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VERTEX-REINFORCED RANDOM WALK ON ARBITRARY GRAPHS

BY STANISLAV VOLKOV

University of Bristol

Vertex-reinforced random walk (VRRW), defined by Pemantle, is a random process in a continuously changing environment which is more likely to visit states it has visited before. We consider VRRW on arbitrary graphs and show that on almost all of them, VRRW visits only finitely many vertices with a positive probability. We conjecture that on all graphs of bounded degree, this happens with probability 1, and provide a proof only for trees of this type.

We distinguish between several different patterns of localization and explicitly describe the long-run structure of VRRW, which depends on whether a graph contains triangles or not.

While the results of this paper generalize those obtained by Pemantle and Volkov for Z^1, ideas of proofs are different and typically based on a large deviation principle rather than a martingale approach.

1. Definition of VRRW and results. Let G be any locally finite graph without loops with the neighbor relation denoted by $\sim$. For any $x \in G$ and $V \subseteq G$, write $x \sim V$ if there exists a site $v \in V$ such that $x \sim v$, and write $x \not\sim V$ otherwise.

For any process $X_0, X_1, X_2, \dots$ taking values in the vertex set of a graph G, we define (shifted) local times

$$Z(t, v) = 1 + \sum_{s=0}^{t} \mathbf{1}_{X_s = v}$$

to be the number of times the site v has been visited by time t, plus 1. Also, for any subset $V \subseteq G$, let the local time at this set be

$$Z(t, V) = \sum_{v \in V} Z(t, v).$$

Define a vertex-reinforced random walk (VRRW) on G with starting point $v \in G$ to be a process $\{X_t \colon t \geq 0\}$ such that $X_0 = v$ and

$$\mathsf{P}(X_{t+1} = x \mid \mathcal{F}_t) = \mathbf{1}_{x \sim X_t} \frac{Z(t, x)}{\sum_{y \sim X_t} Z(t, y)}, \tag{1.1}$$

where $\mathcal{F}_t = \sigma(X_1, X_2, \dots, X_t)$. In other words, moves are restricted to the edges of G with the probability of a move to a neighbor x being proportional to the local time at x at that time.

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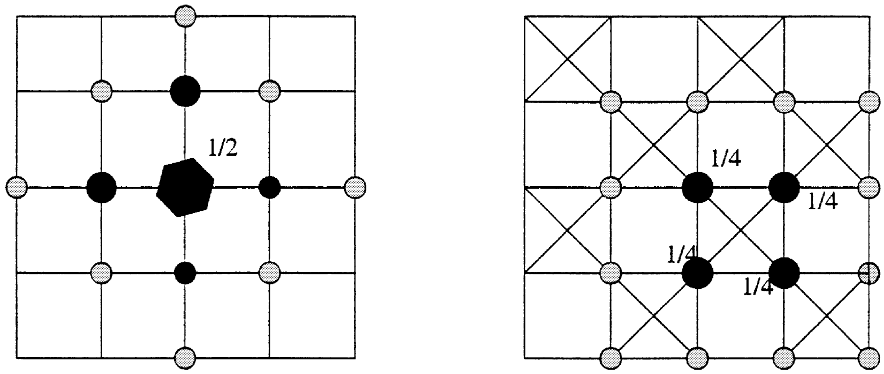

FIG. 1. *"A core and a shell" and uniform localization. On the left: the hexagon is $v = V_1$ while dark circles denote the points of V_2. On the right: dark circles are points of S. The numbers in both pictures numbers represent limiting occupational measures and grey circles are points of B.*

Similar processes can describe a learning behavior, or model a spatial monopolistic competition in economics. The first problem of this kind has been posed by Diaconis and Coppersmith, with the weights being accumulated on edges rather than vertices of a graph, and has been studied later by Davis (1990), Pemantle (1988b), Sellke (1994) and others under various reinforcement conditions.

A more comprehensive overview of the applications and known results for these models can be found in Pemantle and Volkov (1999).

In this paper we show that on practically any locally finite connected infinite graph without loops, VRRW gets stuck at a finite set of points (that is, only finitely many vertices are visited) with a positive probability. All the results obtained are also valid for finite graphs. For earlier results on vertex-reinforced random walks on such graphs see Pemantle (1992) and Benaim (1997). One can also generalize the methodology we develop here for graphs *with* loops.

Further, we will need the following definitions. The second can be found in Bolobás (1979); the first is used in this paper only.

DEFINITION 1. For any set of vertices $S \subseteq G$, the *outer boundary* of S is the set

$$\partial S = \{y \in G \backslash S \colon y \sim S\}.$$

The outer boundary is called *nonembracing* if there exists no point $y \in \partial S$ such that $y \sim x$ simultaneously for all $x \in S$.

DEFINITION 2. A subset $S \subseteq G$ is called a *complete n-partite graph* if it is a disjoint union of nonempty sets $V_1, V_2, \dots, V_n$, $n \geq 2$, such that:

(a) For any $i \in \{1, 2, \dots, n\}$ and any two vertices $x, y \in V_i x \not\sim y$.

(b) For any $i, j \in \{1, 2, \dots, n\}$, $i \neq j$ each $x \in V_i$ is connected to every $y \in V_j$.

We will refer to the sets V_i as *pseudo-vertices* of this graph.

DEFINITION 3. A subgraph $G' \subseteq G$ is called a *trapping* subgraph if it consists of a complete n-partite graph $S = V_1 \cup V_2 \cup \cdots \cup V_n$ and its outer boundary $B = \partial S$ and the following property holds: for any $y \in B$ there exist $i \in \{1, 2, \ldots, n\}$ and $x' \in S \backslash V_i$ such that $y \not\sim V_i \cup \{x'\}$.

REMARK 1. Since the graph G is locally finite, the number of vertices in each of V_i and in all G' is finite.

We start with a general theorem on the localization of VRRW and consider some special cases and examples later.

THEOREM 1.1. *Let $G' = S \cup B$, $S = V_1 \cup V_2 \cup \cdots \cup V_n$, be a trapping subgraph of G. Then for the VRRW which originates on G', with a positive probability there exist a set of positive numbers $\{\alpha_v,\ v \in S\}$ with $\sum_{v \in S} \alpha_v = 1$ such that the following are fulfilled:*

(i) *VRRW never leaves G'.*
(ii) *$Z(t, v)/t \to \alpha_v$ for all $v \in S$ as $t \to \infty$.*
(iii) *$\sum_{v \in V_i} \alpha_v = 1/n$ for all $i \in \{1, 2, \ldots, n\}$.*
(iv) *$\log Z(t, y)/\log t \to (n/(n-1)) \sum_{x \in S,\, x \sim y} \alpha_x$ for all $y \in B$.*

REMARK 2. Property (iii) and the definition of a trapping subgraph insure that the r.h.s. of (iv) is strictly smaller than 1. Therefore, with a positive probability, the (random) limiting occupational measure of VRRW exists and has support on the set S.

Now we shall investigate on which graphs trapping subgraphs exist and their typical shapes. Let us begin with graphs which do not contain triangles. In this case $n = 2$, and the simplest case is when V_1 consists of a single point (see Fig. 1).

COROLLARY 1.2 (A core and a shell localization). *Suppose that a vertex $v \in G$ ("a core") does not belong to any triangle. Let $V_1 = \{v\}$ ("a shell"); $V_2 = \partial V_1$, and on the graph $G \backslash V_1$ the outer boundary B of the set V_2 is nonembracing (the vertex v obviously is connected to all vertices of V_2). If VRRW starts on $G' = V_1 \cup V_2 \cup B$, then with a positive probability there exists a set of $n_2 := |V_2|$ positive numbers $\{\alpha_x,\ x \in V_2\}$, $\sum_{x \in V_2} \alpha_x = 1/2$, such that the following events occur:*

(i) *VRRW never leaves G'.*
(ii) *$Z(t, v)/t \to 1/2$ as $t \to \infty$.*
(iii) *$Z(t, x)/t \to \alpha_x$ for all $x \in V_2$.*
(iv) *$\log Z(t, y)/\log t \to 2 \sum_{x \in V_2,\, x \sim y} \alpha_x$ for all $y \in B$.*

This theorem can be applied to any tree or lattice Z^d. To extend this result for graphs which contain triangles, like a square lattice with two diagonals in alternating squares, we need the following.

DEFINITION 4. A *clique* is a complete subgraph S not contained as a subset of a larger complete subgraph.

[This definition is taken from Tucker (1995).]

REMARK 3. For any $v \in G$ there exist a finite clique containing v; however, it may be not unique.

Now let S be a clique containing v and having $n \geq 3$ vertices (for simplicity, we will denote them as $1, 2, \dots, n$). Let B be the outer boundary of S, and $G' = S \cup B$. It is easy to see that B is nonembracing as soon as S is a clique. However, it is not enough for G' to be a trapping subgraph.

COROLLARY 1.3 (Uniform localization). *Let S be a clique with $n := |S| \geq 3$ vertices, whose outer boundary $B = \partial S$ has the property that none of its vertices is connected to more than $n-2$ sites of S. Then VRRW starting on $G' = S \cup B$ with a positive probability satisfies the following:*

(i) *It never leaves G'.*
(ii) $Z(t, x)/t \to 1/n$ *for all* $x \in S$ *as* $t \to \infty$.
(iii) $\log Z(t, y)/\log t \to |\{x \in S, x \sim y\}|/(n-1)$ *for all* $y \in B$.

(This corresponds to a case when every V_i consists of a single point; see also Figure 1.)

Thus, the limiting occupational measure of VRRW may exhibit different patterns of convergence depending on whether a graph has a complete subgraph of size greater than or equal to 3. This is not very surprising since it is known that for a closely related process, edge-reinforced random walk, having triangles on a graph can lead to substantial difficulties [see Sellke (1994)]. Notice that Corollary 1.3 does not cover the case of a planar triangular lattice.

1.1. *More complicated types of localization.* Let us show how (possibly) one can locate a trapping subgraph.

Iterative procedure 1. Pick any point $v_1 \in G$, consider a clique $S = \{v_1\} \cup \{v_2\} \cup \dots \cup \{v_n\}$ containing this point and let $B = \partial S$. If no point of B is connected to more than $n-2$ points of S, we can immediately apply Corollary 1.3. Otherwise, there exists a vertex in B connected to all points of S but one (as B is nonembracing). Without loss of generality let it be v_1. Set $V_1 := \{y \in G\colon y \sim v_i \text{ for all } i = 2, 3, \dots, n\}$; hence V_1 contains at least two points. Iteratively define V_j for $j = 2, 3, \dots, n$ by

$$V_j := \left\{ y \in G\colon y \sim x \text{ for all } x \in \left(\bigcup_{k=1}^{j-1} V_j \right) \cup \left(\bigcup_{k=j+1}^{n} \{v_k\} \right) \right\}.$$

Let $S := \bigcup_{j=1}^{n} V_j$. If for some j there are two vertices $v' \in V_j$ and $v'' \in V_j$ such that $v' \sim v''$, then the set $1, 2, \dots, j-1, v', v'', j+1, \dots, n$ constitutes a complete subgraph of size $n+1$. In this case we let S be a clique containing this set and start the procedure over. In the other case, when for all j no two points of V_j are connected, S constitutes a complete n-partite graph by construction. Consider $B = \partial S$. If there exists a point $x \in B$ such that $x \sim V_j$ for all j, we can find a complete subgraph having $n+1$ vertices and we start the procedure again with this graph. We will stop the iterative procedure only when all V_j consist of nonconnected vertices and for each $x \in B$ there exists j such that $x \not\sim V_j$. Note that the way we construct S leads to the existence of another $j' \neq j$ such that x is not connected to some point of $V_{j'}$ (otherwise we would have $x \in V_j$).

There is no guarantee that this procedure will ever stop; however, if it does, we call it *successful*. This being the case, the resulting subgraph $S \cup B$ is precisely a trapping subgraph by Definition 3, and consequently Theorem 1.1 applies.

Graphs for which the procedure is not successful must have some unusual properties (in particular, they should contain complete subgraphs of arbitrary high orders).

The corollary of this observation is that if the graph G does not contain triangles, it always contains a trapping subgraph, as any clique on G always consists of two vertices. In this case the complete n-partite graph in a trapping subgraph also has $n = 2$ and if both $|V_1| \geq 2$ and $|V_2| \geq 2$ it is natural to call the localization on $G' := V_1 \cup V_2 \cup B$, $B = \partial(V_1 \cup V_2)$, a *bipartite* localization, in a contrast to a core and a shell localization. This localization can occur on Z^2 (see on the right of Figure 2).

REMARK 4. The proof of Theorem 1.1 will show that its statement remains valid even if VRRW starts outside of a trapping graph G', with (i) replaced by event “VRRW never leaves G' after its first visit to G'.”

A graph G is of bounded degree if there exists a constant K such that any vertex of G is incident to at most K edges of the graph. Now we summarize the facts mentioned above.

COROLLARY 1.4. *Consider a locally finite connected graph without loops. If it contains at least one trapping subgraph, then VRRW visits only finitely many vertices with a positive probability. In particular, this holds for:*

(A) *Any graph which does not contain triangles (including Z^d and trees);*
(B) *Any graph of bounded degree;*
(C) *Any graph on which the size of any complete subgraph is uniformly bounded by some number K.*

The next statement describes a.s. behavior of VRRW on some trees.

THEOREM 1.5. *One any tree of a bounded degree, VRRW visits only finitely many sites with probability 1.*

The proof of this theorem and relevant open problems are presented in Section 4.

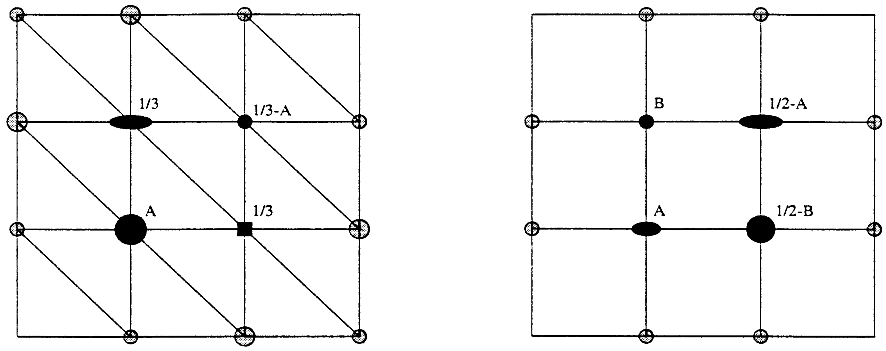

FIG. 2. *General and bipartite localization. Dark circles denote points of V_1, ovals of V_2, a square is V_3 and gray circles are vertices of B. Numbers stand for limiting occupational measures.*

2. Tools. Let ξ_i be iid random variables with $\mathsf{P}\{\xi_i = 1\} = 1 - \mathsf{P}\{\xi_i = 0\} = p$. We start with an elementary fact from large deviation theory [see, e.g., Shiryaev (1989)].

LEMMA 2.1. *For any $0 < p \le a < 1$,*

$$\mathsf{P}\left\{\frac{1}{n}\sum_{i=1}^{n}\xi_i \ge a\right\} \le \exp\{-nH(a, p)\}, \tag{2.1}$$

where

$$H(a, p) = a \log\frac{a}{p} + (1-a)\log\frac{1-a}{1-p} \ge 0. \tag{2.2}$$

Similarly, if $0 < a \le p < 1$ then

$$\mathsf{P}\left\{\frac{1}{n}\sum_{i=1}^{n}\xi_i \le a\right\} \le \exp\{-nH(a, p)\} \tag{2.3}$$

with the same entropy function $H(a, p)$ given by (2.2).

We will be interested in two special cases: when a is just barely larger (resp., smaller) than p and when both a and p are small. Proofs of the following two statements are trivial and they are omitted.

PROPOSITION 2.2. *Let $a = p + \delta$ (resp., $a = p - \delta$) and $\delta > 0$ is small. Then* (2.1) [*resp.*, (2.3)] *holds with*

$$H(a, p) = \frac{\delta^2}{2p(1-p)} + \Theta\left(\frac{\delta^3}{p^2(1-p)^2}\right). \tag{2.4}$$

PROPOSITION 2.3. *Let $a = rp$, $r > 1$ and a and p are small. Then* (2.1) *holds with*

$$H(a, p) = p(r \log r - r + 1) + \Theta(p^2) > 0.$$

We remark that in the proof of the main theorem we will apply these statements not for iid processes, but for those which are stochastically larger (or smaller) than them. Also, to simplify notation, we will write $f = g + o(\triangleleft)$ whenever $f = g + o(g)$.

2.1. *Pólya urn model.* The classical Pólya urn model consists in the following. An urn contains balls of n different colors. At each unit of time a ball is drawn at random and is then replaced together with another ball of the same color. Let $Z(t,i)$ be the number of balls of ith color at time t, and $Z(t) = \sum_i Z(t,i)$ be the total number of balls. Denote the relative distribution of the colors at time t as

$$\bar{\alpha}(t) := \left(\frac{Z(t,1)}{Z(t)}, \frac{Z(t,2)}{Z(t)}, \dots, \frac{Z(t,n)}{Z(t)}\right).$$

LEMMA 2.4. *The vector $\bar{\alpha}(t)$ converges a.s. to some random element in the interior of $(n-1)$-simplex $\Delta \subset \mathsf{R}^n$.*

In fact, even a stronger statement can be made: $\lim_{t\to\infty} \bar{\alpha}(t)$ has a Dirichlet distribution with parameters depending on the initial distribution of the colors [see Pemantle (1988b), Lemma 1] but we will not need it here. We present a very short proof.

SHORT PROOF OF LEMMA 2.4. It is easy to show that, with probability 1, $\lim_{t\to\infty} Z(t,i) = \infty$ for all $i = 1, 2, \dots, n$. Fix any i observe that $\xi_i(t) = \log Z(t) - \log(Z(t,i) - 1)$ constitutes a nonnegative supermartingale with respect to the filtration $\mathscr{F}_t := \sigma(Z(s,i), s \le t, i = 1, 2, \dots, n)$ since

$$\begin{aligned}\mathsf{E}(\xi_i(t+1) - \xi_i(t) \mid \mathscr{F}_t) &= \log\left(1 + \frac{1}{Z(t)}\right) + \frac{Z(t,i)}{Z(t)} \log\left(1 - \frac{1}{Z(t,i)}\right) \\ &\le \log\left(1 + \frac{1}{Z(t)}\right) - \frac{1}{Z(t)} < 0\end{aligned}$$

as soon as $Z(t,i) \ge 2$ (using elementary properties of the logarithm). Therefore, $\xi_i(t)$ must converge a.s. to a random value $\xi_i(\infty) \ge 0$. Let $\bar{\alpha}_i = \exp(-\xi_i(\infty)) \in (0,1]$, then

$$\lim_{t\to\infty} \bar{\alpha}(t) = \bar{\alpha} := (\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n).$$

Since $\sum_{i=1}^n \bar{\alpha}_i = 1$ and all $\bar{\alpha}_i$ are positive, $\bar{\alpha}_i < 1$ for every i. $\square$

3. Proof of Theorem 1.1. Fix small positive numbers ε, $\varepsilon_* \in (0,1)$ and a number $\zeta \in (0,1)$ close to 1, and consider VRRW at the times t_k when the total local time at S is exactly k^m (the constant $m > 1$ will be chosen later). Formally,

$$t_k = \inf\{t > t_{k-1} \colon Z(t,S) \ge k^m\} \tag{3.1}$$

(it is conceivable that $t_k = \infty$). Let $n_i = |V_i|$, $n_S = \sum_{i=1}^n n_i$ be the number of vertices in S, $n_B = |B|$ and $n_O = |\partial G'|$ be the number of points lying outside of G' but connected to G'.

For every $x \in S$ we define the empirical weight $\alpha_x^{(k)} := Z(t_k, x)/k^m$, so that $\sum_{x\in S} \alpha_x^{(k)} \equiv 1$. For $i \in \{1, 2, \dots, n\}$, let $\alpha_i^{(k)} = \sum_{x\in V_i} \alpha_x^{(k)}$ be the empirical weight of the pseudo-vertex V_i. Moreover, we will need "relative-to-V_i" weights. Namely, if $x \in V_i$, then its *relative* weight is $\tilde{\alpha}_x^{(k)} = \alpha_x^{(k)}/\alpha_i^{(k)}$. Naturally, for any i we have $\sum_{x\in V_i} \tilde{\alpha}_x^{(k)} = 1$.

Let $E(k)$ be the event that the following simultaneously happen:

$E_1(k)$: $t_k < \infty$ and VRRW does not visit points outside of G' by time t_k;
$E_2(k)$: $\tilde{\alpha}_x^{(k)} > \varepsilon$ for all $x \in S$;
$E_2'(k)$: $\alpha_i^{(k)} > 1/(n + \varepsilon_*)$ for all $i \in \{1, 2, \dots, n\}$;
$E_3(k)$: $Z(t_k, y) < k^{m\zeta}$ for all $y \in B$;
$E_4(k)$: VRRW behaves "regularly" during the time period $t \in (t_{k-1}, t_k]$.

An exact formulation of $E_4(k)$ will be given later in the proof by equations (3.8), (3.15), (3.17) and (3.19). Also notice that $E_2(k)$ and $E_2'(k)$ combined imply that $\alpha_x^{(k)}$ is bounded below by $\varepsilon_1 := \varepsilon/(n + \varepsilon_*)$.

We will show that $\mathsf{P}(E(k+1) \mid E(k), E(k-1), \dots, E(k_0)) \geq 1 - \gamma(k)$ where the the sequence $\gamma(k)$ is summable given a proper choice of m, ζ, k_0 and a starting configuration on G' which VRRW can achieve with a positive probability. This will imply that with a positive probability, all events $E(k)$, $k \geq k_0$, occur. Then we will prove that on the intersection of all $E(k)$'s the convergences described by Theorem 1.1 indeed take place.

The proof will proceed in eight steps; on each of the steps we will assume that the events described in the previous steps occur (that is, we will consider *conditional* probabilities).

In the first two steps we show that with probability at least $1 - \gamma_1 - \gamma_2$, the time t_{k+1} is finite and VRRW does not visit points outside of G' when $t \in (t_k, t_{k+1}]$ and the number of visits to B during this time period is not too large. Once this is established, the way in which a trapping graph was defined allows us to think of VRRW as a process on a complete graph consisting of n vertices, with some perturbations. Then we obtain that the (conditional) probability of $E_4(k+1)$ is at least $1 - \gamma_3 - \gamma_4 - \gamma_5$, yielding a certain set of inequalities. The latter implies that the number of visits to each of the V_i while $t \in (t_k, t_{k+1}]$ tends to "smooth" the differences between $\alpha_i^{(k)}$'s for different i's. This implies $E_2'(k+1)$ and will be essential for (iii) later. Another implication of $E_4(k+1)$ is that

$$\left|\tilde{\alpha}_x^{(k+1)} - \tilde{\alpha}_x^{(k)}\right| < \frac{\text{const}}{k^{1+\beta}} \tag{3.2}$$

simultaneously for all $x \in S$. hence, if we start from the initial configuration with $\min_{x\in S} \tilde{\alpha}_x^{(k_0)} > 2\varepsilon$ and $k_0 = k_0(\varepsilon)$ being large enough, $E_2(k+1)$ is fulfilled whenever $\bigcap_{l=k_0}^{k} E_2(l)$ occurs. Then, in Step 5, we show that $E_3(k+1)$ occurs with a probability at least $1-\gamma_6$.

We set $\gamma(k)=\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6$ and observe that $\sum_{k>k_0}\gamma(k)<\infty$. As a result, the probability of $\bigcap_{k>k_0} E(k)$ is positive. In the last three steps we show that (ii), (iii) and (iv) take place on this event a.s.

Let us consider VRRW between the moments t_k and t_{k+1} assuming that $E(k_0),\dots,E(k-1)$, $E(k)$ all occur, and let $N=(k+1)^m-k^m=mk^{m-1}+O(k^{m-2})$.

STEP 1. Let $a=2n_Bk^{-m(1-\zeta)}/\varepsilon_1=o(1)$ and consider VRRW during the first $aN\equiv 2n_Bmk^{m\zeta-1}/\varepsilon_1+o(\triangleleft)$ visits to B after time t_k. Let t' be the time of the last of these visits (set it to infinity if this situation is never reached). Suppose that

$$m\zeta-1>0. \tag{3.3}$$

Then the probability that VRRW does not leave G' during the time period $(t_k,t']$ is greater than $1-\gamma_1$, where

$$\gamma_1=1-\left(1-\frac{n_O}{\varepsilon_1k^m}\right)^{aN}=\frac{2n_Bn_Om}{\varepsilon_1^2k^{1+m(1-\zeta)}}+o(\triangleleft). \tag{3.4}$$

STEP 2. We state that the event $\{t'\ge t_{k+1}\}$ has probability close to 1. To show this, notice that the total number of visits to S during the time period $(t_k,\min(t',t_{k+1})]$ does not exceed N; therefore,

$$\begin{aligned}\mathsf{P}(\{t'<t_{k+1}\})\le \mathsf{P}\,&(\text{the number of visits to } B \text{ exceeds } aN\\ &\quad\text{while } S \text{ has been visited } N \text{ times}).\end{aligned}$$

However, each such visit has a probability smaller than

$$p:=\frac{n_Bk^{m\zeta}+N}{\varepsilon_1k^m}=\frac{a}{2}+o(\triangleleft) \tag{3.5}$$

under the assumption that

$$m\zeta>m-1. \tag{3.6}$$

Consider a Bernoulli sequence ξ_i, $i=1,2,\dots,N$ where the probability of a success is p. The number of successes in this sequence is stochastically larger than the number of visits from S to B between times t_k and $t_{k+1}=t_k+N$. Therefore, by Proposition 2.3 with $r=2$ and p and a being indeed small,

$$\mathsf{P}(t'<t_{k+1})<\exp(-C_1k^{m\zeta-1}+o(\triangleleft))=:\gamma_2, \tag{3.7}$$

where $C_1=n_Bm(2\log 2-1)/\varepsilon_1\approx 0.386\,n_Bm/\varepsilon_1$. Consequently, t_{k+1} is finite with probability larger than $1-\gamma_2$ and by that time,

$$\begin{gathered}\text{the number of visits to } B \text{ does not exceed } N_B:=Ck^{m\zeta-1},\\ C=2n_Bm/\varepsilon_1,\end{gathered} \tag{3.8}$$

and γ_2 is small since (3.3) is fulfilled.

STEP 3. Assume now that the events mentioned in the two previous steps occurred, namely $t_{k+1} < \infty$, VRRW does not leave G' and the number of visits to B is at most $N_B = o(N)$ while $t \in (t_k, t_{k+1}]$. Consequently, VRRW on G' can be "almost" coupled with a VRRW on a complete graph of n vertices. Indeed, during the $N - o(N)$ steps after visiting V_i, the set V_j, $j \neq i$, is visited with probability proportional to a total local time at V_j. This justifies the name "pseudo-vertex" used to address V_i's.

Let N_i denote the number of visits to V_i during the time period $(t_k, t_{k+1}]$. Observe that $N_i \le (N + N_B)/2$. Hence, the number of visits to $S \backslash V_i$ lies between $(N - N_B)/2 = N/2 - o(\triangleleft) > N/4$ and N. Every time VRRW is at $S \backslash V_i$, it can go either to $G' \backslash V_i$ or to V_i. The probability of the latter event is at least

$$p = \frac{\varepsilon_1 k^m}{k^m + N + n_B k^{m\zeta} + N_B} = \varepsilon_1 - o(1).$$

Now we want to apply Proposition 2.2. Choose $\delta \in (0, \varepsilon_1/2)$ so small, that the expression in the r.h.s. of (2.4) is a strictly positive and independent of k (one can do this because $p(1-p) \ge \varepsilon_1(1-\varepsilon_1) - o(1)$) and set $a := p - \delta$. After making a comparison with the iid sequence of Bernoulli variables as done in Step 2, we conclude that the probability that during the first $N/4$ departures from $S \backslash V_i$ the set V_i is visited less than $N/4 \times (\varepsilon_1 - \delta)/2 (\ge \varepsilon_1 N/16)$ times is smaller than

$$\gamma^* = \exp\left(-\text{const } \times \frac{N}{4} + o(\triangleleft)\right). \tag{3.9}$$

Taking into account that the probability of the intersection of n (not necessarily independent) events, each having probability at least $1 - \gamma^*$ cannot be less than $1 - n\gamma^*$, we see that this value is a lower bound for the probability of the event

$$\{\varepsilon_1 N/16 \le N_i \le N \text{ for all } i\}. \tag{3.10}$$

From now on we will condition on (3.10). Let m_{ij} denote the number of visits from V_i to V_j, $m_i = \sum_{j \neq i} m_{ij}$ be the total number of departures from V_i to $S \backslash V_i$ and $m_i' = \sum_{j \neq i} m_{ji}$ be that of arrivals. Clearly, $|m_i - m_i'| \le N_B$ and $|m_i - N_i| \le N_B$.

Our goal is to estimate m_{ij}/m_i for a fixed j. Every time a particle departs from V_i and does not go to B, it goes to V_j with probability of order

$$p = \frac{\alpha_j^{(k)}}{\sum_{l \neq i} \alpha_l^{(k)}} + O(k^{-1}) = \frac{\alpha_j^{(k)}}{1 - \alpha_i^{(k)}} + O(k^{-1}).$$

Apply Proposition 2.2 with $a = p + k^{-\beta}$ where

$$\beta \in \left(0, \min\left\{\frac{m-1}{2}, m(1-\zeta), 1\right\}\right) \tag{3.11}$$

so that $\delta = k^{-\beta} + o(\triangleleft)$. We obtain that with probability greater than $1 - \gamma^{**}$ where

$$\gamma^{**} = \exp\left(-\frac{k^{-2\beta}}{2p(1-p)} m_i\right) + o(\triangleleft) \le \exp(-m\varepsilon_1 k^{m-1-2\beta}/8) + o(\triangleleft) \tag{3.12}$$

[we used the lower bound for $N_i = m_i + O(N_B)$ and the fact that $2p(1-p)$ is at most 1/2] the following holds:

$$\frac{m_{ij}}{m_i} \le \frac{\alpha_j^{(k)}}{1 - \alpha_i^{(k)}} + k^{-\beta}. \tag{3.13}$$

Repeating the same arguments for $a := p - k^{-\beta}$ implies the probability that

$$\frac{m_{ij}}{m_i} \ge \frac{\alpha_j^{(k)}}{1 - \alpha_i^{(k)}} - k^{-\beta} \tag{3.14}$$

is also at least $1 - \gamma^{**}$. Consequently, we have

$$\left| \frac{m_{ij}}{m_i} - \frac{\alpha_j^{(k)}}{1 - \alpha_i^{(k)}} \right| < k^{-\beta} \quad \text{for all } i, j \in \{1, 2, \dots, n\} \text{ such that } j \ne i \tag{3.15}$$

with probability exceeding $1 - \gamma_2 - \gamma_3 - \gamma_4$ where

$$\gamma_4 = 2n(n-1)\gamma^{**} + n\gamma^{*} = 2n(n-1)\exp(-m\varepsilon_1 k^{m-1-2\beta}/8) + o(\triangleleft) \tag{3.16}$$

which decays to zero since $2\beta < m - 1$.

STEP 4. We start with the following useful statement, the proof of which is given in Section 3.2.

LEMMA 3.1. *Assume that* (3.15) *takes place and* $0 < \beta < m(1 - \zeta)$. *Then there exists a* (*possibly negative*) *constant* C' *depending only on* $n, n_B, \varepsilon, \varepsilon_1$ *and* m *such that for any* $j \in \{1, 2, \dots, n\}$ *and large* k,

$$\frac{N_j}{N} \ge \frac{\alpha_j^{(k)}(1 - \alpha_j^{(k)})}{1 - 1/n} + C' k^{-\beta}. \tag{3.17}$$

Notice that on any segment, the function $f(\alpha) = \alpha(1-\alpha)/(1-1/n)$ achieves its minimum at one of the endpoints. So, together with

$$\alpha_j^{(k)} = 1 - \sum_{j' \ne j} \alpha_{j'}^{(k)} \le \frac{1 + \varepsilon_*}{n + \varepsilon_*},$$

Lemma 3.1 yields

$$N_i \ge N/(n + \varepsilon_*), \tag{3.18}$$

once k is large enough. This, in turn, implies $E_2'(k+1)$. Note that since $N_B/N = o(k^{-\beta})$, we also obtain that $m_i' \ge N/(n + \varepsilon_*) + o(\triangleleft)$.

Fix $i \in \{1, 2, \dots, n\}$ and $x \in V_i$, and consider all the moments when VRRW goes from $S \backslash V_i$ to V_i. Taking into account that $1/\alpha_i^{(k)} \le n + \varepsilon_*$ and $\tilde{\alpha}_x^{(k)} < 1$, we obtain that at these times the probability p to go to x is bounded from below by

$$\frac{\alpha_x^{(k)} k^m}{\alpha_i^{(k)} k^m + N} \ge \tilde{\alpha}_x^{(k)} - \frac{(n + \varepsilon_*)m}{k} + o(k^{-1})$$

and from above by

$$\frac{\alpha_x^{(k)} k^m + N}{\alpha_i^{(k)} k^m} \le \tilde{\alpha}_x^{(k)} - \frac{(n + \varepsilon_*)m}{k} + o(k^{-1}).$$

Now we can carry through the same arguments that we used in the previous step: we apply Proposition 2.2 with $\delta = k^{-\beta} + ((n+\varepsilon_*)m/k) = k^{-\beta} + o(\triangleleft)$ twice [β is the same as before and in both cases $p \in (\varepsilon, 1 - \varepsilon)$]. Consequently, the number N_x of visits to vertex x satisfies

$$\begin{aligned} m_i' \times (\tilde{\alpha}_x^{(k)} - \delta) \le N_x &\le m_i' \times (\tilde{\alpha}_x^{(k)} + \delta) + N_B \\ &\le m_i' \times (\tilde{\alpha}_x^{(k)} + \delta + (n + \varepsilon_*) N_B / N) \\ &= m_i' \times (\tilde{\alpha}_x^{(k)} + \delta + o(\delta)) \end{aligned} \tag{3.19}$$

with probability at least $1 - 2\gamma^{***}$ where

$$\gamma^{***} = \exp\biggl(-\frac{k^{-2\beta}}{2p(1-p)} m_i'\biggr) + o(\triangleleft) \le \exp(-2mk^{m-1-2\beta}/(n+\varepsilon_*)) + o(\triangleleft). \tag{3.20}$$

Therefore, with probability $1 - \gamma_5$,

$$\gamma_5 := 2\gamma^{***} \times n_S, \tag{3.21}$$

the inequalities in (3.19) hold simultaneously for all $x \in S$ (with i being a function of x—the index of the pseudo-vertex to which x belongs).

Let us show that this implies (3.2). Indeed, for $x \in V_i$,

$$\begin{aligned} \tilde{\alpha}_x^{(k+1)} &= \frac{\alpha_x^{(k)} k^m + N_x}{\alpha_i^{(k)} k^m + N_i} \le \frac{\tilde{\alpha}_x^{(k)} \alpha_i^{(k)} k^m + N_i(\tilde{\alpha}_x^{(k)} + \delta) + N_B}{\alpha_i^{(k)} k^m + N_i} \\ &= \tilde{\alpha}_x^{(k)} + \frac{N_i \delta + N_B}{\alpha_i^{(k)} k^m + N_i} \le \tilde{\alpha}_x^{(k)} + m(n + \varepsilon_*) k^{-1-\beta} + o(k^{-1-\beta}) \end{aligned}$$

since $m - 1 - \beta > m\zeta - 1$. Similarly,

$$\begin{aligned} \tilde{\alpha}_x^{(k+1)} &\ge \frac{\tilde{\alpha}_x^{(k)} \alpha_i^{(k)} k^m + (N_i - N_B)(\tilde{\alpha}_x^{(k)} - \delta)}{\alpha_i^{(k)} k^m + N_i} \\ &\ge \tilde{\alpha}_x^{(k)} - m(n + \varepsilon_*) k^{-1-\beta} + o(k^{-1-\beta}). \end{aligned}$$

Consequently, the bounds given by (3.2) take place.

STEP 5. Now we want to obtain that not only the number of visits to B is smaller than N_B but, in fact, with a probability close to 1, for any $y \in B$ the number of visits N_y to the vertex y does not exceed $(r')^{-1} m\zeta k^{m\zeta-1}$ for some constant $r' > 1$. This will automatically imply $E_3(k+1)$ once $E_3(k)$ holds.

Depending on the graph, there are at most two ways the vertex y can be visited by VRRW: it can be visited either from S or from B (the latter cannot take place on graphs without triangles). Regardless, we will show that the number of visits of the second type is $o(k^{m\zeta-1})$ and therefore negligible.

The probability p to jump to y from B is at most $k^{m\zeta}/(\varepsilon_1 k^m) + o(\triangleleft)$. Setting $r = 2$ and $a = rp$ in Proposition 2.3 yields that the probability of the event "VRRW comes to y *from* B more than $aN_B = o(k^{m\zeta-1})$ times" is smaller than

$$\gamma_* = \exp(-pN_B(\log 4 - 1)) + o(\triangleleft).$$

Suppose that

$$\phi := 2m\zeta - m - 1 > 0. \tag{3.22}$$

Then pN_B is proportional to k^ϕ and $\gamma_* = \gamma_*(k)$ goes to zero as fast as $\exp(-ck^\phi)$, for some constant $c > 0$, whence $\sum_k \gamma_*(k) < \infty$.

Let us concentrate now on the visits from S to y. The probability p to go to y is bounded above by

$$\frac{k^{m\zeta}}{((n-1)/(n+\varepsilon_*))k^m} + o(\triangleleft) = \frac{n+\varepsilon_*}{n-1} k^{-m(1-\zeta)} + o(\triangleleft).$$

Recall the definition of trapping subgraph. One of the conditions on it required that there be at most $n-1$ different pseudo-vertices V_i's connected to y (at least by one edge) and there is some x' belonging to one of them such that $x' \not\sim y$. Therefore, according to (3.18), the time spent on those vertices of S from which VRRW is allowed to go directly to y is at most

$$N_{\sim y} = N\left(1 - \frac{1}{n+\varepsilon_*}\right) - N_{x'}.$$

Let V_i be the pseudo-vertex to which x' belongs. By (3.18) and (3.19),

$$\frac{N_{x'}}{N} = \frac{N_{x'}}{N_i} \times \frac{N_i}{N} \ge \left(\tilde{\alpha}_{x'}^{(k)} - \delta\right)\frac{1}{n+\varepsilon_*} \ge \frac{\varepsilon/2}{n+\varepsilon_*}$$

and consequently,

$$N_{\sim y} \le \frac{n-1+\varepsilon_* - \varepsilon/2}{n+\varepsilon_*} N.$$

Next,

$$pN_{\sim y} = \left\{\frac{1}{\zeta}\left(1 - \frac{\varepsilon/2 - \varepsilon_*}{n-1}\right)\right\} m\zeta k^{m\zeta-1} + o(\triangleleft).$$

If

$$\varepsilon_* < \varepsilon/2 \tag{3.23}$$

and

$$1 - \frac{\varepsilon/2 - \varepsilon_*}{n-1} < \zeta < 1, \tag{3.24}$$

then the expression in the curly brackets is smaller than 1. As a result, we can pick some $r' > 1$ such that $r'\{1 - (\varepsilon/2 - \varepsilon_*)/(n-1)\}/\zeta < 1$. We apply Proposition 2.3 and obtain that with probability $1 - \gamma_{**}$,

$$\gamma_{**} = \exp(-C_2 k^{m\zeta - 1} + o(\triangleleft)), \tag{3.25}$$

where $C_2 = C_2(r', \varepsilon, \varepsilon_*, m, n) > 0$, the number of visits to y from S does not exceed $r' pN_{\sim y}$. By the same arguments, this [the statement that the number of visits to y is smaller than $(r')^{-1} m\zeta k^{m\zeta-1}$] holds simultaneously for all $y \in B$ with probability at least $1 - \gamma_6$, where

$$\gamma_6 = (\gamma_* + \gamma_{**}) n_B \tag{3.26}$$

is small as soon as (3.3) is fulfilled. Therefore, $E_3(k+1)$ occurs with probability close to 1.

Before proceeding to the next step, it is natural to ask whether the constants $m, \zeta, \varepsilon, \varepsilon_*, \beta$ and r' can simultaneously satisfy conditions (3.3), (3.6), (3.11), (3.23), (3.22) and (3.24). The answer is positive. Indeed, for any positive ε smaller than $1/n$, we can take $\varepsilon_* = \varepsilon/4$, $m = 2$, $r' = 1 + \varepsilon/(4n-4)$, $\zeta = 1 - (\varepsilon/32(n-1)^2)$ and $\beta = (\varepsilon/32(n-1)^2)$. It is easy to check that all the conditions are fulfilled with this set of parameters.

STEP 6. In this and the following steps we condition on the event

$$\text{Loc} \geq \bigcap_{k > k_0} E(k).$$

As we know by now, $\mathsf{P}(\text{Loc}) \geq \prod_k (1 - \gamma(k)) > 0$ since $\sum \gamma(k) < \infty$. The event Loc automatically implies part (i) of Theorem 1.1.

From (3.17) we obtain that

$$\alpha_j^{k+1} = \frac{\alpha_j^{(k)} k^m + N(j)}{k^m + N} \geq f\left(\alpha_j^{(k)}\right) + (mC' - 1)k^{-1-\beta}, \tag{3.27}$$

where $f(\alpha) = \alpha[1 + m(1 - n\alpha)/((n-1)k)]$. Since $\sum_{i=1}^n \alpha_i^{(k)} \equiv 1$, the sequence of random variables $\{\eta_k\}_k$ defined by

$$\eta_k := 1 - n \min_{i \in \{1,2,\ldots,n_1\}} \alpha_i^{(k)}$$

in nonnegative. The observation that the function $f(\alpha)$ is increasing for all $\alpha \in (0, 1)$ when $k > 3m$ and inequality (3.27) yield the following formula:

$$\eta_{k+1} \leq \eta_k \left(1 - \frac{m}{k}\right) + C'' k^{-1-\beta} \tag{3.28}$$

with the constant C'' being independent of k and η_k.

PROPOSITION 3.2. *Let a nonnegative sequence* $\{\eta_k\}_{k=1}^{\infty}$ *satisfy condition* (3.28) *with* $0 < \beta < m$. *Then* $\eta_k \to 0$ *as* $k \to \infty$.

PROOF. The case $C'' \le 0$ is straightforward, so we will assume that $C'' > 0$. Let $\mu_k = \eta_k - hk^{-\beta}$ where the constant $h > 0$ will be chosen later. Then (3.28) yields

$$\mu_{k+1} \le \left(1 - \frac{m}{k}\right)\mu_k - \frac{(m-\beta)h - C''}{k^{1+\beta}} + o(k^{-1-\beta}) \le \left(1 - \frac{m}{k}\right)\mu_k$$

as soon as $h > C''/(m-\beta)$ and k is large. If all μ_k are positive, then the equation above implies that $\lim_{k\to\infty} \mu_k = 0$. On the other hand, if $\mu_l \le 0$ for some l then it follows that $\mu_k \le 0$ for all $k > l$. In any case, since $hk^{-\beta} \to 0$ and $\eta_k \ge 0$, we have

$$\liminf_{k\to\infty} \eta_k \ge 0$$

and

$$\limsup_{k\to\infty} \eta_k = \limsup_{k\to\infty} (\mu_k + hk^{-\beta}) = \limsup \mu_k \le 0$$

and the proposition is proved. □

Therefore, on Loc we have $\lim_{k\to\infty} \alpha_i^{(k)} = 1/n$ for every $i \in \{1, 2, \dots, n\}$. On the other hand, (3.2) implies convergence of $\tilde{\alpha}_x^{(k)} = \alpha_x^{(k)}/\alpha_i^{(k)}$ for all $x \in V_i$. Combining this with $Z(t_k, B) = o(Z(t_k, S))$ and $(t_{k+1} - t_k)/t_k \to 0$ as $k \to \infty$, we obtain parts (ii) and (iii) of the theorem.

STEP 7. The only statement left unproved thus far is part (iv).

For its proof, pick a very small $\nu > 0$. From Step 6 and (3.17) and (3.19) it follows that on Loc there exists some $k_1 \ge k_0$ such that for any $k \ge k_1$,

$$\left| \frac{Z(t_{k+1}, x) - Z(t_k, x)}{k^{m+1} - k^m} - \alpha_x \right| < \frac{\nu}{ns} \quad \text{for every } x \in S \tag{3.29}$$

and for any $t \ge k_1^m$

$$\begin{aligned} &\frac{Z(t, x)}{Z(t, S)} \ge \nu \quad \text{for every } x \in S, \\ &\left| \frac{Z(t, V_i)}{Z(t, S)} - \frac{1}{n} \right| < \nu \quad \text{for every } i \in \{1, 2, \dots, n\} \end{aligned} \tag{3.30}$$

and

$$Z(t, y) < 2(Z(t, S))^{\zeta} \quad \text{for every } y \in B.$$

Consider a new sequence of times $\{s_j\}$ where

$$s_j = \inf\{t > s_{j-1}\colon Z(t, S) \ge (1+\nu)^j\}.$$

Since $\{s_j\}$ grows much faster than the sequence $\{t_k\}$, one can obtain from (3.34) that

$$\left|\frac{Z(S_{j+1},x)-Z(S_j,x)}{(1+\nu)^{j+1}-(1+\nu)^j}-\alpha_x\right|<\frac{2\nu}{n_S} \tag{3.31}$$

for all $x \in S$ and all j greater than some $j_1 > m\log_{1+\nu} k_1$.

Fix some $y \in B$ and let $N_{\sim y} = N_{\sim y}(j)$ be the number of visits to the points $x \in S$ from which VRRW can go to y between the moments s_j and s_{j+1}. Also let N'_y be the number of visits to y from S, N''_y be the number of visits to y from B, and $N_y = N_y(j) = N'_y + N''_y$ be the total number of visits to y during this time interval. Inequality (3.31) yields

$$(\alpha_{\sim y}-2\nu)N \le N_{\sim y} \le (\alpha_{\sim y}+2\nu)N, \tag{3.32}$$

where $N = N(j) = \nu(1+\nu)^j$ is the total number of visits to S during this period (maybe plus or minus 1) and

$$\alpha_{\sim y} = \sum_{x\in S,\, x\sim y}\alpha_x.$$

The probability that VRRW will jump to y from some $x \in V_i$ such that $x \sim y$ when $t \in (s_j, s_{j+1}]$ is

$$\begin{aligned}
&\frac{Z(t,y)}{Z(t,S\backslash V_i)+\sum Z(t,y')}\\
&\qquad\ge \frac{Z(s_j,y)}{(1-(1/n)+\nu)Z(s_{j+1},S)+2n_B(Z(s_{j+1},S))^{\zeta}} \qquad (3.33)\\
&\qquad\ge \frac{L(j)}{(1-1/n+2\nu)(1+\nu)^{j+1}} =: p = p(j)
\end{aligned}$$

as soon as $Z(s_j, S)$ is large enough (the sum on the l.h.s. ranges over all $y' \in B$ such that $y' \sim x$). Here we used (3.30) and set $L(j) := Z(s_j, y)$ for simplicity.

Consider an iid sequence of Bernoulli zero–one random variables $\xi_1, \xi_2, \ldots, \xi_{N_{\sim y}}$ with $\mathsf{P}(\xi_i = 1) = p$. Then N'_y is stochastically larger than $\Xi := \xi_1 + \cdots + \xi_{N_{\sim y}}$ (and $N_y \ge N'_y$). Consequently, from Chebyshev's inequality we obtain that

$$\mathsf{P}(A(j)^c) \le \frac{1}{\nu^2 pN_{\sim y}}, \tag{3.34}$$

where the event $A(j)$ is

$$A(j) := \{N_y - pN_{\sim y} > -\nu pN_{\sim y}\}.$$

Also observe that (3.32) implies

$$\left(\frac{n}{n-1}\alpha_{\sim y}-8\nu\right)\frac{\nu}{1+\nu}L(j) \le pN_{\sim y} \le C_2L(j) \tag{3.35}$$

by using the following obvious inequalities:

$$1 < \frac{n}{n-1} \le 2, \quad \frac{1}{1+x} \ge 1 - x, \quad \frac{n}{n-1}\alpha_{\sim y} < 1,$$

where C_2 is a constant not depending on j.

Let us show that $L(j) \to \infty$ as $j \to \infty$. If the contrary were true, there would exist some j_2 and $\overline{L}$ such that $L(j) \equiv \overline{L}$ as soon as $j \ge j_2$. The probability of the event $B(j) = \{$VRRW does not jump to y at all when $t \in (s_j, s_{j+1}]\}$ is at most $(1-p)^{N_{\sim y}}$ and, because of (3.35), it satisfies

$$\limsup_{j\to\infty} \mathsf{P}(B(j)) \le \limsup_{j\to\infty} \exp(-pN_{\sim y}) \le C_3 < 1$$

for some constant $C_3 = C_3(n, \alpha_{\sim y}, \nu, \overline{L})$. Consequently, $\mathsf{P}(\cap_{j>j_2} B(j)) = 0$ and we obtain a contradiction with the assumption that $L(j)$ is bounded. In particular, together with (3.35) this implies that the r.h.s. of (3.34) goes to zero and the events $A(j)$ occur infinitely often.

Let us introduce two recursively defined integer-valued sequences of stopping times:

$$j'_k = \inf\{j > j''_{k-1}\colon A(j-1) \text{ occurs}\},$$
$$j''_k = \inf\{j > j'_k\colon A(j-1)^c \text{ occurs}\},$$

where $k = 0, 1, 2\dots$ with the exception of j'_0 which is the smallest $j \ge j_1$ for which (3.33) holds. So,

$$j_1 \le j'_0 < j''_0 < j'_1 < j''_1 < j'_2 < j''_2 < \cdots.$$

Notice, that j'_k is finite as soon as j''_{k-1} is finite [since the $A(j)$'s occur infinitely often]. We cannot make a similar statement about j''_k; in fact, we will show that a.s. there exists a number $\bar{k}$ such that $t''_{\bar{k}} = \infty$. To prove this, observe that (3.35) yields

$$L(j+1) \ge L(j) \times \left(1 + (1-\nu)\left(\frac{n}{n-1}\alpha_{\sim y} - 8\nu\right)\frac{\nu}{1+\nu}\right) = L(j) \times R(\nu)$$

whenever the $A(j)$ takes place. Suppose that ν is so small that $R = R(\nu) > 1$. Then, by the construction of the sequences $\{j'_k\}$ and $\{j''_k\}$, there exists a constant $C_4 > 0$ such that $L(j'_k) \ge C_4 R^k$ for all $k \ge 0$. Let $C(k) = \{t''_k < \infty\}$ be the event that t''_k is finite. Then, according to (3.34),

$$\mathsf{P}(C(k)) = \mathsf{P}\left(\bigcup_{l=0}^{\infty} A(k+l)^c\right) \le \sum_{l=0}^{\infty} \frac{1}{\nu^2 C_4 R^{k+l}} = \frac{R-1}{\nu^2 C_4 R^{k-1}},$$

whence $\sum_k \mathsf{P}(C(k)) < \infty$. We remark that, in fact, we should be considering $\mathsf{P}(A(j)^c \mid \text{Loc})$ rather than the probability mentioned in (3.34) which conditions only *on part* of Loc up to a certain moment of time. However, since for any two events A and B we have $\mathsf{P}(A^c \mid B) \le \mathsf{P}(A^c)/\mathsf{P}(B)$ and the probability of any subset of Loc is uniformly bounded form zero by $\mathsf{P}\,(\text{Loc}) > 0$, the sum above must be corrected by a multiplicative constant smaller than $\mathsf{P}(\text{Loc})^{-1}$, so it will remain bounded. Therefore, $\mathsf{P}(C(k)$ occurs infinitely often$) = 0$ and for some $\bar{k}$ we have $t''_{\bar{k}} = \infty$. On this event, $L(j) \geq C_4 R^{\bar{k}+j-j'_{\bar{k}}}$ and hence

$$\liminf_{t\to\infty} \frac{\log Z(t, y)}{\log t} \geq \liminf_{j\to\infty} \frac{\log L(j-1)}{\log(1+\nu)^j} \geq \frac{\log R(\nu)}{\log(1+\nu)}. \tag{3.36}$$

Since $\nu > 0$ can be chosen arbitrarily small, we let $\nu \to 0$ and obtain

$$\liminf_{t\to\infty} \frac{\log Z(t, y)}{\log t} \geq \lim_{\nu\to 0} \frac{\log R(\nu)}{\log(1+\nu)} = \frac{n}{n-1}\alpha_{\sim y}.$$

STEP 8. To complete the proof, it suffices to show that for any α_* such that

$$\frac{n}{n-1}\alpha_{\sim y} < \alpha_* < \zeta \tag{3.37}$$

and any sufficiently small $\nu > 0$ such that $\alpha_*(1+\nu) < \zeta$, we have

$$\limsup_{t\to\infty} \frac{\log Z(t, y)}{\log t} \leq \alpha_*(1+\nu). \tag{3.38}$$

We will use similar arguments as in Step 7; however, here we also have to estimate the number of visits to y from B. According to (3.30), the probability to go to y at time $t > s_j$ is at most

$$p_t := \frac{Z(t, y)}{(1-(1/n)-\nu)(1+\nu)^j} = \frac{Z(t, y)}{\Phi}$$

$[\Phi = \Phi(j) = (1 - 1/n - \nu)(1+\nu)^j]$ whenever $X_{t-1} \in S$ and no larger than

$$\frac{Z(t, y)}{\nu(1+\nu)^j} = C_5 p_t$$

when $X_{t-1} \in B$. Besides, the total number of visits to B between times s_j and s_{j+1} does not exceed $2(1+\nu)^{(j+1)\zeta}$. As soon as ν is small, C_5 is a constant larger than 1 and we notice that if, instead of being at B, X_{t-1} were at S consecutively $2C_5$ times, the probability of visiting y would be larger. Therefore, $Z(s_{j+1}, y)$ is stochastically smaller than a random variable $\psi(N_*)$, where

$$N_* := (\alpha_{\sim y} + 3\nu)N \geq N_{\sim y} + 2C_5 \times 2(1+\nu)^{(j+1)\zeta} = N_{\sim y} + o(N_{\sim y})$$

and $\psi(t)$ is a Markov process defined by

$$\mathsf{P}(\psi(t+1) = k+1 \mid \psi(t) = k) = 1 - \mathsf{P}(\psi(t+1) = k \mid \psi(t) = k) = \frac{\psi(t)}{\Phi(j)},$$

$$\psi(0) := Z(s_j, y) \equiv L(j).$$

Let $T_i = \inf\{t\, \psi(t) = \psi(0) + i\}$ and $\Delta_i = T_{i+1} - T_i$, $i = 0, 1, 2, \ldots$. It is easy to see that the Δ_i's constitute a sequence of independent random variables and for each i, Δ_i has a geometric distribution with parameter

$$p_i = \frac{L(j) + i}{\Phi}$$

so that $\mathsf{P}(\Delta_i = k) = p_i(1-p_i)^{k-1}$, $k = 1, 2, \ldots$.

Let $W = \nu\alpha_* L(j)$. The event $B(j) := \{\psi(N_*) \ge L(j) + W\}$ equals the event $\{\Psi(W) \le N_*\}$ where

$$\Psi(W) = \sum_{i=0}^{W-1} \Delta_i.$$

Hence, by Chebyshev's inequality,

$$\begin{aligned} \mathsf{P}(B(j)) = \mathsf{P}(\Psi(W) \le N_*) &\le \mathsf{P}(|\Psi(W) - \mathsf{E}\Psi(W)| \ge \mathsf{E}\Psi(W) - N_*) \\ &\le \frac{\mathsf{Var}(\Psi(W))}{(\mathsf{E}\Psi(W) - N_*)^2}. \end{aligned} \tag{3.39}$$

Notice that

$$\begin{aligned} \mathsf{E}\Psi(W) &= \sum_{i=0}^{W-1} \frac{1}{p_i} = \Phi \sum_{i=0}^{W-1} \frac{1}{L(j)+i} \ge \Phi \int_{L(j)}^{L(j)+W} \frac{1}{x}\,dx \\ &= \Phi \log\left(1 + \frac{W}{L(j)}\right) \ge \Phi(\nu\alpha_* - \nu^2) \end{aligned}$$

and

$$\begin{aligned} \mathsf{Var}(\Psi(W)) &= \sum_{i=0}^{W-1} \frac{1-p_i}{p_i^2} \le \sum_{i=0}^{W-1} \frac{1}{p_i^2} \le \Phi^2 \int_{L(j)-1}^{L(j)+W-1} \frac{1}{x^2}\,dx \\ &= \Phi^2 \frac{W}{(L(j)-1)(L(j)+W-1)} \le \Phi^2 \frac{W}{L(j)^2} = \frac{\nu\alpha_*\Phi^2}{L(j)} \end{aligned}$$

for large j. Furthermore,

$$\begin{aligned} \mathsf{E}\Psi(W) - N_* &\ge (1 - 1/n)\nu(1+\nu)^j \left[\alpha_* - \nu - \frac{n}{n-1}\nu\alpha_* - \frac{n}{n-1}(\alpha_{\sim y} + 3\nu)\right] \\ &\ge \frac{1}{2}\nu(1+\nu)^j \left[\alpha_* - \frac{n}{n-1}(\alpha_{\sim y} - 9\nu)\right] = C_6(\nu)(1+\nu)^j, \end{aligned}$$

where $C_6 = C_6(\nu)$ is a constant which is positive as soon as $9\nu < \alpha_* - (n/(n-1))\alpha_{\sim y}$. Consequently, (3.39) implies

$$\mathsf{P}(B(j)) \le \frac{C_7}{L(j)}. \tag{3.40}$$

However, from Step 7 we know that there exists $a > 1$ such that $L(j+1) \ge aL(j)$ and therefore $\sum_j \mathsf{P}(B(j)) < \infty$. Thus, the Borel–Cantelli lemma yields that the $B(j)$'s occur only finitely many times, so that $L(j+1) \ge L(j) + W \ge L(j)(1 + \nu\alpha_*)$ for large j. Hence,

$$\limsup_{t\to\infty} \frac{\log Z(t,y)}{\log t} \le \limsup_{j\to\infty} \frac{\log L(j)}{\log(1+\nu)^j} \le \frac{\log(1+\nu\alpha_*)}{\log(1+\nu)} \le \alpha_*(1+\nu)$$

as soon as $\nu < 1$. Once again, notice that in (3.40) we ignored conditioning on a "future" part of Loc; however, we were allowed to do so since $\alpha_*(1+\nu) < \zeta$

implies $B(j)^c \subseteq$ Loc and therefore $\mathsf{P}(B(j) \mid \mathrm{Loc}) = 1 - \mathsf{P}(B(j)^c \mid \mathrm{Loc}) = 1 - \mathsf{P}(B(j)^c)/\mathsf{P}(\mathrm{Loc}) \le (1 - \mathsf{P}(B(j)^c))/\mathsf{P}(\mathrm{Loc}) = \mathsf{P}(B)/\mathsf{P}(\mathrm{Loc})$. Consequently, (3.40) should be corrected by a multiplicative constant which is at most $\mathsf{P}(\mathrm{Loc})^{-1} < \infty$. Therefore, (3.38) is established and the proof is complete. □

REMARK 5. The proof of Theorem 1.2 shows that by choosing a proper initial configuration, the limiting distribution of $\{\alpha_x\}$ can be arbitrary close to any element of the interior of the $|S| - n$-dimensional set $\Delta_{n_1} \times \Delta_{n_2} \times \cdots \times \Delta_{n_n}$ where Δ_{n_i} is a $n_i - 1$-simplex. Therefore, the set of all possible limiting weights is dense on this set.

3.1. *A core and a shell localization of VRRW with an irregular vertex.* Here we will derive an implication of the proof of Theorem 1.1 that is essential for the proof of Theorem 1.5. Consider "a core and a shell" configuration described in Corollary 1.2. Suppose that there exist an extra point v^* lying outside of the graph G and connected to a point v, "the core," only. Recall that $Z(t, x)$ is a local time at vertex $x \in G$. At the same time, let $Z(t, v^*)$ be an arbitrary sequence with $Z(t, v^*) \ge A$ for all $t \ge 0$ (thus, v^* is an "irregular" vertex). Let *VRRW** be a reinforced random walk on $G \cup \{v^*\}$ obeying the law (1.1) with $X(0) = v$.

Denote by $\varphi(A, \{Z(t, v^*)\})$ the probability never to leave $G^* := G' \cup \{v^*\}$ for a particular sequence $\{Z(t, v^*)\}_{t=0}^{\infty}$ and let $\varphi(A) = \inf \varphi(A, \{Z(t, v^*)\})$ where the infimum is taken over all sequences such that $Z(t, v^*) \ge A$. Then the following statement holds.

LEMMA 3.3. *For VRRW* described above,*

$$\lim_{A \to \infty} \varphi(A) = 1. \tag{3.41}$$

PROOF. First, we will explain how Corollary 1.2 (the localization of a regular VRRW) can be proved directly using the ideas from the proof of Theorem 1.1 but not as its corollary. Second, we will modify the arguments to accommodate for the presence of the irregular point.

Recall that the numbers of vertices in B and $\partial G'$ are n_B and n_O, respectively, and consider a regular VRRW on G starting at v. Fix some $0 < \varepsilon < 1/(2n_2)$ and take the "snapshots" of VRRW at the moments t_k when the joint number of visits to V_2 is $\lfloor k^m \rfloor$, $k \ge k_0$ where $m > 1$ is some constant. Redefine $E(k)$ to be the intersection of the events:

$E_1(k)$: VRRW does not visit points outside of G'.
$E_2(k, \varepsilon)$: $\tilde{\alpha}_x^{(k)} > \varepsilon$ for all vertices $x \in V_2$.
$E_3(k)$: $Z(t_k, y) < k^{m\zeta}$ for all $y \in B$.
$E_4(k)$: VRRW behaves "regularly" during the time period $t \in (t_{k-1}, t_k]$.

Using the same arguments as in the proof of Theorem 1.1, one can show that

$$\mathsf{P}(E(k) \mid E(k-1), E(k-2), \dots, E(k_0)) > 1 - \gamma(k)$$

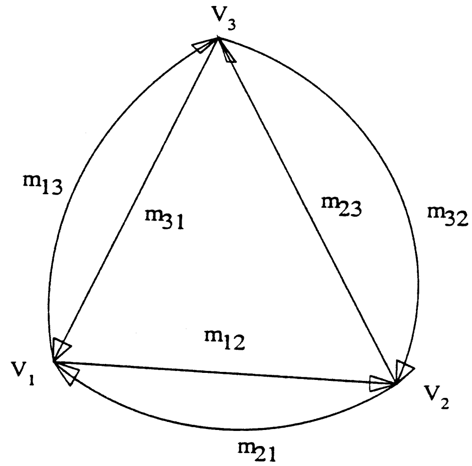

FIG. 3. *Illustration of VRRW on a complete graph with $n = 3$.*

with $\sum_{k>k_0} \gamma(k) < \infty$ when the initial configuration is "proper" in the sense that all $\alpha_x^{(k)} > 2\varepsilon$ and $k_0 = k_0(\varepsilon)$ is so large that

$$\sum_{k>k_0} \frac{\text{const}\ (\varepsilon, n_2)}{k^{1+\beta}} < \varepsilon.$$

Since $\lim_{l\to\infty} \sum_{k>l} \gamma(k) = 0$, the probability that VRRW never leaves G' is not only positive, but can be made arbitrary close to 1 by choosing a large enough k_0 on the event $E_1(k_0) \cap E_2(k_0, 2\varepsilon) \cap E_3(k_0)$.

Next, recall that we are actually interested in VRRW on G with v^*. Hence, one can carry through the arguments presented above, though with a couple of corrections. First, we are not guaranteed that $t_k < \infty$ for all k, due to the fact that *VRRW* may get stuck jumping between v and v^* [if $Z(t, v^*)$ grows with t faster than linear, for example]. However, this is also a localization. Second, (iii) and (iv) in Corollary 1.2 do not necessarily hold anymore, but, as the proof of Theorem 1.1 shows, for a localization this is not important. As a result, we obtain that for any $\delta > 0$ there exists $k_1(\varepsilon, \delta/5)$ such that

$$\mathsf{P}\biggl(\bigcap_{k>k_0} E(k) \mid E_{k_0}\biggr) > 1 = \frac{\delta}{5} \tag{3.42}$$

as soon as $k_0 \geq k_1(\varepsilon, \delta/5)$.

The last step consists in proving that either a "proper" starting configuration is achieved by VRRW* with a probability converging to one as A grows or VRRW* gets stuck at v and v^*. To achieve this goal, we pick an arbitrarily small $\delta > 0$ and show that $\phi(A) > 1 - \delta$ whenever $A \geq A_0$ for some $A_0 = A_0(\delta)$.

Consider a Pólya urn model with balls of n_2 different colors. By Lemma 2.4, the relative distribution of colors $\bar{\alpha}(t)$ converges a.s. to some random vector

$\bar{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{n_2})$ in the interior of $(n_2 - 1)$-simplex. Consequently, there exists $\varepsilon = \varepsilon(\delta) > 0$ so small that

$$\mathsf{P}\Bigl(\min_i \alpha_i > 2\varepsilon\Bigr) \ge 1 - \frac{\delta}{5}. \tag{3.43}$$

Since $\bar{\alpha}(t) \to \bar{\alpha}$ a.s., $\bar{\alpha}(t)$ also converges to $\bar{\alpha}$ in distribution and therefore there exist some $t_0 = t_0(\varepsilon, \delta/5)$ such that for any $t \ge t_0$,

$$\Bigl|\mathsf{P}\Bigl(\min_i \alpha_i > 2\varepsilon\Bigr) - \mathsf{P}\Bigl(\min_i \alpha_i(t) > 2\varepsilon\Bigr)\Bigr| \le \frac{\delta}{5}. \tag{3.44}$$

Next, notice that the probability that VRRW does not visit $x \in V_2$ even once while visiting v for the first $A^{2/3}$ times is at least

$$\Bigl(1 - \frac{n_2}{A}\Bigr)^{A^{2/3}} = 1 - \frac{n_2}{A^{1/3}} + o(A^{-1/3}). \tag{3.45}$$

Hence VRRW makes a large number of steps to v from v^* and back before it hits V_2 for the first time. Similarly, VRRW does not visit B during its first $A^{1/3}$ visits to V_2 with probability exceeding

$$\Bigl(1 - \frac{n_B}{A^{2/3}}\Bigr)^{A^{1/3}} = 1 - \frac{n_B}{A^{1/3}} + o(A^{-1/3}). \tag{3.46}$$

At the same time, when VRRW visits V_2 from v, conditional on not going to B, the distribution of $Z(t, x)$, $x \in V_2$, coincides with that of a Pólya urn model with balls of n_2 colors.

Let $\varepsilon = \varepsilon(\delta)$ be as given in equation (3.43), $t_0 = t_0(\varepsilon, \delta/5)$ is taken from (3.44) and $k_1 = k_1(\varepsilon, \delta/5)$ from (3.42). Set

$$A_0(\delta) = \Bigl(\max\Bigl\{\frac{5n_2}{\delta}, \frac{5n_B}{\delta}, t_0, k_1\Bigr\}\Bigr)^3$$

and consider VRRW* with $A \ge A_0$. On the event that it does not get stuck at $\{v\} \cup \{v^*\}$, formulas (3.43), (3.44), (3.45) and (3.46) imply that with probability $1 - 4\delta/5$ there will be a moment of time t when the set V_2 has been visited exactly $A^{1/3}$ times, B has not been visited at all and $Z(t, x)/A^{1/3} > 2\varepsilon$ for all $x \in V_2$. Combined with (3.42), this yields that $\varphi(A) \ge 1 - \delta$. Since $\delta > 0$ was arbitrary, (3.46) follows. □

3.2. *VRRW on a complete graph.*

PROOF OF LEMMA 3.1. In this section we omit the superscript$^{(k)}$ on the alphas to make notation less cumbersome. We will also write $x \approx y$ whenever $x = y + O(k^{-\beta}N)$ in the sense that $|x - y| \le Ck^{-\beta}N$ for large k with a constant $C > 0$ not depending on k or the state of VRRW.

Recall that m_{ij} $(i \ne j)$ denotes the number of steps from V_i to V_j between the moments t_k and t_{k+1}. Set $m_{ii} \equiv 0$. As before, $m_i = \sum_j m_{ij}$ is the number of times VRRW leaves the pseudo-vertex V_i for S, and N_i is the total number of visits to V_i from both S and B. Since the number of times VRRW comes to V_i should match the number of times VRRW leaves it, we have

$$\begin{aligned} m_i &= \sum_j m_{ij} \approx \sum_j m_{ji} \quad \text{for all } i \in \{1, 2, \dots, n\}; \\ N &\approx \sum_{i,j} (m_{ij} + m_{ji}), \end{aligned} \tag{3.47}$$

where we take into account the fact that $N_B = o(\delta N)$ where $\delta = k^{-\beta} + o(\triangleleft)$ and that there is a possibility of going from V_i to V_j through B without changing m_{ij}.

Observe that (3.20) can be rewritten as

$$m_{ij} \approx \frac{\alpha_j}{1-\alpha_i} m_i \quad \text{for all } i \text{ and } j \neq i. \tag{3.48}$$

By summing over i (3.48) we obtain

$$m_j \approx \alpha_j \left(m^* - \frac{m_j}{1-\alpha_j} \right),$$

where

$$m^* = \sum_i \frac{m_i}{1-\alpha_i}.$$

Consequently, $m_j \approx \alpha_j(1-\alpha_j)m^*$, whence

$$\frac{m_j}{\sum_{i=1}^l m_i} \approx \frac{\alpha_j(1-\alpha_j)}{\sum_{i=1}^n \alpha_i(1-\alpha_i)} = \frac{\alpha_j(1-\alpha_j)}{(1-\sum_{i=1}^n \alpha_i^2)} \geq \frac{\alpha_j(1-\alpha_j)}{1-1/n},$$

where we use the fact that $\sum \alpha_i = 1$ and, therefore, $\sum \alpha_i^2 \leq 1/n$. □

4. The problem of universality. A natural question arises after seeing Theorem 1.1, Corollary 1.2 and Corollary 1.3: since VRRW gets stuck with a positive probability, does this imply that it must get stuck *somewhere* with probability 1? Unfortunately, we can not answer this question in all cases (in particular, for Z^d, $d \geq 2$). This is, in part, due to the non-Markovian nature of the process and the impossibility of using Kolmogorov's zero–one law directly.

Moreover, in general, this is not true. Indeed, consider a tree that on level n has K_n branches coming out of each vertex with $\sum K_n^{-1} < \infty$ VRRW on this tree can make infinitely many steps without ever coming back to any vertex it has previously visited, with a positive probability. Therefore the probability that VRRW visits only finitely many vertices is smaller than 1, though it is positive by Corollary 1.2. Consequently, the "traditional" a.s.-recurrence versus a.s.-transience dichotomy does not hold here.

However, we conjecture that on a very broad class of graphs of bounded degree, VRRW visits a.s. only finitely many points. We also believe that on any periodic graph the number of different kinds of subgraphs on which VRRW can get stuck is finite and is given by the set of all possible trapping subgraphs, or perhaps, a slightly broader class of subgraphs (for example, Theorem 1.1 implies that there are at least two different patterns of localization on $\mathbb{Z}^2$). The latter is a generalization of the conjecture in Pemantle and Volkov (1999) that on $\mathbb{Z}^1$ VRRW will eventually get stuck at the set of exactly five points. Unfortunately, at this time we do not have a proof for any of the above statements.

For trees of bounded degree (including $\mathbb{Z}^1$) the fact that VRRW localizes (eventually gets stuck at a finite set) follows from the following arguments.

PROOF OF THEOREM 1.5. Embed the tree G in the plane such that the root (the vertex where VRRW starts) is at the top, the vertices adjacent to the root are placed one level below, etc. We say that VRRW moves up or down depending whether it gets closer or farther from the root, respectively. A vertex v_c is called a child of a vertex v (and v is called the parent of v_c) if v_c is adjacent to v and v lies on the path connecting v_c to the root.

Since VRRW is on a tree, a possible localization is "a core and a shell" as described by Corollary 1.2. Moreover, each vertex is incident to no more than K edges, whence the number of different trapping subgraphs G' with outer boundaries $\partial G'$ is finite. This observation will play an important role later in the proof.

Denote by $A(v)$ the local time at the parent of v when VRRW visits vertex v for the first time and set $A(v) = \infty$ if VRRW never hits v. Consider the sum

$$S_A = \sum_{v \in G} \frac{1}{A(v)}.$$

Two cases are possible: $S_A < \infty$ and $S_A = \infty$. We state that in the second case VRRW gets stuck at some "core and shell" configuration almost sure, which, in turn, yields a contradiction since $1/A(v) \neq 0$ only for finitely many v. Indeed, as follows from Borel–Cantelli lemma, $S_A = \infty$ implies that there exist infinitely many times when VRRW makes four consecutive steps down through the vertices it has never visited before. Let v' be the last visited of these four vertices; therefore the local times at each of its three direct ancestors equal 2. Consider a trapping subgraph

$$G' = \{v'\} \cup \partial\{v'\} \cup \partial(\partial\{v'\})$$

with v' being "a core." It is easy to see that there are only finitely many nonisomorphic combinations $(G' \cup \partial G', \{Z(t, x), x \in G' \cup \partial G'\})$ since the local time on any vertex of $G' \cup \partial G'$ is at most 2. Consequently, there exists some constant $\gamma > 0$ depending on K only, such that the probability that VRRW gets stuck on G' is at least γ.

Let N be the number of times when VRRW makes four consecutive steps down on vertices not visited before. Then the probability not to end up with a core and a shell localization is at most $(1-\gamma)^N$. On the other hand, $S_A = \infty$ insures that there are infinitely may such times and therefore the above probability must be zero.

To show a localization in the other case ($S_A < \infty$), we construct a subcritical branching process on the vertices of G. Suppose that VRRW visit infinitely many vertices. Then there exists an infinite sequence $v_1, v_2, \dots$ of distinct vertices of G to which VRRW goes, ordered by the times of the first visits to them. The finiteness of S_A yields

$$\lim_{i\to\infty} A(v_i) = \infty. \tag{4.1}$$

Consider VRRW* described in Section 3.1 on all possible trapping subgraphs of G with the property that their "cores" (points v) are connected to no more than $K-1$ vertices. There are only finitely many such subgraphs, hence Lemma 3.3 implies the existence of a number A_0 such that the probability ever to leave any such subgraph does not exceed ν, for any $\nu > 0$, as soon as the local time at the "irregular" vertex v^* is at least $A \ge A_0 = A_0(\nu)$. We fix $\nu = 1/K^2$ and A_0 corresponding to this ν.

According to (4.1), there exist i_0 such that $A(v_i) \ge A_0$ for $i \ge i_0$. Let t_0 be the time of the first visit to v_{i_0}. We construct a branching process on the vertices of G as follows. The process starts at time t_0 and initially consists of the subset of vertices visited by time t_0 lying on the boundary of the set,

$$\left\{v_i \colon i \le i_0 \text{ and there exists } x \in G\Big\backslash\left(\bigcup_{i=1}^{i_0}\{v_i\}\right) \text{ such that } x \sim v_i\right\}.$$

There may be up to t_0 such vertices.

We say that a vertex v "dies" if VRRW never visits any of its great-grandchildren (points connected to v by three consecutive edges all going down). In the other case, at the very moment when some great-grandchild v_{gg^c} of v is visited for the first time, we say that v splits into a set of vertices consisting of all descendants of v (on the graph) which are adjacent to at least one point of G not visited by VRRW by this time. The cardinality of this set cannot exceed $(K-1)^2 - 1 + (K-2) + (K-1) = K^2 - 2$.

Consequently, each point v of the vertex-branching process constructed above either splits into at most $K^2 - 2$ new particles or dies with probability at least $1-\nu$ regardless of the history of VRRW. The latter follows from the observation that once v is visited for the first time with v_p being its parent, we can couple the VRRW on the subtree of G consisting of v and all its descendents with a *VRRW** for which an "irregular" vertex is $v^* = v_p$. Since $A(v) \ge A_0$, by Lemma 3.3 the probability of visiting any great-grandchild of v does not exceed ν.

Therefore, the vertex-branching process is stochastically smaller than a Galton–Watson process in which each particle either branches into exactly K^2-2 particles with probability ν or dies with the complimentary probability, independently of the others. As the expected number of direct descendants of a single particle for this process is $0 \times (1-\nu) + (K^2-2) \times \nu < 1$, the Galton–Watson process is subcritical and therefore dies out [e.g., see Athreya and Ney (1972)]. This implies that the vertex-branching process also dies out and this is equivalent to a loalization of VRRW. $\square$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRISTOL
BRISTOL, BS8 1TW
UNITED KINGDOM
E-MAIL: S.Volkov@bristol.ac.uk