EDITORIAL

FUNDAMENTALS OF THE THEORY OF SAMPLING

III. DISTRIBUTION OF SAMPLE m TH MOMENTS ABOUT THE ORIGIN OF THE PARENT POPULATION

As in section I, we shall be concerned with the $\binom{s}{r}$ possible samples, each consisting of r variates, that can be selected from the parent population of s variates $x_1, x_2, \ldots x_r, \ldots x_s$. The m th moment of each sample, computed in each case about the origin of the parent population, may be written

$$\begin{cases} z_1 = \frac{1}{r}\left\{x_1^m + x_2^m + x_3^m + \cdots + x_r^m\right\} \\ z_2 = \frac{1}{r}\left\{x_2^m + x_3^m + x_4^m + \cdots + x_{r+1}^m\right\} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ z_{\binom{s}{r}} = \frac{1}{r}\left\{x_{s-r+1}^m + x_{s-r+2}^m + x_{s-r+3}^m + \cdots + x_s^m\right\} \end{cases}$$

If we write $\frac{x_i^m}{r} = y_i$, it will be observed that the above distribution may be written

$$\begin{cases} z_1 = y_1 + y_2 + y_3 + \cdots + y_r \\ z_2 = y_2 + y_3 + y_4 + \cdots + y_{r+1} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ z_{\binom{s}{r}} = y_{s-r+1} + y_{s-r+2} + y_{s-r+3} + \cdots + y_s \end{cases}$$

and therefore may be regarded as a distribution of the algebraic sums of the respective samples withdrawn from the parent population $y_1, y_2, \ldots y_s$, i. e. $\frac{x_1^m}{r}, \frac{x_2^m}{r}, \ldots \frac{x_s^m}{r}$. Consequently, since

$$\mu'_{n:y} = \frac{\Sigma y_i^n}{N} = \frac{1}{N}\Sigma\frac{x_i^{mn}}{r^n} = \frac{1}{r^n}\mu'_{mn:x}$$

it follows from formulae 1, 2, . . . of section I that

$$(1)\quad M_x = r\,M_y = \mu'_{m:x}$$

$$(2)\quad \mu_{2:x} = s\{\rho_1-\rho_2\}\,\mu_{2:y} = s\{\rho_1-\rho_2\}\{\mu'_{2:y}-M_y^2\}$$

$$= s\{\rho_1-\rho_2\}\left\{\frac{\mu'_{2m:x}}{r^2} - \left(\frac{\mu'_{m:x}}{r}\right)^2\right\}$$

$$= \frac{s}{r^2}\{\rho_1-\rho_2\}\{\mu'_{2m:x} - (\mu'_{m:x})^2\}$$

$$(3)\quad \mu_{3:x} = s\{\rho_1-3\rho_2+2\rho_3\}\,\mu_{3:y} = s\{\rho_1-3\rho_2+2\rho_3\}\{\mu'_{3:y}-3M_y\mu'_{2:y}+2M_y^3\}$$

$$= \frac{s}{r^3}\{\rho_1-3\rho_2+2\rho_3\}\{\mu'_{3m:x}-3\mu'_{2m:x}\mu'_{m:x}+2(\mu'_{m:x})^3\}$$

$$(4)\quad \mu_{4:x} = \frac{s}{r^4}\{\mu'_{4m:x}-4\mu'_{3m:x}\mu'_{m:x}+6\mu'_{2m:x}(\mu'_{m:x})^2-3(\mu'_{m:x})^4\}$$

$$\{\rho_1-7\rho_2+12\rho_3-6\rho_4\}+3\frac{s^2}{r^4}\{\mu'_{2m:x}-(\mu'_{m:x})^2\}^2\{\rho_2-2\rho_3+\rho_4\}$$

etc.

For the case of sampling from an unlimited supply, we have, permitting s to approach infinity, that corresponding to formulae (18) of section I

$$
(5)\begin{cases}
M_{\bar{x}} = \mu'_{m:x} \\
\mu_{2:\bar{x}} = \frac{1}{r}\left\{\mu'_{2m:x} - (\mu'_{m:x})^2\right\} \\
\mu_{3:\bar{x}} = \frac{1}{r^2}\left\{\mu'_{3m:x} - 3\mu'_{2m:x}\mu'_{m:x} + 2(\mu'_{m:x})^3\right\} \\
\mu_{4:\bar{x}} = \frac{1}{r^3}\left\{\mu'_{4m:x} - 4\mu'_{3m:x}\mu'_{m:x} + 6\mu'_{2m:x}(\mu'_{m:x})^2 - 3(\mu'_{m:x})^4\right\} \\
\qquad + \frac{3r^{(2)}}{r^4}\left\{\mu'_{2m:x} - (\mu'_{m:x})^2\right\}
\end{cases}
$$

etc.

The distribution of sample means may be obtained by placing $m = 1$, yielding

$$
(6)\begin{cases}
M_{\bar{x}} = M_x \\
\mu_{2:\bar{x}} = \frac{1}{r}\mu_{2:x} \\
\mu_{3:\bar{x}} = \frac{1}{r^2}\mu_{3:x} \\
\mu_{4:\bar{x}} = \frac{1}{r^3}\mu_{4:x} + \frac{3(r-1)}{r^3}\mu^2_{2:x}
\end{cases}
$$

etc.

These results may be written corresponding to formulae (19) of section I,

$$
(7)\begin{cases}
\mu_{2:\bar{x}} = \frac{1}{r}\mu_{2:x} \\
\mu_{3:\bar{x}} = \frac{1}{r^2}\mu_{3:x} \\
\mu_{4:\bar{x}} - 3\mu^2_{2:\bar{x}} = \frac{1}{r^3}\left\{\mu_{4:x} - 3\mu^2_{2:x}\right\} \\
\mu_{5:\bar{x}} - 10\mu_{3:\bar{x}}\mu_{2:\bar{x}} = \frac{1}{r^4}\left\{\mu_{5:x} - 10\mu_{3:x}\mu_{2:x}\right\} \\
\mu_{6:\bar{x}} - 15\mu_{4:\bar{x}}\mu_{2:\bar{x}} - 10\mu^2_{3:\bar{x}} + 30\mu^3_{2:\bar{x}} = \frac{1}{r^5}\left\{\mu_{6:x} - 15\mu_{4:x}\mu_{2:x} - 10\mu^2_{3:x} + 30\mu^3_{2:x}\right\}
\end{cases}
$$

etc.

The distribution of sample means withdrawn from an infinite parent population is therefore characterized by means of the semi-invariant relation

$$\lambda_{n:z} = \frac{\lambda_{n:x}}{r^{n-1}} \tag{8}$$

and the standard semi-invariants by the relation

$$\gamma_{n:z} = \frac{\gamma_{n:x}}{r^{n/2-1}} \tag{8-a}$$

An interesting result is obtained by considering the special case of formulae (8) for which $n = 2$, and assuming that the parent population is normal. Since for a normal distribution

$$\mu_{2n+1} = 0$$

$$\mu_{2n} = 1 \cdot 3 \cdot 5 \; . \; . \; . \; (2n-1)\sigma^{2n}$$

and for any distribution

$$(9)\quad \begin{cases} \mu_2' = \mu_2 + M^2 \\ \mu_3' = \mu_3 + 3M\mu_2 + M^3 \\ \mu_4' = \mu_4 + 4M\mu_3 + 6M^2\mu_2 + M^4 \end{cases}$$

etc.

it follows that for a normal distribution

$$(10)\quad \begin{cases} \mu_2' = \sigma^2 + M^2 \\ \mu_3' = 3M\sigma^2 + M^3 \\ \mu_4' = 3\sigma^4 + 6\sigma^2 M^2 + M^4 \\ \mu_5' = 15M\sigma^4 + 10M^3\sigma^2 + M^5 \end{cases}$$

etc.

and therefore for the distribution of sample second moments about a fixed point in the case of withdrawals from an unlimited "normal" supply, we have, from (5)

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1 See formulae 23 and 24, page 117, Vol. I, No. 1, of ANNALS.

$$
(11)\left\{
\begin{aligned}
M_z &= \mu'_{2:x} = \sigma_x^2 + M_x^2 \\
\mu_{2:z} &= \frac{1}{r}\left\{ \mu'_{4:x} - (\mu'_{2:x})^2 \right\} \\
&= \frac{2\sigma^2}{r}\left\{ \sigma_x^2 + 2M_x^2 \right\} \\
\mu_{3:z} &= \frac{1}{r^2}\left\{ \mu_{6:x} - 3\mu'_{4:x}\mu'_{2:x} + 2(\mu'_{2:x})^3 \right\} \\
&= \frac{8\sigma_x^4}{r^2}\left\{ \sigma_x^2 + 3M_x^2 \right\} \\
\mu_{4:z} &= \frac{48\sigma_x^6}{r^3}\left\{ \sigma_x^2 + 4M_x^2 \right\} + \frac{12\sigma_x^4}{r^2}\left\{ \sigma_x^2 + 2M^2 \right\}^2 \\
\mu_{5:z} &= \frac{384\sigma_x^8}{r^4}\left\{ \sigma_x^2 + 5M_x^2 \right\} + \frac{160\sigma_x^6}{r^3}\left\{ \sigma_x^2 + 2M_x^2 \right\}\left\{ \sigma_x^2 + 3M_x^2 \right\} \\
\mu_{6:z} &= \frac{3840\sigma_x^{10}}{r^5}\left\{ \sigma_x^2 + 6M_x^2 \right\} + \frac{160\sigma_x^8}{r^4}\left\{ 13\sigma_x^4 + 78\sigma_x^2 M_x^2 + 108M_x^4 \right\} \\
&\quad + \frac{120\sigma_x^6}{r^3}\left\{ \sigma_x^2 + 2M_x^2 \right\}^3
\end{aligned}
\right.
$$

In terms of semi-invariants[1]

$$
(12)\left\{
\begin{aligned}
M_z &= \sigma_x^2 + M_x^2 \\
\lambda_{2:z} &= \frac{2\sigma_x^2}{r}(\sigma_x^2 + 2M_x^2) \\
\lambda_{3:z} &= \frac{2^2\cdot 2!\,\sigma_x^4}{r^2}(\sigma_x^2 + 3M_x^2) \\
\lambda_{4:z} &= \frac{2^3\cdot 3!\,\sigma_x^6}{r^3}(\sigma_x^2 + 4M_x^2) \\
\lambda_{5:z} &= \frac{2^4\cdot 4!\,\sigma_x^8}{r^4}(\sigma_x^2 + 5M_x^2) \\
\lambda_{6:z} &= \frac{2^5\cdot 5!\,\sigma_x^{10}}{r^5}(\sigma_x^2 + 6M_x^2)
\end{aligned}
\right.
$$

1 Formulae (21), Section I. Page 116, Vol. I, No. 1, of ANNALS.

Apparently the general expression is

$$\lambda_{n:x} = \frac{2^{n-1}(n-1)!\,\sigma_x^{2n}}{r^{n-1}}\left\{1 + n\left(\frac{M_x}{\sigma_x}\right)^2\right\} \tag{13}$$

If the parent population be normal, and if furthermore $M_x = 0$, then

$$\lambda_{n:x} = \frac{2^{n-1}(n-1)!\,\sigma_x^{2n}}{r^{n-1}} \tag{14}$$

and the standardized semi-invariants would likewise be

$$\gamma_{n:x} = \frac{\lambda_{n:x}}{(\lambda_{2:x})^{n/2}} = \left(\frac{2}{r}\right)^{\frac{n}{2}-1} \cdot (n-1)! \tag{15}$$

Again, since

$$\gamma_{3:x} = \alpha_{3:x} = \frac{2^{3/2}}{r^{1/2}}$$

formula (15) may be written.

$$\gamma_{n:x} = \left(\frac{\alpha_3}{2}\right)^{n-2} \cdot (n-1)! \tag{16}$$

On page 196 of Vol. I, No. 2 of the ANNALS it was shown that the standard moments for Pearson's Type III function

$$y = y_0 \left(1 + \frac{\alpha_3}{2}t\right)^{\frac{4}{\alpha_3^2}-1} e^{-\frac{2}{\alpha_3}t}$$

are defined by the recurring relation

$$\alpha_{n+1} = n\left(\alpha_{n-1} + \frac{\alpha_3\,\alpha_n}{2}\right),$$

so that $\alpha_4 = 3\left(1+\frac{\alpha_3^2}{2}\right)$

$$\alpha_5 = 2\alpha_3\left(5+3\,\frac{\alpha_3^2}{2}\right)$$

$$\alpha_6 = 5\left(3+13\,\frac{\alpha_3^2}{2}+6\,\frac{\alpha_3^4}{4}\right)$$

etc.

The standard semi-invariants of Type III are

$$\gamma_4 = \left(\frac{\alpha_3}{2}\right)^2\cdot 3!$$

$$\gamma_5 = \left(\frac{\alpha_3}{2}\right)^3\cdot 4!$$

$$\gamma_6 = \left(\frac{\alpha_3}{2}\right)^4\cdot 5!$$

etc.

Comparing these results with formula (16) it appears, therefore, that if the parent population be normal *and its mean zero*, the distribution of sample second moments computed about the fixed mean of the parent population will be Pearson't Type III, **for r finite. As r** approaches infinity, the Type III distribution will approach the Normal Curve as a limit.

To illustrate: If from an infinite population of spherical balls whose diameters formed a normal distribution characterized by M_x and σ_x, samples of r balls each were withdrawn, then if the average area be determined for the balls in each sample, the distribution of these areas, from formula (13), would be described by the relation

$$\lambda_{n:x} = \frac{2^{n-1}(n-1)!\,\sigma_x^{2n}}{r^{n-1}}\left\{1+n\left(\frac{M_x}{\sigma_x}\right)^2\right\}$$

and if one could conceive of negative diameters of the balls so that $M_x = 0$, then the distribution of areas would be Type III.

If one were to succeed in finding the function whose n th semi-invariant agrees with the above expression, then the law of distribution for the sample areas would be available. Again by likewise investigating the cases of formulae (5) where $m = 3, 4, 5$, etc., other semi-invariant relations can be found, and these in turn may lead to the discovery of new and important frequency functions. At all events, such sample moments and semi-invariants will generally permit one to express as an infinite series, such as the Gram-Charlier series, the unknown law of distribution.

SECTION IV

The problem of the distribution of sample moments about the origin of the parent population[1] is unfortunately often confused with the problem of the distribution of sample moments computed about the means of the respective samples. The latter problem is more briefly termed sampling *about the mean.* If M_1 and M_2 designate the means of the first two samples respectively, and z_1 and z_2 the second moments of these two samples computed about M_1 and M_2 respectively, then for $m = 2$

$$z_1 = \frac{\sum^{r:1} (x - M_1)^2}{r}$$

$$z_2 = \frac{\sum^{r:2} (x - M_2)^2}{r}$$

where, as before, $\sum^{r:i}$ indicates that the summation extends over the r variates occurring in the i th sample.

In order to sum all values of z_i and z_i^n it is necessary to obtain first another expression for the second moment about the mean, which, although of value in algebraic manipulations, is practically of no value in arithmetic computation. Thus,

$$z_i = \frac{\sum^{r:i} (x - M_i)^2}{r}$$

$$= \frac{\sum^{r:i} x^2 - 2M_i \sum^{r:i} x + rM_i^2}{r}$$

1 Also referred to as the distribution of sample moments *about a fixed point.*

$$= \frac{\overset{r:i}{\Sigma} x^2 - \frac{(\overset{r:i}{\Sigma} x)^2}{r}}{r}$$

$$= \frac{r\overset{r:i}{\Sigma} x^2 - [\overset{r:i}{\Sigma} x^2 + 2\overset{r:i}{\Sigma} x_i x_j]}{r^2}$$

(17) $$z_i = \frac{1}{r^2}\left[(r-1)\overset{r:i}{\Sigma} x^2 - 2\overset{r:i}{\Sigma} x_i x_j\right],$$

where $\overset{r:i}{\Sigma} x_i x_j$ designnates the sum of all the terms formed by taking the products of all the variates in the i th sample two at a time.

Then $$M_z = \frac{\Sigma z_i}{\binom{s}{r}} = \frac{1}{r^2}\left[(r-1)\frac{r}{s}\overset{s}{\Sigma} x^2 - 2\frac{r^{(2)}}{s^{(2)}}\overset{s}{\Sigma} x_i x_j\right]$$

by employing the method employed in section I. The above reduces easily as follows:

(18) $$\begin{aligned} M_z &= \frac{1}{r^2}\left[(r-1)\frac{r}{s}\overset{s}{\Sigma} x^2 - \frac{r^{(2)}}{s^{(2)}}\left\{(\overset{s}{\Sigma} x)^2 - \Sigma x^2\right\}\right] \\ &= \frac{s(r-1)}{r(s-1)} \cdot \left\{\frac{\overset{s}{\Sigma} x^2}{s} - \left(\frac{\overset{s}{\Sigma} x}{s}\right)^2\right\} \\ &= \frac{s(r-1)}{r(s-1)}\,\mu_{2:x} \end{aligned}$$

Whereas the expected value of a sample mean is equal to the mean of the parent population and the expected value of a sample n th moment about a fixed point is equal to the n th moment of the parent population[1], it appears that the expected value of a sample second moment is less than the second moment of the parent population.

A slight digression at this point is desirable. In formula (6) of Section III we found that for the distribution of sample means withdrawn from an infinite parent population,

$$\mu_{2:\bar{x}} = \frac{1}{r}\mu_{2:x}$$

That is, the *standard error of the mean*

1 Formula (1), Section III.

(19) $$\sigma_M = \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{r}} ,$$

where σ_x denotes the standard deviation of the infinite parent population. By formula (18) above it appears that the expected value of the second sample moment is for $s = \infty$

$$M_2 = \frac{r-1}{r} \mu_{2:x} = \frac{r-1}{r} \sigma_x^2$$

Designating the square root of the expected sample moment by σ'_x, we have that

$$\sigma'_x = \sigma_x \sqrt{\frac{r-1}{r}} , \text{ or } \sigma_x = \sigma'_x \sqrt{\frac{r}{r-1}}$$

and therefore formula (19) may be written

(20) $$\sigma_M = \frac{\sigma'_x}{\sqrt{r-1}}$$

Since the probable error is defined as .6745 σ, we have that the *probable error of the mean*

(21) $$P.E._M = .6745 \frac{\sigma_x}{\sqrt{r}} = .6745 \frac{\sigma'_x}{\sqrt{r-1}}$$

It should be observed that the expressions for both the standard and probable errors of the mean are expected values when σ' is employed. If one obtains but a single sample and computes its mean and standard deviation, he still has no accurate knowledge regarding the true value of the standard deviation of the parent population. Consequently even the expression

$$P.E._M = .6745 \frac{\sigma'_x}{\sqrt{r-1}}$$

is merely an approximation. So far as I know, the true value of the probable error of the mean has never been found—even upon the assumption that the parent population is normal. Since we have shown that for $s \to \infty$ the skewness of the samples is only $\frac{1}{\sqrt{r}}$ times the skewness of the parent population, the fact that the parent population is not normal is of no importance compared to the fact that where only functions of the single sample are available, *these* must be substituted as the expected values of the corresponding functions of the unknown parent population

Returning to our problem of describing further the distribution of sample second moments about the mean:

Corresponding to formula (17), one can show by employing symmetric functions that

$$(22)\qquad \begin{aligned} z_i^2 = \frac{1}{r^4}\Big\{&(r-1)^2 \sum^{r:i} x^4 - 4(r-1)\sum^{r:i} x_i^3 x_j + 2(r^2 - 2r + 3)\sum^{r:i} x_i^2 x_j^2 \\ &- 4(r-3)\sum^{r:i} x_i^2 x_j x_k + 24\sum^{r:i} x_i x_j x_k x_l\Big\} \end{aligned}$$

and therefore

$$(23)\qquad \begin{aligned} \mu_{2:z} &= \frac{\sum z_i^2}{\binom{s}{r}} - M_z^2 \\ &= \frac{s(r-1)(s-r)}{r^3(s-1)^2(s-2)(s-3)}\cdot\Big\{(s-1)(rs-s-r-1)\,\mu_{4:x} \\ &\quad + \left[(3-r)s^2 - 6s + 3r + 3\right]\mu_{2:x}^2\Big\} \end{aligned}$$

For $s = \infty$ this becomes

$$(24)\qquad \begin{aligned} \mu_{2:z} &= \frac{r-1}{r^3}\left[(r-1)\,\mu_{4:x} - (r-3)\,\mu_{2:x}^2\right] \\ &= \frac{(r-1)\sigma^4}{r^3}\left[(r-1)\,\alpha_{4:x} - (r-3)\right] \end{aligned}$$

In a thesis, C. H. Richardson[1] has shown that when $s = \infty$

1 Submitted in 1927 to the University of Michigan. The balance of this section is a synopsis of one part of this thesis.

$$(25)\quad \mu_{3:x}=\frac{(r-1)\sigma^6}{r^5}\Big[(r-1)^2\alpha_{6:x}-3(r-1)(r-5)\,\alpha_{4:x}$$

$$-2\,(3r^2-6r+5)\,\alpha_3^2+2\,(r^2-12r+15)\Big]$$

$$(26)\quad \mu_{4:x}=\frac{(r-1)\sigma^8}{r^7}\Big[(r-1)^3\alpha_{8:x}-8(r-1)(3r^2-6r+7)\,\alpha_{5:x}\,\alpha_{3:x}$$

$$+(3r^4-12r^3+42r^2-60r+35)\alpha_{4:x}^2-4(r-1)^2(r-7)\,\alpha_{6:x}$$

$$-6\,(r^4-7r^3+49r^2-105r+70)\,\alpha_{4:x}$$

$$+16\,(6r^3-27r^2+50r-35)\,\alpha_{3:x}^2$$

$$+3\,(r^4-9r^3+93r^2-255r+210)\Big]$$

$$(27)\quad \mu_{5:x}=\frac{(r-1)\sigma^{10}}{r^9}\Big[(r-1)^4\,\alpha_{10:x}-5\,(r-1)^3\,(r-9)\,\alpha_{8:x}$$

$$-40(r-1)^2(r^2-2r+3)\,\alpha_{7:x}\,\alpha_{3:x}$$

$$+10(r-1)(r^4-4r^3+18r^2-28r+21)\,\alpha_{6:x}\,\alpha_{4:x}$$

$$-10(3r^5-27r^4+162r^3-450r^2+595r-315)\alpha_{4:x}^2$$

$$-20\,(r-2)(3r^4-24r^3+80r^2-140r+105)\,\alpha_{4:x}\,\alpha_{3:x}^2$$

$$+10\,(5r^5-64r^4+572r^3-2070r^2+3255r-1890)\,\alpha_{3:x}^2$$

$$-4\,(5r^5-86r^4+1050r^3-4620r^2+8505r-5670)$$

$$-10(r-1)\,(r^4-7r^3+65r^2-161r+126)\,\alpha_{6:x}$$

$$-2\,(15r^4-60r^3+130r^2-140r+63)\,\alpha_{5:x}^2$$

$$-80\,(6r^4-36r^3+97r^2-126r+63)\,\alpha_{5:x}\,\alpha_{3:x}\Big]$$

$$
\begin{aligned}
(28)\ \mu_{6:x} = \frac{(r-1)\sigma^{12}}{r^{11}} \Big[& (r-1)^5 \alpha_{12:x} - 6(r^5-15r^4+50r^3-70r^2+45r-11)\alpha_{10:x} \\
& - 20(3r^5-15r^4+38r^3-54r^2+39r-11)\alpha_{9:x}\,\alpha_{3:x} \\
& + 15(r^6-6r^5+31r^4-84r^3+127r^2-102r+33)\alpha_{6:x}\,\alpha_{4:x} \\
& - 15(r^6-9r^5+96r^4-394r^3+729r^2-621r+198)\alpha_{8:x} \\
& + 2(5r^6-30r^5+165r^4-460r^3+735r^2-630r+231)\alpha^2_{6:x} \\
& - 120(r^6-11r^5+81r^4-294r^3+567r^2-567r+231)\alpha_{6:x}\alpha_{4:x} \\
& + 15(r^7-8r^6+51r^5-258r^4+815r^3-1540r^2+1645r-770)\alpha^3_{4:x} \\
& + 20(5r^6-75r^5+828r^4-3938r^3+9009r^2-9891r+4158)\alpha_{6:x} \\
& + 5(9r^7-159r^6+2436r^5-26130r^4+135885r^3-35941r^2 \\
& \qquad + 474390r-24980)\alpha_{4:x} \\
& - 5(9r^7-126r^6+1413r^5-11214r^4+47355r^3-107730r^2 \\
& \qquad + 127575r-62370)\alpha^2_{4:x} \\
& - 24(5r^5-25r^4+70r^3-110r^2+93r-33)\alpha_{7:x}\,\alpha_{5:x} \\
& + 480(2r^5-15r^4+52r^3-96r^2+90r-33)\alpha_{7:x}\,\alpha_{3:x} \\
& - 40(3r^6-39r^5+206r^4-616r^3+1113r^2-1113r+462)\alpha_{6:x}\,\alpha^2_{3:x} \\
& + 24(30r^5-225r^4+820r^3-1610r^2+1638r-693)\alpha^2_{5:x} \\
& - 120(3r^6-33r^5+172r^4-530r^3+987r^2-1029r+462)\alpha_{5:x}\alpha_{4:x}\alpha_{3:x} \\
& + 40(51r^6-435r^5+1896r^4-5218r^3+9191r^2-9387r+4158)\alpha_{5:x}\alpha_{3:x} \\
& + 600(3r^6-45r^5+294r^4-1076r^3+2317r^2-2751r+1386)\alpha_{4:x}\alpha^2_{3:x} \\
& - 80(21r^6-558r^5+5012r^4-22820r^3+57445r^2-76230r \\
& \qquad + 41580)\,\alpha^2_{3:x} \\
& + 40(9r^6-105r^5+564r^4-1830r^3+3745r^2-4445r+2310)\,\alpha^4_{3:x} \\
& - 5(3r^7-59r^6+1136r^5-15642r^4+96135r^3-290115r^2 \\
& \qquad + 429030r-249480)\Big]
\end{aligned}
$$

If the parent population be normal, that is if

$$\alpha_{2n+1} = 0$$

$$\alpha_{2n} = \frac{(2n)!}{2^n \cdot n!}$$

the preceding formulae yield on reduction

$$\text{(29)} \quad \mu_{2:x} = \frac{2(r-1)}{r^2}\,\mu_{2x}^{2}$$

$$\text{(30)} \quad \mu_{3:x} = \frac{8(r-1)}{r^3}\,\mu_{2:x}^{3}$$

$$\text{(31)} \quad \mu_{4:x} = \frac{12(r-1)(r+3)}{r^4}\,\mu_{2.x}^{4}$$

$$\text{(32)} \quad \mu_{5:x} = \frac{32(r-1)(5r+7)}{r^5}\,\mu_{2:x}^{5}$$

$$\text{(33)} \quad \mu_{6:x} = \frac{40(r-1)(3r^2+46r+47)}{r^6}\,\mu_{2:x}^{6}$$

These may be written in turn

$$\text{(34)} \quad \left\{ \begin{aligned} \alpha_{3:x} &= \frac{4}{\sqrt{2(r-1)}} \\ \alpha_{4:x} &= \frac{3(r+3)}{r-1} \\ \alpha_{5:x} &= \frac{8(5r+7)}{\sqrt{2}\,(r-1)^{3/2}} \\ \alpha_{6:x} &= \frac{5(3r^2+46r+47)}{(r-1)^2} \end{aligned} \right.$$

For the corresponding standard semi-invariants

$$\gamma_3 = \alpha_3 = \frac{4}{\sqrt{2(r-1)}}$$

$$\gamma_4 = \alpha_4 - 3 = \frac{12}{r-1} = \left(\frac{\alpha_3}{2}\right)^2 \cdot 3!$$

$$\gamma_5 = \alpha_5 - 10\alpha_3 = \frac{96}{\sqrt{2}\,(r-1)^{3/2}} = \left(\frac{\alpha_3}{2}\right)^3 \cdot 4\,!$$

$$\gamma_6 = \alpha_6 - 15\,\alpha_4 - 10\alpha_3^2 + 30 = \frac{480}{(r-1)^2} = \left(\frac{\alpha_3}{2}\right)^4 \cdot 5\,!$$

These results show that so far as the sixth standard semi-invariant the distribution of sample second moments about the mean is Type III, irrespective of the mean of the parent population.

It is to be regretted that many of the results presented here have never been generalized for moments of any order. The methods presented have been chosen for two reasons: first, they permit one with no knowledge of calculus to achieve somewhat of an understanding into the theory of sampling; and secondly, they yield results of sampling from a finite parent population--a problem of considerable practical importance.

The results of sampling from an infinite population may be obtained more readily and with far greater elegance and rigor by employing the method of semi-invariants.

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