notin B_r] = 0.$$

Thus the first moment measure sequence of $\{P_t^N(\mu, \cdot)\}_{N\ge 1}$ is tight in $M_1(R^d)$. Recall that (cf. [11]) for any $Q \in M_1(M_1(R^d))$, the corresponding first moment measure Q_1 on R^d is defined by

$$Q_1(B) = \int_{M_1(R^d)} Q(d\mu)\mu(B)$$

for any Borel set $B \subset R^d$. Then by Lemma 3.2.8 in [11], and noticing $M_1(R^d)$ is closed in $M(R^d)$, we see that $\{P_t^N(\mu, \cdot)\}_{N\ge 1}$ is tight in $M_1(M_1(R^d))$.

Thus by Lemma 2.3 and (3.6), we conclude that $\{P_t^N(\mu, \cdot)\}_{N\ge 1}$ has a unique weak limit point $P_t(\mu, \cdot)$ in $M_1(M_1(R^d))$ which satisfies (3.5). The measurability of $\mu \to P_t(\mu, \cdot)$ is obvious because $T_t F(\mu)$ is measurable in μ for all $F \in C_p(M_1(R^d))$. $\square$

With the kernel probability $P_t(\mu, d\nu)$, we can now extend $T_t F$ to all bounded Borel functions F on $M_1(R^d)$ by setting

$$T_t F(\mu) = \int F(\nu) P_t(\mu, d\nu). \tag{3.7}$$

LEMMA 3.3. *$\{T_t\}_{t\geq 0}$ is a weakly continuous Feller semigroup on $C_b(M_1(R^d))$.*

PROOF. The semigroup property follows from Lemma 3.1(i) and the monotone class argument. By Lemma 3.1(ii), (iii) and Lemma 2.3,

$$|T_t F(\mu) - F(\mu)| \to 0, t \to 0 \quad \text{for all } F \in C_u(M_1(R^d)).$$

That is, $P_t(\mu, \cdot)$ converges weakly to δ_μ when $t \to 0$.

Therefore, $T_t F(\mu) = \int F(\nu) P_t(\mu, d\nu)$ converges to $F(\mu)$ for all $F \in C_b(M_1(R^d))$, that is, $\{T_t;\ t \geq 0\}$ is weakly continuous. Note that for each n, $\{V_t^n; t \geq 0\}$ is a weak Feller semigroup. Hence if $F = F_{f,n} \in C_p(M_1(R^d))$, then

$$T_t F(\mu) = \int V_t^n f d\mu^n \in C_b(M_1(R^d)),$$

which extends to all $F \in C_u(M_1(R^d))$ immediately by Lemma 3.1(iii) and Lemma 2.3.

Therefore, $P_t(\mu, \cdot)$ is continuous in μ w.r.t. the weak topology of $M_1(R^d)$, which implies the weak Feller property of $\{T_t; t \geq 0\}$. $\square$

REMARK 3.4. If each $\{V_t^n; t \geq 0\}$ is strongly Fellerian, then by Lemma 3.1(ii)$'$ and Lemma 2.3, the weak continuity is strengthened by

$$\|T_t F - F\| \to 0, \qquad t \to 0 \text{ for all } F \in C_u(M_1(R^d)).$$

We are going to extend the semigroup to the whole space $M(R^d)$. To this end, we note that each $m \in M(R^d)$ with $m(R^d) > 0$ is uniquely expressed as $m = r\mu$ with $r > 0$ and $\mu \in M_1(R^d)$. Thus we may identify $M(R^d)$ with $\{(0, \infty) \times M_1(R^d)\} \cup \{0\}$. We define

$$\rho(m_1, m_2) = |r_1 - r_2| + \min\{r_1, r_2\} \cdot \rho(\mu_1, \mu_2)$$

for $m_1 = r_1\mu_1$ and $m_2 = r_2\mu_2$ [with the convention that $0 = 0\mu$ for arbitrary $\mu \in M_1(R^d)$]. Then, using the fact that ρ is a metric on $M_1(R^d)$ and $\rho(\mu_1, \mu_2) \leq 1$, one can check that ρ is a metric on $M(R^d)$ and its induced topology coincides with the weak topology. For $r \geq 0$, we denote by $J_r(\mu) = r\mu$, then J_r is a measurable map from $M_1(R^d)$ to $M(R^d)$.

We extend the kernel probability measure $P_t(\mu, \cdot)$ to $M(R^d)$ by setting

$$P_t(m, B) = P_t(\mu, J_r^{-1}(B)) \tag{3.8}$$

for $m = r\mu \in M(R^d)$ and Borel set $B \subset M(R^d)$. Here

$$P_t(m, B) = I_B(0) \quad \text{if } m = 0.$$

PROPOSITION 3.6. *Let*

$$T_t^X F(m) = \int_{M(R^d)} F(\nu) P_t(m, d\nu) \tag{3.9}$$

for bounded Borel function F on $M(R^d)$. Then $\{T_t^X\}_{t\geq 0}$ is a weakly continuous Feller semigroup on $C_b(M(R^d))$.

PROOF. Clearly $\{T_t^X\}_{t\geq 0}$ is a semigroup. If F is a bounded uniformly continuous function on $M(R^d)$ w.r.t. the metric ρ, then one can easily check the weak continuity and Feller property. Thus the weak continuity and Feller property holds for all $F \in C_b(M(R^d))$ because the family of all bounded uniformly continuous functions is a convergence determining class. □

REMARK 3.7. Note that $M_1(R^d)$ is a Polish space with the weak topology (not necessarily with the metric specified in Remark 2.2). By Kolmogorov's extension theorem, there is an $M_1(R^d)$-valued Markov process $(\overline{X}_t)_{t\geq 0}$ associated to the semigroup $\{T_t\}_{t\geq 0}$. For $\mu \in M_1(R^d)$ and $m = r\mu$, let $X_t = r\overline{X}_t$ with $\overline{X}_0 = \mu$, then $P_t(m, \cdot)$ specified in (3.8) is the law of X_t and $X = (X_t)_{t\geq 0}$ is a Markov process on $M(R^d)$.

In the next section we shall prove that there exists a version of $(\overline{X}_t)_{t\geq 0}$ which has continuous sample paths. For a moment we assume that we have already taken such a continuous version. Then we may conclude the following assertions.

PROPOSITION 3.8. *Assume $(\overline{X}_t)_{t\geq 0}$ has continuous sample paths and let $X = (X_t)_{t\geq 0}$ be as in the above remark.*

(i) *For each r and k, if $X_0 = (r/k)\sum_{i=1}^k \delta_{x_i} \in M_r(R^d)$, then the distribution of X coincides with rX^k, where X^k is specified in Lemma* 2.1. *In particular, the support $S = (S_t)$ of (X_t) evolves according to the law of (Y_t^k) with initial value $(y_1, \dots, y_k)$ with the rule that paths in R^d which meet must coalesce.*

(ii) *The above property* (i) *and the weak Feller property implies the uniqueness of $(X_t)_{t\geq 0}$.*

PROOF. (i) Without loss of generality, we may assume $r = 1$. For $F = F_{f,n} \in C_p(M_1(R^d))$, following the argument of (3.2) and (3.3), we can show that

$$T_t F_{f,n}(\nu) = T_t^k F_{f,n}(\nu) \qquad \forall\ \nu = \frac{1}{k}\sum_{i=1}^k \delta_{y_i}.$$

This shows that the two continuous Markov processes $(\overline{X}_t)$ and $(X_t^k) = (\Xi_k(Y_t^k))$ have the same distribution. Thus the desired assertion follows from the behavior of $(\Xi_k(Y_t^k))$.

(ii) Suppose there is another $M_1(R^d)$-valued Markov process $\overline{X}'$ satisfies the above property (i) and has the weak Feller property. For an arbitrary $\mu \in M_1(R^d)$, let $\overline{X}'_0 = \mu_N$ be as in the proof of Lemma 3.1(iii); then the semigroup $\{T'_t\}$ of $\overline{X}'$ must satisfy that for any $f \in C_b^\infty((R^d)^n)$,

$$T'_t F_{f,n}(\mu_N) = T_t^N F_{f,n}(\mu_N) = F_{V_t^n f,n}(\mu_N) \quad \text{a.s.}$$

Note that $\mu_N \to \mu$ weakly a.s.; hence from the weak Feller property of $\{T'_t\}$, we obtain

$$T'_t F_{f,n}(\mu_N) \to T'_t F_{f,n}(\mu) \qquad \text{a.s.,} \quad N \to \infty.$$

Also from the weak Feller property of $\{V_t^n\}_t$, we obtain that as $N \to \infty$,

$$F_{V_t^n f,n}(\mu_N) \to F_{V_t^n f,n}(\mu) \quad \text{a.s.}$$

Therefore,

$$T'_t F_{f,n}(\mu) = T_t F_{f,n}(\mu).$$

With Lemma 2.3, we get

$$T'_t = T_t,$$

which shows that $\overline{X}'$ and $\overline{X}$ have the same finite-dimensional distribution. If in addition $\overline{X}'$ has continuous sample paths, then the two processes have the same law on the space $C([0,\infty); M_1(R^d))$. □

4. Continuous $M_1(R^d)$-valued process. For a metric space E, we denote by D_E all cadlag functions from $[0,\infty)$ to E. It is known that D_E equipped with the Skorohod metric is a Polish space provided E is a Polish space.

LEMMA 4.1. *Let X^k be as in Lemma* 2.1. *Denote by A^k the generator of X^k. Then for $F_{f,n} \in C_p^\infty(M_1(R^d))$ and $\mu = (1/k)\sum_{i=1}^k \delta_{y_i}$, we have*

$$A^k F_{f,n}(\mu) = F_{A_n f,n}(\mu). \tag{4.1}$$

PROOF. From the definition of the generator, we have

$$\begin{aligned} A^k F_{f,n}(\mu) &= \lim_{t\to 0+} \frac{1}{t} E\big[F_{f,n}(X_t^k) - F_{f,n}(X_0^k) | X_0^k = \mu\big] \\ &= \frac{1}{k^n} \sum_{j_1,\dots,j_n=1}^{k} \lim_{t\to 0+} \frac{1}{t} E\big[f(Y_{0t}(y_{j_1}),\dots,Y_{0t}(y_{j_n})) \\ &\qquad\qquad - f(y_{j_1},\dots,y_{j_n})\big] \\ &= \frac{1}{k^n} \sum_{j_1,\dots,j_n=1}^{k} A_n f(y_{j_1},\dots,y_{j_n}) \\ &= \langle \mu^n, A_n f\rangle = F_{A_n f,n}(\mu). \end{aligned}$$

□

Given $\mu \in M_1(R^d)$, let $(X_t^N)_{t\ge 0}$ with $X_0^N = \mu_N$ be the process specified at the beginning of the proof of Lemma 3.2. We consider $X^N = (X_t^N)$ as a random variable takes value in $D_{M_1(R^d)} \subset D_{M_1(R^d_\triangle)}$ and denote by P^N its distribution on $D_{M_1(R^d_\triangle)}$.

LEMMA 4.2. *$\{P^N\}_{N\geq 1}$ form a tight family in $D_{M_1(R^d_\triangle)}$.*

PROOF. Define a function family on $M_1(R^d_\triangle)$ as follows:

$$C_p^\infty(M_1(R^d_\triangle)) = \{F_{g,n}(\cdot) | g = f + c,\ c \text{ is constant,}$$
$$f \in C_b^\infty((R^d)^n) \cap C_0((R^d)^n)\}.$$

It is easy to prove that $C_p^\infty(M_1(R^d_\triangle))$ is an algebra and separates points in $M_1(M_1(R^d_\triangle))$.

Let $F_{f,n} \in C_p^\infty(M_1(R^d_\triangle))$. Denote by $(Z_t)_{t\geq 0}$ the coordinate process on $D_{M_1(R^d_\triangle)}$. By the above lemma,

$$F_{f,n}(Z_t) - \int_0^t F_{A_n f,n}(Z_s)\,ds, \qquad t \geq 0$$

is a P^N-martingale for each N. Denote by E^N the expectation of P^N. Clearly we have

$$\sup_N E^N\left[\sup_{s\leq t} |F_{A_n f,n}(Z_s)|\right] < \infty.$$

Therefore, by Corollary 3.6.3 of [11], the family

$$P^N \circ (F_{f,n}(Z.))^{-1}, \qquad N \geq 1$$

is tight in D_R for each $F_{f,n} \in C_p^\infty(M_1(R^d_\triangle))$. Note that $M_1(R^d_\triangle)$ is a compact metric space. Hence by Theorem 3.6.4 of [11], $\{P^N\}_{N\geq 1}$ is tight in $D_{M_1(R^d_\triangle)}$. $\square$

For a metric space E, we denote by C_E the space of all continuous functions from $[0,\infty)$ to E.

LEMMA 4.3. *$\{P^N\}$ converges weakly to a probability measure P_μ on $C_{M_1(R^d_\triangle)}$ such that its finite-dimensional distribution is uniquely determined by the semigroup $\{T_t\}_{t\geq 0}$ specified in* (3.7) *and the initial measure $\mu \in M_1(R^d)$.*

PROOF. Note that

$$T_t F_{f,n}(\nu) = T_t^N F_{f,n}(\nu) \qquad \forall\, \nu \in M_{1,N}(R^d),\ n \leq N.$$

By Lemmas 3.3, 2.3 and 4.2, combining with the Markov property, $\{P^N\}$ is convergent to P_μ on $D_{M_1(R^d_\triangle)}$ according to finite-dimensional distributions following from a standard method (cf. [11]).

Let $\{f_n\}_{n\geq 1}\subset C_b(R^d_\triangle)$ be the family of functions used in Remark 2.2 for the definition of the metric ρ. We define the following metric which is also compatible with the weak topology on $M_1(R^d_\triangle)$:

$$d(\mu,\nu)=\sum_{n=1}^{\infty}\frac{1}{a_n}\left(1\wedge\left|\int f_n\,d\mu-\int f_n\,d\nu\right|\right), \tag{4.2}$$

where

$$a_n=2\{\max\{2^n,1+b_n\}\}^{12},$$
$$b_n=\max_{x\in R^d}\Bigg\{2|f_n(x)|+\left|\tfrac{1}{2}\sum_{p,q=1}^{d}a^{pq}(x,x)\partial_{pq}f_n(x)+\sum_{p=1}^{d}b^p(x)\partial_p f_n(x)\right|$$
$$+\sum_{p=1}^{d}\left(\sum_{q=1}^{d}\sigma_{pq}(x)\partial_q f_n(x)\right)^2\Bigg\},$$

$(\sigma_{pq}(\cdot,\cdot)):=\sigma(\cdot,\cdot)$ is a nonnegative symmetric matrix with bounded entries such that $\sigma(\cdot,\cdot)^2=a(\cdot,\cdot)$, and $\partial_i f$ denotes the ith direction derivative of f, $\partial_{ij}f=\partial_i(\partial_j f)$.

Given $\mu\in M_1(R^d)$, let $(X^N_t)_{t\geq 0}$ with $X^N_0=\mu_N$ be the process specified at the beginning of the proof of Lemma 3.2. Note that for any function $f\in C^2_b(R^d)$,

$$\int f\,dX^N_t:=\langle X^N_t,f\rangle=\frac{1}{N}\sum_{j=1}^{N}f(Y_{0t}(z_j)),$$

where $(Y_{0t}(x))_t$ is an A_1-diffusion with coeficients satisfying assumptions (1.1)–(1.3).

By Itô's rule, we can prove

$$\begin{aligned}f(Y_{0t}(x))-f(x)&=M^f_t(x)+\int_0^t A_1 f(Y_{0s}(x))\,ds\\&=M^f_t(x)+N^f_t(x),\end{aligned}$$

where $M^f_t(x)$ is a martingale with

$$\langle M^f_\cdot(x)\rangle_t=\int_0^t\sum_{p=1}^{d}\left(\sum_{q=1}^{d}\sigma_{pq}(Y_{0s}(x))\partial_q f(Y_{0s}(x))\right)^2 ds.$$

[In the above we have taken the convention $(\sigma(z):=\sigma(z,z))$.]

Let f_n be as in (4.2); then

$$\int f_n\,dX^N_t-\int f_n\,dX^N_s=\frac{1}{N}\sum_{j=1}^{N}\left[M^{f_n}_t(z_j)-M^{f_n}_s(z_j)+N^{f_n}_t(z_j)-N^{f_n}_s(z_j)\right].$$

Since x^p $(p > 1)$ is a convex function on R_+,

$$\begin{aligned}
&\left|\int f_n\,dX_t^N - \int f_n\,dX_s^N\right|^p \\
&\quad\le \left\{\frac{1}{N}\sum_{j=1}^N \left|M_t^{f_n}(z_j) - M_s^{f_n}(z_j) + N_t^{f_n}(z_j) - N_s^{f_n}(z_j)\right|\right\}^p \\
&\quad\le \frac{1}{N}\sum_{j=1}^N \left|M_t^{f_n}(z_j) - M_s^{f_n}(z_j) + N_t^{f_n}(z_j) - N_s^{f_n}(z_j)\right|^p \\
&\quad\le \frac{2^p}{N}\sum_{j=1}^N \{|M_t^{f_n}(z_j) - M_s^{f_n}(z_j)|^p + |N_t^{f_n}(z_j) - N_s^{f_n}(z_j)|^p\}.
\end{aligned}$$

Taking $p = 3$, by Hölder's inequality, we obtain

$$\begin{aligned}
d(X_t^N, X_s^N)^3 &\le \left(\sum_{n=1}^\infty \frac{1}{a_n^{3/4}}\right)^2 \left(\sum_{n=1}^\infty \frac{1}{a_n^{3/2}}\left|\int f_n dX_t^N - \int f_n dX_s^N\right|^3\right) \\
&\le c\biggl(\sum_{n=1}^\infty \frac{1}{a_n^{3/2}}\sum_{j=1}^N \frac{1}{N}\{|M_t^{f_n}(z_j) - M_s^{f_n}(z_j)|^3 \\
&\qquad + |N_t^{f_n}(z_j) - N_s^{f_n}(z_j)|^3\}\biggr),
\end{aligned}$$

where $c = 2^3(\sum_{j=1}^\infty 1/a_n^{3/4})^2$.

Taking expectations at both sides of the above formula, and using the Burkholder–Davies–Gundy inequality and the consistency of $\{A_k\}_{k\ge 1}$, we obtain

$$\begin{aligned}
&E^N\big(d(X_t^N, X_s^N)^3\big) \\
&\quad\le c\biggl(\sum_{n=1}^\infty \frac{1}{a_n^{3/2}}\sum_{j=1}^N \frac{1}{N}E^N\{\left|M_t^{f_n}(z_j) - M_s^{f_n}(z_j)\right|^3 \\
&\qquad\qquad + \left|N_t^{f_n}(z_j) - N_s^{f_n}(z_j)\right|^3\}\biggr) \\
&\quad= c\biggl(\sum_{n=1}^\infty \frac{1}{a_n^{3/2}}E^N\{\left|M_t^{f_n}(z_1) - M_s^{f_n}(z_1)\right|^3 \\
&\qquad\qquad + \left|N_t^{f_n}(z_1) - N_s^{f_n}(z_1)\right|^3\}\biggr) \\
&\quad= c\biggl(\sum_{n=1}^\infty \frac{1}{a_n^{3/2}}\biggl\{\int_{R^d} P\big[\left|M_t^{f_n}(x) - M_s^{f_n}(x)\right|^3\big|Y_{00}(x) = x\big]\mu(dx) \\
&\qquad\qquad + \int_{R^d} P\big[\left|N_t^{f_n}(x) - N_s^{f_n}(x)\right|^3\big|Y_{00}(x) = x\big]\biggr\}\biggr)\mu(dx)
\end{aligned}$$

$$
\begin{aligned}
&\le c\biggl(\sum_{n=1}^{\infty}\frac{1}{a_n^{3/2}}\sup_{x\in R^d}\bigl\{P\bigl[|M_t^{f_n}(x)-M_s^{f_n}(x)|^3|Y_{00}(x)=x\bigr]\\
&\qquad\qquad + P\bigl[|N_t^{f_n}(x)-N_s^{f_n}(x)|^3|Y_{00}(x)=x\bigr]\bigr\}\biggr)\\
&\le c\biggl(\sum_{n=1}^{\infty}\frac{1}{a_n^{3/2}}\bigl\{c_1 b_n^{3/2}|t-s|^{3/2}+b_n^3|t-s|^3\bigr\}\biggr)\\
&\le c\biggl(\sum_{n=1}^{\infty}\frac{1}{a_n^{3/4}}\frac{c_1 b_n^{3/2}+b_n^3}{a_n^{3/4}}(|t-s|^3\vee|t-s|^{3/2})\biggr)\\
&\le c\biggl(\sum_{n=1}^{\infty}\frac{1}{a_n^{3/4}}c_2(|t-s|^3\vee|t-s|^{3/2})\biggr)\\
&= c_3(|t-s|^3\vee|t-s|^{3/2}),
\end{aligned}
$$

where c_1, c_2, c_3 are constants. With the convergence we have just proved, we get

$$E_\mu\bigl(d(Z_t, Z_s)^3\bigr)\le c_3(|t-s|^3\vee|t-s|^{3/2}),$$

where E_μ is the expectation related to P_μ and (Z_t) is the coordinate process on $D_{M_1(R^d_\triangle)}$. Now by Kolmogorov's continuity criterion, we see that (Z_t) has a continuous realization on $C_{M_1(R^d_\triangle)}$. So far we have proved the lemma. □

LEMMA 4.4. *Suppose $\mu\in M_1(R^d)$. Define*

$$\sigma=\inf\{t|Z_t\notin M_1(R^d)\}.$$

Then

$$P_\mu\{\sigma<\infty\}=0.$$

PROOF. In order to obtain $P_\mu[\sigma<\infty]=0$, it suffices to prove

$$\lim_{n\to\infty}P_\mu\Bigl[\inf_{0\le s\le T}\langle Z_s, f_n\rangle\Bigr]=1\qquad\forall\ 0<T<\infty. \tag{4.3}$$

Here T is constant and $f_n\in C_0(R^d)\cap C_b^\infty(R^d)$ such that

$$f_n(x)=\begin{cases}1, & |x|\le n,\\ 0, & |x|>n+1.\end{cases}$$

In fact, if $P_\mu[\sigma<\infty]>0$, then we can choose $T\in(0,\infty)$ satisfying

$$P_\mu[\sigma<T]>0.$$

Note that on $[\sigma<T]$,

$$\inf_{0\le s\le T}\langle Z_s, f_n\rangle\le\inf_{o\le s\le T}Z_s(R^d)<1.$$

Then we have that

$$\lim_{n\to\infty} P_\mu\Big[\inf_{0\le s\le T}\langle Z_s, f_n\rangle\Big] \le P_\mu\Big[\inf_{0\le s\le T} Z_s(R^d)\Big] < 1,$$

which contradicts (4.3).

Now we are in the position to prove (4.3).

On $D_{M_1(R^d_\triangle)}$, the Skorohod topology is weaker than the local uniform topology; but when $Z \in C_{M_1(R^d_\triangle)}$, a sequence $\{Z^n\}_n \subset D_{M_1(R^d_\triangle)}$ converges to Z for the Skorohod topology if and only if it converges to Z locally uniformly (refer to [21], Proposition 1.17, page 292, for a similar proof). By this fact, on $C_{M_1(R^d_\triangle)}$, the Skorohod topology is just the locally uniform one. Note that $\{P^N\}_N$ converges weakly to P_μ on $C_{M_1(R^d_\triangle)}$. We see that $\inf_{0\le s\le T}\langle Z_s, f_n\rangle$ is a bounded continuous function on $C_{M_1(R^d_\triangle)}$ and

$$\lim_{N\to\infty} P^N\Big[\inf_{0\le s\le T}\langle Z_s, f_n\rangle\Big] = P_\mu\Big[\inf_{0\le s\le T}\langle Z_s, f_n\rangle\Big].$$

However, by the consistency, we have

$$\begin{aligned}
P^N\Big[\inf_{0\le s\le T}\langle Z_s, f_n\rangle\Big] &\ge P^N\Big[\inf_{0\le s\le T} Z_s(B_n)\Big]\\
&= P\bigg[\inf_{0\le s\le T}\bigg\{\sum_{i=1}^N \frac{I_{[Y_{0s}(z_i)\in B_n]}}{N}\bigg\}\bigg]\\
&\ge P\bigg[\sum_{i=1}^N \frac{1}{N}\inf_{0\le s\le T} I_{[Y_{0s}(z_i)\in B_n]}\bigg]\\
&= P\Big[\inf_{0\le s\le T} I_{[Y_{0s}(z_1)\in B_n]}\Big]\\
&= \int_{R^d} P_x\Big[\inf_{0\le s\le T} I_{[Y_{0s}(x)\in B_n]}\Big]\mu(dx)\\
&\ge \int_{R^d} P_x[\tau_n \ge T]\mu(dx),
\end{aligned}$$

where P_x is the law of $(Y_{0s}(x))$, $B_n = \{x \in R^d \mid |x| \le n\}$ and

$$\tau_n = \inf\{t \mid Y_{0t}(x) \notin B_n\}.$$

Therefore,

$$\begin{aligned}
\liminf_{n\to\infty} P_\mu\Big[\inf_{0\le s\le T}\langle Z_s, f_n\rangle\Big] &\ge \liminf_{n\to\infty}\int_{R^d} P_x[\tau_n \ge T]\mu(dx)\\
&\ge \int_{R^d}\liminf_{n\to\infty} P_x[\tau_n \ge T]\mu(dx) = 1.
\end{aligned}$$

This proves (4.3). □

THEOREM 4.5. *Let $(\overline{X}_t)_{t\ge 0}$ be the coordinate process on $C_{M_1(R^d)}$. Then there exists a unique family of probability measures $\{P_\mu\}_{\mu\in M_1(R^d)}$ on $C_{M_1(R^d)}$ such* *that* $(\overline{X}_t)_{t\ge 0}$ *becomes a continuous strong Markov process associated with* $\{T_t\}_{t\ge 0}$.

PROOF. With Lemmas 4.3 and 4.4, we conclude that $(\overline{X}_t)$ $(\overline{X}_0 = \mu \in M_1(R^d))$ has indeed a $M_1(R^d)$-valued continuous version. Let P_μ be its law on $C_{M_1(R^d)}$. Then $\{P_\mu\}_{\mu\in M_1(R^d)}$ is the desired family of probabilities. By a routine argument (cf., e.g., Sharp [30], Theorem 7.4(v), page 31), the strong Markov property of $(\overline{X}_t)$ follows from the Feller property and the continuity of sample paths. The uniqueness of $\{P_\mu\}_{\mu\in M_1(R^d)}$ follows from Proposition 3.8(ii). □

From the above proof, we have the following.

PROPOSITION 4.6. *For the metric specified in* (4.2), $\overline{X}$ *has an* α*-Hölder continuous version on* $C_{M_1(R^d)}$ *for* $\alpha \in (0, \frac{1}{2})$.

REMARK 4.7. As we did in Remark 3.7, for $\mu \in M_1(R^d)$ and $m = r\mu$, let $X_t = r\overline{X}_t$ with $\overline{X}_0 = \mu$, then $X = (X_t)_{t\ge 0}$ is a continuous strong Markov process on $M(R^d)$ associated with the Feller semigroup $\{T_t^X\}_{t\ge 0}$.

5. Generator of $\overline{X}$. Denote by A with domain $D(A)$ the (weak) generator of $(\overline{X})$. That is,

$$AF(\mu) = \lim_{t\to 0+}\frac{1}{t}E_\mu\big[F(\overline{X}_t) - F(\overline{X}_0)\big] = \lim_{t\to 0+}\frac{1}{t}\big[T_tF(\mu) - F(\mu)\big] \tag{5.1}$$

and

$$D(A) = \big\{F \in C_b(M_1(R^d)) \mid AF(\mu) \text{ exists for all } \mu \in M_1(R^d)\big\}.$$

LEMMA 5.1. *If* $F = F_{f,n} \in C_p^\infty(M_1(R^d))$, *then* $F_{f,n} \in D(A)$ *and*

$$AF_{f,n}(\mu) = F_{A_nf,n}(\mu). \tag{5.2}$$

PROOF. Let $F = F_{f,n} \in C_p^\infty(M_1(R^d))$. Then

$$\begin{aligned}&\left|\frac{T_tF(\mu) - F(\mu)}{t} - F_{A_nf,n}(\mu)\right|\\ &\quad\le \int_{(R^d)^n}\left|\frac{1}{t}(V_t^nf - f) - A_nf\right|(x)\mu^n(dx) \to 0.\end{aligned}$$

□

For a function F on $M(R^d)$, we define for $\mu \in M(R^d)$ and $x \in R^d$,

$$\frac{\delta F(\mu)}{\delta\mu(x)} = \frac{d}{d\varepsilon}F(\mu + \varepsilon\delta_x)\bigg|_{\varepsilon=0},$$

provided the derivative exists. It is easy to see [11] that for $F = F_{f,n}$ with $f \in C_b^\infty((R^d)^n)$, we have

$$\frac{\delta F_{f,n}(\mu)}{\delta\mu(x)} = \sum_{j=1}^n \int f(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_n) \prod_{i\neq j} \mu(dx_i),$$

$$\frac{\delta^2 F_{f,n}(\mu)}{\delta\mu(x)\delta\mu(y)} = \sum_{j\neq k} \int f(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_{k-1}, y, x_{k+1}, \dots, x_n) \prod_{i\neq j,k} \mu(dx_i).$$

THEOREM 5.2. *For $F = F_{f,n}$ with $n \geq 2$ and $f \in C_b^\infty((R^d)^n)$, we have*

$$AF(\mu) = \int A_1 \frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) - 2\int \overline{A}_1\Big(\frac{\delta^2 F(\mu)}{\delta\mu(\cdot)\delta\mu(y)}\Big)(x)\mu^2(dxdy) + \int \overline{A}_2 \frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu^2(dxdy) \qquad \forall\, \mu \in M_1(R^d),$$

where $\overline{A}_1$, $\overline{A}_2$ are the diffusion parts of A_1, A_2, respectively.

PROOF. With Lemma 5.1, we have

$$\begin{aligned}
AF(\mu) &= F_{A_n f, n}(\mu) = \int A_n f\, d\mu^n \\
&= \int \Big[\frac{1}{2}\sum_{i,j=1}^n \sum_{p,q=1}^d a^{pq}(x_j, x_i)\frac{\partial^2 f}{\partial x_i^p \partial x_j^q}(x_1, \dots, x_n) \\
&\qquad + \sum_{i=1}^n \sum_{p=1}^d b^p(x_i)\frac{\partial f}{\partial x_i^p}(x_1, \dots, x_n)\Big]\mu^n(dx_1\cdots dx_n) \\
&= \sum_{i=1}^n \int \Big[\frac{1}{2}\sum_{p,q=1}^d a^{pq}(x_i, x_i)\frac{\partial^2 f}{\partial x_i^p \partial x_i^q}(x_1, \dots, x_n) \\
&\qquad + \sum_{p=1}^d b^p(x_i)\frac{\partial f}{\partial x_i^p}(x_1, \dots, x_n)\Big]\mu^n(dx_1\cdots dx_n) \\
&\quad + \int \Big[\frac{1}{2}\sum_{i\neq j}\sum_{p,q=1}^d a^{pq}(x_j, x_i)\frac{\partial^2 f}{\partial x_i^p \partial x_j^q}(x_1, \dots, x_n)\Big]\mu^n(dx_1\cdots dx_n) \\
&= \sum_{i=1}^n \int \Big[\frac{1}{2}\sum_{p,q=1}^d a^{pq}(x_i, x_i)\frac{\partial^2 f}{\partial x_i^p \partial x_i^q}(x_1, \dots, x_n) \\
&\qquad + \sum_{p=1}^d b^p(x_i)\frac{\partial f}{\partial x_i^p}(x_1, \dots, x_n)\Big]\Big(\prod_{k\neq i}\mu(dx_k)\Big)\mu(dx_i)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i\neq j}\int\sum_{p,\,q=1}^{d} a^{pq}(x_j, x_i)\frac{\partial^2 f}{\partial x_i^p \partial x_j^q}(x_1,\dots,x_n)\\
&\qquad\times\Big(\prod_{k\neq i,\,j}\mu(dx_k)\Big)\mu^2(dx_i dx_j)\\
&=\int A_1\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx)\\
&+\frac{1}{2}\sum_{i\neq j}\int\sum_{p,\,q=1}^{d}\Big\{a^{pq}(x_i, x_i)\frac{\partial^2 f}{\partial x_i^p \partial x_i^q}(x_1,\dots,x_n)\\
&\qquad+a^{pq}(x_j, x_j)\frac{\partial^2 f}{\partial x_j^p \partial x_j^q}(x_1,\dots,x_n)\\
&\qquad+a^{pq}(x_i, x_j)\frac{\partial^2 f}{\partial x_i^p \partial x_j^q}(x_1,\dots,x_n)\Big\}\\
&\qquad\times\Big(\prod_{k\neq i,\,j}\mu(dx_k)\Big)\mu^2(dx_i dx_j)\\
&-\frac{1}{2}\sum_{i\neq j}\int\sum_{p,\,q=1}^{d}\Big\{a^{pq}(x_i, x_i)\frac{\partial^2 f}{\partial x_i^p \partial x_i^q}(x_1,\dots,x_n)\\
&\qquad+a^{pq}(x_j, x_j)\frac{\partial^2 f}{\partial x_j^p \partial x_j^q}(x_1,\dots,x_n)\Big\}\\
&\qquad\times\Big(\prod_{k\neq i,\,j}\mu(dx_k)\Big)\mu^2(dx_i dx_j)\\
&=\int A_1\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx)+\int\overline{A}_2\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu^2(dxdy)\\
&-2\int\overline{A}_1\Big(\frac{\delta^2 F(\mu)}{\delta\mu(\cdot)\delta\mu(y)}\Big)(x)\mu^2(dxdy). \qquad \square
\end{aligned}
$$

6. Comparison with other superprocesses. (I) Suppose that $\{Y_t\}_{t\geq 0}$ is an $M(R^d)$ or $M_1(R^d)$ valued Markov process. Denote by $\{P_t\}_{t\geq 0}$ the semigroup of $Y=\{Y_t\}_{t\geq 0}$. For F in an appropriate domain $D(L)$, the generator (L_t) of Y or $\{P_t\}_{t\geq 0}$ has the following form [11]:

$$
L_t F(\mu)=\int_{R^d} L(t,\mu,dx)\frac{\delta F(\mu)}{\delta\mu(x)}+\frac{1}{2}\int_{(R^d)^2}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}Q(t,\mu,dx,dy),
$$

where $L(t, \cdot, \cdot)$ generates a deterministic evolution on $M(R^d)$ $(M_1(R^d))$ and

$$Q(t, \mu, dx, dy)$$

is a symmetric signed measure on $R^d \times R^d$ satisfying

$$\int_{(R^d)^2} f(x)f(y)Q(t, \mu, dx, dy) \geq 0 \qquad \forall\, f \in C_b(R^d).$$

If we specialize $L(t, \mu, dx)$ and $Q(t, \mu, dx, dy)$, then we can obtain Dawson–Watanabe superprocesses and Fleming–Viot processes, respectively [11].

(a) Dawson–Watanabe process:

$$\begin{aligned}
L(t, v) &= Lv + R(t, v), \qquad v \in D(L),\\
Q(t, \mu, dx, dy) &= c(x)\delta_x(dy)\mu(dx),\\
L_t F(\mu) &= \int_{R^d} L\left(t, \frac{\delta F(\mu)}{\delta\mu(\cdot)}\right)\mu(dx)\\
&\quad + \frac{1}{2}\int_{R^d} c(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(x)}\mu(dx),
\end{aligned}$$

where L is the generator of a Feller semigroup on $C_0(R^d)$ which describes the spatial motion, $R(\cdot, \cdot)$ satisfies certain regular conditions and is called the interaction term.

(b) Fleming–Viot process:

$$\begin{aligned}
L(t, v) &= Lv + R(t, v), \qquad v \in D(L),\\
Q(t, \mu, dx, dy) &= \gamma(x)(\delta_x(dy)\mu(dx) - \mu(dx)\mu(dy)), \qquad \gamma > 0,\\
L_t F(\mu) &= \int_{R^d} L\left(t, \frac{\delta F(\mu)}{\delta\mu(\cdot)}\right)\mu(dx)\\
&\quad + \frac{1}{2}\int_{(R^d)^2} \frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\gamma(x)\big(\delta_x(dy)\mu(dx) - \mu(dx)\mu(dy)\big).
\end{aligned}$$

In strict sense, (a) and (b) should be called the Dawson–Watanabe process with interaction term and Fleming–Viot process with interaction term, respectively; the usual Dawson–Watanabe process or Fleming–Viot process corresponds to the case when $R = 0$.

(II) The homogeneous Ornstein–Uhlenbeck superprocesses based on R^d can be described as follows (see [12]).

Let $\{V_t\}_{t\geq 0}$ be a semigroup associated with a Feller process on R^d. Define $P_t(\mu, \cdot)$ as a Markov transition function in $M(R^d)$ such that

$$\int_{M(R^d)} P_t(\mu, dv)\langle v, f\rangle = \langle \mu, V_t f\rangle \qquad \forall\, f \in C_b(R^d).$$

We say $(P_t(\mu, \cdot))$ is a supertransition function over $(V_t)_{t\geq 0}$. $(P_t(\mu, \cdot))_{t\geq 0,\, \mu\in M(R^d)}$ is an Ornstein–Uhlenbeck supertransition function in

$M(R^d)$ if for any $f \in C_b(R^d)$,

$$(6.1)\quad \int_{M(R^d)} P_t(\mu, d\nu) e^{\sqrt{-1}\langle \nu, f\rangle} = \exp\Big\{\sqrt{-1} < \mu,\ V_t f > -\tfrac{1}{2}\int_0^t Q(V_s f)\, ds\Big\}.$$

Here Q is a positive semidefinite quadratic form on $M(R^d)$.

Conclusion. Comparing with Theorem 5.2, we see that the generator A of $\overline{X}$ differs from both the types (a) and (b) of (I). Furthermore, $\overline{X}$ has interaction term $\overline{A}_2$ that generally cannot be determined by a semigroup of a Markov process on R^d. Comparing with (6.1), we see that $\overline{X}$ is not an O–U superprocess.

In a word, $\overline{X}$ or X is a new type of superprocess. This new type of superprocess is closely related to stochastic flows, and hence is of interest to be studied further. The present paper is just a beginning.

7. Stochastic coalescence of $\overline{X}$. The process $\overline{X}$ determines a stochastic coalescence which is similar to those of [32] (see [1]) in some sense.

When the initial value of the process $\overline{X}$ is an atomic measure distributing its mass equally among k points $y_1, \dots, y_k \in R^d$, if we divide mass 1 into r fragmentations,

$$\overline{X}_t\big(\{Y_{0t}(y_{i_1})\}\big), \dots, \overline{X}_t(\{Y_{0t}(y_{i_r})\})$$

at time t, where

$$\big\{Y_{0t}(y_{i_l}) | 1 \le l \le r,\ 1 \le i_1 < \cdots < i_r \le k\big\}$$

is the set of all distinct points among $\{Y_{0t}(y_i)\}_{1\le i\le k}$, then we can see $\overline{X}$ corresponds to a stochastic coalescence.

More generally, when the initial value of $\overline{X}$ is an atomic measure, the process $\overline{X}$ corresponds also to a stochastic coalescence.

Let $\mu = \sum_{i=1}^{\infty} a_i \delta_{y_i} \in M_1(R^d)$, $a_i \ge 0$, $\sum a_i = 1$. Note that for each natural number n, there is a one-to-one map from $C([0,\infty), R^d)^n$ to $C([0,\infty), (R^d)^n)$ as follows:

$$\big((\omega_1(t))_{t\ge 0}, \dots, (\omega_n(t))_{t\ge 0}\big) \to \big(\omega_1(t), \dots, \omega_n(t)\big)_{t\ge 0}.$$

Thus for any natural number sequence $(i_1, \dots, i_n)$, the law of

$$\big(Y_{0t}(y_{i_1}), \dots, Y_{0t}(y_{i_n})\big)_{t\ge 0}.$$

can be viewed as a probability on $C([0,\infty), R^d)^n$.

Now by the consistency of the stochastic flows and Kolmogorov's extension theorem, we can see that there exists a unique probability P on

$$E := C\big([0,\infty), R^d\big)^\infty = \{\omega = (\omega_1, \omega_2, \dots) | \omega_i \in C([0,\infty), R^d)\ \forall\ i\}$$

under which $(\omega_{i_1}(t), \dots, \omega_{i_n}(t))_{t\ge 0}$ evolves according to the law of

$$(Y_{0t}(y_{i_1}), \dots, Y_{0t}(y_{i_n}))_{t\ge 0}$$

for any natural number sequence $(i_1, \dots, i_n)$.

Define a process $Y = (Y_t)_{t \geq 0}$ on E as follows:

$$Y_t(\omega) = \sum a_i \delta_{\omega_i(t)} \qquad \forall\, t \geq 0,\ \omega \in E.$$

Then we can easily see that Y has the semigroup $\{T_t;\, t \geq 0\}$ on $C_p(M_1(R^d))$ and the same distribution as the process $\overline{X} = (\overline{X}_t)_{t \geq 0}$ with $\overline{X}_0 = \mu$.

Therefore the process $\overline{X}$ corresponds to a stochastic coalescence.

8. Examples arising from isotropic flows. An isotropic stochastic flow on R^d, denoted by $U = \{Y_{st}(x);\, 0 \leq s \leq t,\ x \in R^d\}$, describes the motion of a particle dropped in an isotropic mean zero Gaussian field. Here $Y_{st}(x)$ denotes the position of time t $(\geq s)$ of a particle which is at x at time s. The mutual variation for the motion of two points under the flow is to be given by

$$d\langle Y^i_{0t}(y), Y^j_{0t}(z)\rangle = b^{ij}(Y_{0t}(y) - Y_{0t}(z))\, dt, \tag{8.1}$$

where b: $R^d \to \{$symmetric $d \times d$ nonnegative definite real matrices$\}$ is the covariance tensor of U. The isotropic condition on U is

$$b(x) = G' b(Gx) G \qquad \forall\, G \in O(d), \tag{8.2}$$

where $O(d)$ is the group of real orthogonal $d \times d$ matrices, and G' stands for the transpose of G. We always suppose that $b(0)$ is the identical matrix, $b(x)$ is continuous, and all entries in $b(x)$ converge to 0 as $|x|$ goes to ∞. It is known that the covariance tensor $b(\cdot)$ has to be the following form (cf. [5, 9, 34]). For $d = 1$,

$$b(x) = \int_{R^1} e^{\sqrt{-1}xy} F(dy), \tag{8.3}$$

where $F(\cdot)$ is a probability measure with $F(dy) = F(-dy)$, and with no atom at 0.

For $d \geq 2$, for $x \neq 0$,

$$b^{pq}(x) = \big(B_L(|x|) - B_N(|x|)\big) x^p x^q / (|x|^2) + B_N(|x|)\delta^{pq}, \tag{8.4}$$

where

$$B_L(r) = A_d \left\{ \int_0^\infty \left[\frac{J_{d/2}(rs)}{(rs)^{d/2}} - \frac{J_{(d+2)/2}(rs)}{(rs)^{(d-2)/2}} \right] \Phi_1(ds) + (d-1) \int_0^\infty \frac{J_{d/2}(rs)}{(rs)^{d/2}} \Phi_2(ds) \right\},$$

$$B_N(r) = A_d \left\{ \int_0^\infty \frac{J_{d/2}(rs)}{(rs)^{d/2}} \Phi_1(ds) + \int_0^\infty \left[\frac{J_{(d-2)/2}(rs)}{(rs)^{(d-2)/2}} - \frac{J_{d/2}(rs)}{(rs)^{d/2}} \right] \Phi_2(ds) \right\},$$

$$A_d = 2^{(d-2)/2} \Gamma(d/2),$$

and J_m denotes the Bessel function of the first kind of order m, and Φ_1 and Φ_2 are any positive finite measures on $(0, \infty)$ such that

$$\frac{1}{d}\big[\Phi_1((0,\infty)) + (d-1)\Phi_2((0,\infty))\big] = 1.$$

Let $(Y_{0t}(y_1), \dots, Y_{0t}(y_k))$ be a motion of k points in R^d under a stochastic flow satisfying (8.1). This must be a diffusion process with generator A_k, where for any smooth function f on $(R^d)^k$,

$$\text{(8.5)} \qquad A_k f(z_1, \dots, z_k) = \frac{1}{2}\sum_{i,\,j=1}^{k}\sum_{p,\,q=1}^{d} b^{pq}(z_j - z_i)\frac{\partial^2}{\partial z_i^p \partial z_j^q} f(z_1, \dots, z_k).$$

The above observation suggests a way of constructing isotropic flows via martingale problem methods. To this end, denote by D_k the set of k-tuples of distinct points in $(R^d)^k$. Note that Ω_k is the path in R^d having the property that two particles have to stay together whenever they meet. It is known that, for $d \geq 2$ and for every choice of measures Φ_1 and Φ_2, and for every $s \geq 0$ and $z = (z_1, \dots, z_k) \in D_k$, there is a unique probability measure $P_{s,z}$ on Ω_k, under which the canonical process $\{(Y_{st}(z_1), \dots, Y_{st}(z_k)),\ t \geq s\}$ is a diffusion process on $(R^d)^k$ with zero drift, such that the mutual quadratic variations satisfy (8.1). Moreover the map $(s, z) \to P_{s,z}$ is measurable and the family $\{P_{s,z}\}$ is the strong Markov. The same conclusions hold when $d = 1$, provided F specified in (8.3) is not atomic with a finite number of atoms. See [9] and references therein for details of the above results.

The $\{P_{s,z};\ s \geq 0,\ z \in D_k\}$ are referred to as the law of the k-point motion. They have the obvious consistency property in the sense that the law of any k components of the n-point motion ($k \leq n$) has the law of k-point motion.

Given some strong conditions on the moment of $Y_{st}(x) - Y_{st}(y)$ for fixed $t > s$ as $y \to x$, one can construct a stochastic flow of continuous mappings (Totoki's theorem; cf. [14]).

When the measures Φ_1 and Φ_2 have finite second moments, then $b(\cdot)$ has two order-continuous derivatives. Using the reproducing kernel Hilbert space method under this condition, one can construct an isotropic stochastic flow by solving a Lipschitz type Itô stochastic differential equations (see [2, 5, 9, 24]). By Theorem 6.3.4 in [33], we can get a stochastic flow satisfying the conditions in the introduction.

However in general a stochastic flow may fail to be spatially continuous. Harris [19] constructed a stochastic flow of monotone (hence spatially measurable) mappings in R^1. But in R^d for $d \geq 2$, the only result was given by Darling [9] who constructed a stochastic flow of mappings by the martingale problem approach, which were not shown to be spatially measurable, from a consistent set of laws for k-point motions.

Though $\{A_k + \varepsilon I_{kd}\}_{k\geq 1}$ satisfies conditions (1.1)–(1.3) for each $\varepsilon > 0$, where I_{kd} denotes the $kd \times kd$ identity matrix, it is a pity that the Feller property of A_k-diffusion generally is not known at present for each k.

In a survey paper [9], Darling proposed a problem from the point of view that the real object of interest in stochastic flows is not the family of mappings $\{Y_{st}\}$ but the way that an initial measure on R^d is transported under the flow. It is stated as follows.

PROBLEM ([9], Problem 2.5). Given the generators $\{A_k;\, k \geq 1\}$ for all the k-point motions, which are consistent, construct a Markov process $\{X_t\}$ in $M(R^d)$ such that for each k, if the initial value of the process is an atomic measure distributing its mass equally among k points, the support of $\{X_t\}$ evolves according to the diffusion with the generator A_k with the rule that paths in R^d which meet must coalesce.

The main results of this paper (see Proposition 3.8 and Theorem 4.5) are applicable to certain isotropic stochastic flows constructed by the reproducing kernel Hilbert space method, which realizes the above point of view.

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ACADEMY OF MATH AND SYSTEM SCIENCES
INSTITUTE OF APPLIED MATHEMATICS
ACADEMIA SINICA
BEIJING, 100080
P.R. CHINA

DEPARTMENT OF PROBABILITY
AND STATISTICS
PEKING UNIVERSITY
BEIJING, 100871
P.R. CHINA
E-MAIL: nan_kai@yahoo.com