ON CERTAIN PROPERTIES OF FREQUENCY DISTRIBUTIONS OBTAINED BY A LINEAR FRACTIONAL TRANSFORMATION OF THE VARIATES OF A GIVEN DISTRIBUTION

By

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Considerable evidence has been presented by R. A. Fisher[1] to show that, by an appropriate transformation $z = f(r)$ of small sample correlation coefficients $r\,(-1 \leqq r \leqq 1)$ distributed in accord with a decidedly skew frequency curve, values of z are obtained which are distributed nearly in a normal distribution. In fact, the approach of the distribution of z to normality seems sufficiently rapid to justify the use of the probable error of z in many applications as if it were normally distributed. Such a change in the character of the distribution of an important statistic suggests the further study of properties of the distribution of variables obtained by applying rather simple transformations to variates distributed from -1 to $+1$ in accord with a given frequency function. In a previous paper,[2] the writer has dealt with a similar problem when each variate of a given unimodal distribution of any finite range is replaced by a given power of the variate.

Consider a positive unimodal continuous frequency function

[1] Metron, Vol. 1, Part 4 (1921) pp. 3-32.
[2] Proceedings of the National Academy, Vol. 13, No. 12 (1927), pp. 817-820.

$y = \psi(x)$ of a system of variates $x_1, x_2, \ldots x_n$ with a range of -1 to $+1$, with $\psi(-1) = \psi(1) = 0$, with a single mode at some point, say at $x = b$ $(-1 < b < 1)$, and with the derivative $\psi'(x)$ continuous. More precisely, we assume that $\psi(x)$ is positive except at the end points of the interval -1 to $+1$, where it is zero, and that $\psi'(x)$ changes from positive to negative at $x = b$, and is non-negative or non-positive at any point $x = a$ according as a is less or greater than b.

It is the main object of the present paper to consider certain properties of the distribution of variates $u_i = (ex_i + f)/(gx_i + h)$ obtained by a linear fractional transformation of the x's, where e, f, g, and h are real numbers so selected that $u = (ex + f)/(gx + h)$ is continuous from $x = -1$ to $x = 1$.

When $g = 0$, we have the case of the linear transformation which simply has an effect equivalent to a change of origin and of unit of measurement. As we are not in the present problem much interested in such a simple transformation, we shall, in general, assume $g \neq 0$. Moreover, we take g positive, since this involves no loss of generality.

We shall, except as otherwise stated, restrict our considerations to the interval for u that corresponds to $-1 \leqq x \leqq 1$, and to such transformations that the derivative of u with respect to x is finite for each value of x and that u increases when x increases. These restrictions require that

$$\frac{du}{dx} = \frac{he - fg}{(gx + h)^2}$$

where $g < |h|$ and where the determinant

$$he - fg = \begin{vmatrix} e & f \\ g & h \end{vmatrix} > 0 \tag{1}$$

Starting then with

(2) $$u = \frac{ex+f}{gx+h},$$

we have

(3) $$x = \frac{f-hu}{gu-e}$$

Next, let

(4) $$v = \phi(u)$$

be the frequency function of the new variates u. Then we may write[1]

(5) $$v = \phi(u) = \psi\left(\frac{f-hu}{gu-e}\right) \cdot \frac{he-fg}{(gu-e)^2}.$$

Since $he-fg > 0$, we know that v is positive throughout the interval in which we are interested except that $V = 0$ at the end points. From (5) it seems that the new distribution function may possibly become infinite when $u = e/g$, but the question then arises as to whether e/g is an admissible value of u.

We shall prove that e/g is not an admissible value of u by showing that u cannot take the value e/g within the interval $u = (f-e)/(h-g)$ to $u = (e+f)/(g+h)$ wherein u lies when $-1 \leqq x \leqq 1$. In this connection we shall also establish some inequalities that will be found useful in the consideration of certain properties of the new distribution. Consider first the cases in which $g+h$ is positive.

Then since $eh > fg$, we have $eh + eg > fg + eg$.

Divide by $g(g+h)$, and we have $\frac{e}{g} > \frac{f+e}{g+h}$ Hence,

[1] cf. Annals of Mathematics, vol. 23, No. 4 (1922), pp. 293-4.

e/g is too large when $g+h$ is positive to be an admissible value of u.

Consider next the cases in which $g+h$ is negative. In this case, $h<0$ since $g>0$. Hence $g-h>0$. Then since $eh>fg$, we have $eh-eg>fg-eg$. Divide by the positive number $g(g-h)$ This gives $\frac{-e}{g}>\frac{f-e}{g-h}$ and $\frac{e}{g}<\frac{e-f}{g-h}$.
Hence, when $g(g+h)<0$, e/g is too small to be an admissible value of u.

To summarize with $g>0$, we have shown that:

(a) When $g+h$ is positive, e/g is too large to be an admissible value of u.

(b) When $g+h$ is negative, e/g is too small to be an admissible value of u.

Returning now to the consideration of our frequency function $v=\psi\left(\frac{f-hu}{gu-e}\right)\cdot\frac{he-fg}{(gu-e)^2}$ in (5), we obtain

$$(6)\quad \frac{dv}{du}=\frac{(he-fg)^2}{(gu-e)^4}\,\psi'\left(\frac{f-hu}{gu-e}\right)-\frac{2g(he-fg)}{(gu-e)^3}\,\psi\left(\frac{f-hu}{gu-e}\right).$$

When u takes the value $(eb+f)/(gb+h)$ into which variates at the mode $x=b$ are transformed, we know that $\psi'\left(\frac{f-hu}{gu-e}\right)=\psi'(b)=0$.

By making use of the fact that $he-fg>0$, and the propositions (a) and (b) relating to the inadmissibility of e/g as a value of u in an examination of the right hand member of (6) for $u=(eb+f)/(gb+h)$ we establish the following proposition in regard to the sign of the derivative dv/du for the value of u which corresponds to the modal value of x

When $g+h\neq 0$, dv/du is positive or negative at $u=(eb+f)/(gb+h)$ according as $g+h$ is positive or negative.

The truth of this proposition follows readily by applying (a) and (b) to (6), remembering that g is positive and that $\psi'(b)$ vanishes.

We shall show next in case $g+h > 0$, that dv/du is non-negative for all admissible values of u less than $(eb+f)/(gb+h)$. To see this from (6), note first that $\psi'[(f-hu)/(gu-e)]$ remains non-negative for $(f-hu)/(gu-e) < b$ or for u less than $(eb+f)/(gb+h)$, and note second that $g/(gu-e)^3$ is negative since e/g is too large to be an admissible value of u under the condition $g+h > 0$.

Next, in case $g+h < 0$, dv/du is non-positive for all values of $u > (eb+f)/(gb+h)$. To see this from (6), note first that $\psi'[(f-hu)/(gu-e)]$ remains non-positive for $(f-hu)/(gu-e) > b$ or for $u > \frac{(eb+f)}{(gb+h)}$, and note second that $g/(gu-e)^3$ is positive when $g+h < 0$ because in this case $u > e/g$.

To summarize, when $g+h \neq 0$, we state the

Theorem I. *When the derivative* dv/du is *positive for the value of* u *into which variates at the modal value* $x=b$ *transform, then* dv/du *is non-negative for all smaller values of* u. *Similarly, when* dv/du *is negative for the value of* u *into which variates at the modal value* $x=b$ *transform, then* dv/du *is non-positive for all larger values of* u.

Finally, we wish to inquire about a modal value for the frequency function $v=\phi(u)$ in (5). To this end, consider first the case in which dv/du is positive at $u=(eb+f)/(gb+h)$. At a point between $u=(eb+f)/(gb+h)$ and the upper bound of u, that is $(e+f)/(g+h)$, a maximum value of v occurs. To see this, note when $u = (e+f)/(g+h)$ that $dv/du = \psi'(1)(g+h)^4/(he-fg)^2$ which is negative, or zero since $\psi'(1)$ is negative or zero. If it is negative, there is a maximum where the sign of the continuous first derivative changes from positive to negative. If dv/du is zero at $u = (e+f)/(g+h)$, it follows also that there is at least one maximum of $v = \phi(u)$ between $u = (eb+f)/(gb+h)$ and $u = (e+f)/(g+h)$ since $v = 0$ at $u = (e+f)/(g+h)$ and v must have changed from an increasing positive function at $u = (eb+f)/(gb+h)$ to a decreasing function before becoming zero at $u = (e+f)/(g+h)$. Similarly, it may be shown that there is a mode at a value of $u < (eb+f)/(gb+h)$ whenever dv/du is negative at $u = (eb+f)/(gb+h)$.

We may then state the following:

Theorem II. *Given a unimodal continuous positive function* $y = \psi(x)$ *of variates* x, *with a range from* -1 *to* $+1$, *with a mode at* $x = b\ (-1 < b < 1)$, *with* $\psi(-1) = \psi(1) = 0$, *and with the derivative* $\psi'x$ *continuous from* $x = -1$ *to* $x = 1$, *then the frequency distribution* $v = \phi(u)$ *of variates* $u = (ex+f)/(gx+h)$ $(g > 0)$ *has a mode at a value of* $u > (eb+f)/(gb+h)$ *when* $g+h > 0$. *It has a mode at a value of* $u < (eb+f)/(gb+h)$ *when* $g+h < 0$.

Since we have so restricted our transformation $u = \dfrac{(ex+f)}{(gx+h)}$ that the order of corresponding values is preserved, the transformation carries the median of the distribution of x's into the median of the distribution of u's, and we may state the following:

Corollary. *If* $y = \psi(x)$ *has its median and mode coincident at* $x = b$, *the frequency distribution* $v = \phi(u)$ *of* $u = (ex+f)/(gx+h)$ *has a modal value greater or less than its median according as* $g+h$ *is greater or less than zero.*

Thus far we have imposed the condition $g < |h|$. Let us next consider the cases in which $h = -g$ and $h = g$ instead of requiring that $g < |h|$. Consider first the case $h = -g$. In this case

$$u = \frac{1}{g} \cdot \frac{ex+f}{x-1} \tag{7}$$

and

$$\frac{du}{dx} = \frac{he-fg}{(gx+h)^2} = -\frac{e+f}{g(x-1)^2}. \tag{8}$$

Both u and du/dx become infinite as x approaches 1. Suppose e and f so chosen that u is an increasing function of x for the interval $-1 \leqq x < 1$, then u in (7) is an increasing function of x for the larger interval $-\infty < x < 1$; and it follows, for the case $h = -g$, that e/g is too small to be an admissible value of u when $-1 \leqq x < 1$, since it is the value of u when $x = -\infty$.

For the case $h = g$, we have

$$u = \frac{ex+f}{g(x+1)} \tag{9}$$

and

$$\frac{du}{dx} = \frac{e-f}{g(x+1)^2}. \tag{10}$$

Since u in (9) is an increasing continuous function of x for the interval $-1 < x < \infty$ wherever e and f are so selected that it is increasing for the sub-interval $-1 < x \leqq 1$, it follows, for $h = g$, that e/g, the value of u when $x = \infty$, is too large to be an admissible value of u when $-1 < x \leqq 1$. By making use of the fact that e/g is too small or too large to be an admissible value of u according as $h = -g$ or $+g$, we readily obtain the following results from an examination of (6): The derivative dv/du given in (6) is positive at the point $u = (eb+f)/(gb+h)$ when $h = g$, and it is negative at this point when $h = -g$

Moreover it readily follows as in the case where $g < |h|$ that when the derivative dv/du is positive for the value of u into which the modal $x = b$ transforms, then dv/du is non-negative for all smaller values of u, and when dv/du is negative for the value of u into which the modal value $x = b$ transforms, it is non-positive for all larger values of u.

Next, for the case $h = g$, a mode occurs for a value of $u > (eb+f)/(gb+h)$. This may be seen by noting that as x approaches 1 and as u takes corresponding values dv/du in (6) approaches the value $16\,g^2\,\psi'(1)/(e-f)^2$ which is negative or zero. The analysis given above for the corresponding case $g < |h|$ may be applied, with the conclusions stated in Theroem II by replacing $g+h > 0$ by $h = g$ and $g+h < 0$ by $h = -g$

The question very naturally arises as to whether there exists a linear fractional transformation $u = (ex+f)/(gx+h)$ that will transform almost any distribution with the properties of $y = \psi(x)$ into a new distribution $v = \phi(u)$ with a mode at a previously assigned point $u = c$ within the range of admissible values of u. To insure a mode for $v = \phi(u)$ at $u = c$, it is, of course, sufficient that there exist values of e, f, g, and h that make the continuous function

$$(11)\quad \frac{dv}{du} = \frac{(he-fg)^2}{(gu-e)^4}\,\psi'\left(\frac{f-hu}{gu-e}\right) - \frac{2g(he-fg)}{(gu-e)^3}\,\psi\left(\frac{f-hu}{gu-e}\right)$$

change sign from positive to negative at $u = c$

Since the only restrictions on e, f, g, and h are that they shall be real, and that g and $he - fg$ shall be positive, it seems that the requirement that dv/du shall change from positive to negative at an assigned value of u could probably be satisfied for some important classes of relatively simple functions. As a simple example, take the quadratic function $\psi(x) = Ax^2 + Bx + C$, which, when subjected to the conditions on $\psi(x)$, becomes $\psi(x) = 3(1-x^2)/4$.

The mode is in this case at $x = 0$. The problem we propose is to find the linear fractional transformation $u = (ex+f)/(gx+h)$ that will transform $\psi(x)$ into $v = \phi(u)$ with a mode at an assigned $u = c$. In this case (11) becomes

$$\frac{dv}{du} = -\frac{3}{2}\frac{he-fg}{(gu-e)^5}\left\{(he-fg)(f-hu) - g\left[(g^2-h^2)u^2 + 2u(fh-eg) + e^2 - f^2\right]\right\}. \tag{12}$$

To facilitate the examination of (12), make $h = g$. Then (12) reduces to

$$\frac{dv}{du} = -\frac{3}{2}\frac{g^2(e-f)^2}{(gu-e)^5}(e+2f-3gu). \tag{13}$$

Since $g+h > 0$, we have $gu - e < 0$, and consequently the coefficient of $(e+2f-3gu)$ is positive. To provide for the change of sign of (13) at $u = c$, select e, f, and g so that $e + 2f = 3cg$. To make (13) positive at $u = c - \delta$ and negative at $u = c + \delta$, where δ is arbitrarily small and positive, we may assign to g any positive value and to e any value greater than cg, for then f is less than e, which is the condition $he - fg > 0$ when $h = g$. While there are thus an infinite number of ways in which we may select a linear fractional transformation so that, when applied to special functions, it will give a new distribution with a mode at an assigned point, no general proposition is proved that assures an assigned modal value of $\psi(x)$.

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