so that

$$s'_0 = 7,\ s'_1 = -8,\ \text{and}\ s'_2 = 70;$$

consequently

$$\mu_1 = 2650 - \frac{8}{7} = 2649.$$

$$\mu_2 = \frac{70}{7} - \left(-\frac{8}{7}\right)^2 = 9.$$

The mean deviation is consequently ± 3.

2. In an alternative experiment the result is either "yes", which counts 1, or "no", which counts 0. Out of $m + n$ repetitions the m have given "yes", the n "no". What then is the expression for the law of errors in half-invariants?

Answer: $\mu_1 = \frac{m}{m+n},\ \mu_2 = \frac{mn}{(m+n)^2},\ \mu_3 = \frac{mn(n-m)}{(m+n)^3},\ \mu_4 = \frac{mn(m^2 - 4mn + n^2)}{(m+n)^4}$.

3. Determine the law of errors, in half-invariants, of a voting in which a voters have voted for a motion $(+1)$, c against (-1), while b have not voted (0), and examine what values for a, b, and c give the nearest approximation to the typical form.

$$\mu_1 = \frac{a-c}{a+b+c},\ \mu_2 = \frac{ab + 4ca + bc}{(a+b+c)^2},\ \mu_3 = \frac{(c-a)(ab + 8ca + bc - b^2)}{(a+b+c)^3},$$

$$\mu_4 = -\frac{((a+c)(a+b+c) - 4(a-c)^2)(a+b+c)(2a - b + 2c) + 6(a-c)^4}{(a+b+c)^4}.$$

Disregarding the case when the vote is unanimous, the double condition $\mu_3 = \mu_4 = 0$ is only satisfied when one sixth of the votes is for, another sixth against, while two thirds do not give their votes. If μ_3 is to be $= 0$, without a being $= c$, $b^2 - b(a+c) - 8ac$ must be $= 0$. But then $\mu_4 = -2\mu_2 \left(\frac{a-c}{a+b+c}\right)^2$, which does not disappear unless two of the numbers a, b, and c, and consequently μ_2, are $= 0$.

4. Six repetitions give the quite symmetrical and almost typical law of errors, $\mu_1 = 0$, $\mu_2 = \frac{1}{3}$, $\mu_3 = \mu_4 = \mu_5 = 0$, but $\mu_6 = -\frac{2}{9}$. What are the observed values?

Answer: $-1, 0, 0, 0, 0, +1$.

VII. RELATIONS BETWEEN FUNCTIONAL LAWS OF ERRORS AND HALF-INVARIANTS.

§ 24. The multiplicity of forms of the laws of errors makes it impossible to write a Theory of Observations in a short manner. For though these forms are of very different value, none of them can be considered as absolutely superior to the others. The functional form which has been universally employed hitherto, and by the most prominent writers, has in my opinion proved insufficient. I shall here endeavour to replace it by the half-invariants.

But even if I should succeed in this endeavour, I am sure that not only the functional laws of errors, but even the curves of errors and the tables of frequency are too important and natural to be put completely aside without detriment.

Moreover, in proposing a new plan for this theory, I have felt it my duty to explain as precisely and completely as possible its relation to the old and commonly known methods. I therefore consider it a matter of great importance that even the half-invariants, in their very definition, present a natural transition to the frequencies and to the functional law of errors.

If in the equation (18)

$$n e^{\frac{\mu_1}{1!}\tau + \frac{\mu_2}{2!}\tau^2 + \cdots} = e^{o_1\tau} + \ldots + e^{o_n\tau}$$

some of the o_i's are exactly repeated, it is of course understood that the term $e^{o_i\tau}$ must be counted not once but as often as o_i is repeated. Consequently, this definition of the half-invariants may, without any change of sense, be written

$$\Sigma\varphi(o_i)\cdot e^{\frac{\mu_1}{1!}\tau + \frac{\mu_2}{2!}\tau^2 + \frac{\mu_3}{3!}\tau^3 + \cdots} = \Sigma\varphi(o_i)\, e^{o_i\tau} \tag{28}$$

where the frequencies $\varphi(o_i)$ are given in the form of the functional law of errors. For continuous laws of errors the definition must be written

$$e^{\frac{\lambda_1}{1!}\tau + \frac{\lambda_2}{2!}\tau^2 + \frac{\lambda_3}{3!}\tau^3 + \cdots}\cdot\int_{-\infty}^{+\infty}\varphi(o)\,do = \int_{-\infty}^{+\infty}\varphi(o)\,e^{o\tau}\,do. \tag{29}$$

Thus, if we know the functional law of errors and if we can perform the integrations, the half-invariants may be found. If, inversely, we know the λ_i, then it may be possible also to determine the functional law of errors $\varphi(o)$.

Example 1. Let $\varphi(o)$ be a sum of typical functional laws of errors,

$$\varphi(o) = \Sigma h_i e^{-\frac{1}{2}\left(\frac{o-m_i}{n_i}\right)^2},$$

then $\int_{-\infty}^{+\infty}\varphi(o)\,do = \sqrt{2\pi}\,\Sigma n_i h_i$ and

$$\int_{-\infty}^{+\infty}\varphi(o)\,e^{o\tau}\,do = \Sigma h_i\int_{-\infty}^{+\infty} e^{-\frac{1}{2n_i^2}\left((o-m_i)^2 - 2n_i^2\tau o\right)}\,do$$

$$= \Sigma h_i e^{m_i\tau + \frac{n_i^2}{2}\tau^2}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{o-m_i-n_i^2\tau}{n_i}\right)^2}\,do,$$

and consequently

$$e^{\frac{\lambda_1}{1!}\tau + \frac{\lambda_2}{2!}\tau^2 + \frac{\lambda_3}{3!}\tau^3 + \cdots} = \frac{\Sigma h_i n_i e^{m_i\tau + \frac{n_i^2}{2}\tau^2}}{\Sigma h_i n_i}.$$

By aid of the formulæ (19) that express $\frac{s_r}{s_0}$ as functions of the λ (or μ) it is not difficult to compute the principal half-invariants. The inverse problem, to compute the m_i, n_i, and h_i by means of given half-invariants is very difficult, as it results in equations of a high degree, even if only a sum of two typical functional laws of errors is in question.

Example 2. What are the half-invariants of a pure binomial law of errors? The observation r being repeated $\beta_n(r)$ times, we write

$$2^n e^{\frac{\mu_1}{\underline{|1}}\tau+\frac{\mu_2}{\underline{|2}}\tau^2+\frac{\mu_3}{\underline{|3}}\tau^3+\cdots} = \beta_n(0)+\beta_n(1)e^{\tau}+\ldots+\beta_n(n)e^{n\tau} = (1+e^{\tau})^n,$$

consequently

$$\left(\mu_1-\frac{n}{2}\right)\tau+\frac{\mu_2}{\underline{|2}}\tau^2+\frac{\mu_3}{\underline{|3}}\tau^3+\ldots = n\log\cos\frac{\tau\sqrt{-1}}{2}.$$

Here the right hand side of the equation can be developed by the aid of Bernoullian numbers into a series containing only the even powers of τ, consequently

$$\mu_1 = -\frac{n}{2} \text{ and } \mu_{2r+1} = 0, \quad (r>0)$$

further

$$\mu_2 = \frac{n}{4}, \ \mu_4 = -\frac{n}{8}, \ \mu_6 = \frac{n}{4}, \ \mu_8 = -\frac{17}{16}n, \ \mu_{10} = \frac{31}{4}n, \ \ldots$$

Example 3. What are the half-invariants of a complete binomial law of errors (the complete terms of $(p+q)^n$)? Here

$$e^{\frac{\mu_1}{\underline{|1}}\tau+\frac{\mu_2}{\underline{|2}}\tau^2+\frac{\mu_3}{\underline{|3}}\tau^3+\cdots} = \left(\frac{p+qe^{\tau}}{p+q}\right)^n.$$

From this we obtain by differentiation with regard to τ

$$\mu_1+\frac{\mu_2}{\underline{|1}}\tau+\frac{\mu_3}{\underline{|2}}\tau^2+\frac{\mu_4}{\underline{|3}}\tau^3+\ldots = \frac{nqe^{\tau}}{p+qe^{\tau}},$$

by further differentiation

$$\mu_{s+1}+\frac{\mu_{s+2}}{\underline{|1}}\tau+\ldots = \frac{d^s\frac{nqe^{\tau}}{p+qe^{\tau}}}{d\tau^s};$$

putting $\tau=0$ we get

$$\mu_1 = \frac{np}{p+q}$$

$$\mu_2 = \frac{npq}{(p+q)^2}$$

$$\mu_3 = \frac{npq(p-q)}{(p+q)^3}$$

$$\mu_4 = \frac{npq}{(p+q)^2}\left(\left(\frac{p-q}{p+q}\right)^2 - \frac{2pq}{(p+q)^2}\right)$$

$$\mu_5 = \frac{npq}{(p+q)^2}\left(\left(\frac{p-q}{p+q}\right)^3 - \frac{8pq(p-q)}{(p+q)^3}\right)$$

. . . .

Example 4. A law of presumptive errors is given by its half-invariants forming a geometrical progression, $\lambda_r = ba^r$. Determine the several observations and their frequencies. Here the left hand side of the equation (18) is

$$s_0 e^{b\frac{a\tau}{\underline{|1}} + b\frac{(a\tau)^2}{\underline{|2}} + b\frac{(a\tau)^3}{\underline{|3}} + \cdots} = s_0 e^{-b} e^{be^{a\tau}},$$

but this is $= s_0 e^{-b}\left(1 + be^{a\tau} + \frac{b^2}{\underline{|2}} e^{2a\tau} + \frac{b^3}{\underline{|3}} e^{3a\tau} + \ldots\right)$ and has also the form of the right side of (18). Thus the observed values are 0, a, $2a$, $3a$, ... and the relative frequency of ra is $\frac{b^r}{\underline{|r}} = \varphi(r)$. This law of errors is nearly related to the binomial law, which can be considered as a product of two factors of this kind,

$$\frac{b^r}{\underline{|r}} \cdot \frac{d^{n-r}}{\underline{|n-r}} = \frac{1}{\underline{|n}} \beta_n(\nu) b^r d^{n-r}.$$

It is perhaps superior to the binomial law as a representative of some skew laws of errors.

Example 5. A law of errors has the peculiarity that all half-invariants of odd order are $= 0$, while all even half-invariants are equal to each other, $\lambda_{2r} = 2a$. Show that all the observations must be integral numbers, and that for the relative frequencies

$$\varphi(0) = e^{-2a}\left(1 + \left(\frac{a}{\underline{|1}}\right)^2 + \left(\frac{a^2}{\underline{|2}}\right)^2 + \cdots\right)$$

$$\varphi(\pm r) = e^{-2a}\left(\frac{a^r}{\underline{|0}\,\underline{|r}} + \frac{a^{r+2}}{\underline{|1}\,\underline{|r+1}} + \frac{a^{r+4}}{\underline{|2}\,\underline{|r+2}} + \cdots\right).$$

Example 6. Determine the half-invariants of the law of presumptive errors for the irrational values in the table of a function, in whose computation fractions under $\frac{1}{2}$ have been rejected and those over $\frac{1}{2}$ replaced by 1:

$$\lambda_{2r+1} = 0,\ \lambda_2 = \tfrac{1}{12},\ \lambda_4 = -\tfrac{1}{120},\ \lambda_6 = \tfrac{1}{252},\ \cdots$$

§ 25. As a most general functional form of a continuous law of errors we have proposed (6)

$$\Phi(x) = k_0\varphi(x) - \frac{k_1}{\underline{|1}} D\varphi(x) + \frac{k_2}{\underline{|2}} D^2\varphi(x) - \frac{k_3}{\underline{|3}} D^3\varphi(x) + \ldots,$$

where $\varphi(x) = e^{-\frac{1}{2}\left(\frac{x-m}{n}\right)^2}$.

Now it is a very remarkable thing that we can express the half-invariants without any ambiguity as functions of the coefficients k_i, and vice versa.

By (29) we get

$$s_0 e^{\frac{\lambda_1}{\underline{|1}}\tau + \frac{\lambda_2}{\underline{|2}}\tau^2 + \cdots} = \int_{-\infty}^{+\infty} e^{o\tau} \left(k_0 \varphi(o) - \frac{k_1}{\underline{|1}} D\varphi(o) + \frac{k_2}{\underline{|2}} D^2 \varphi(o) \ldots\right) do,$$

where $s_0 = nk_0\sqrt{2\pi}$. By means of the lemma

$$\int e^{o\tau} D^r \varphi(o)\, do = e^{o\tau}\left\{(D^{r-1}\varphi(o) - \tau D^{r-2}\varphi(o) + \ldots + (-\tau)^{r-1}\varphi(o)\right\} + (-\tau)^r \int e^{o\tau}\varphi(o)\, do,$$

which is easily demonstrated for any $\varphi(o)$ by differentiating with regard to o only, we have in this particular case, where $\varphi(o)$ and every $D^i\varphi(o)$ is $= 0$, if $o = \pm\infty$,

$$\int_{-\infty}^{+\infty} e^{o\tau} D^r e^{-\frac{1}{2}\left(\frac{o-m}{n}\right)^2} do = (-\tau)^r \int_{-\infty}^{+\infty} e^{o\tau - \frac{1}{2}\left(\frac{o-m}{n}\right)^2} do = (-\tau)^r n\sqrt{2\pi}\, e^{m\tau + \frac{n^2}{2}\tau^2}.$$

Consequently, the relation between the half-invariants on one side and the coefficients k of the general functional law of errors on the other, is

$$k_0 e^{\frac{\lambda_1}{\underline{|1}}\tau + \frac{\lambda_2}{\underline{|2}}\tau^2 + \frac{\lambda_3}{\underline{|3}}\tau^3 + \cdots} = \left(k_0 + \frac{k_1}{\underline{|1}}\tau + \frac{k_2}{\underline{|2}}\tau^2 + \frac{k_3}{\underline{|3}}\tau^3 \ldots\right) e^{m\tau + \frac{1}{2}n^2\tau^2}. \qquad (30)$$

If we write here $\lambda'_1 = \lambda_1 - m$ and $\lambda'_2 = \lambda_2 - n^2$, the computation of one set of constants by the other can, according to (17), be made by the formulæ (19) and (21). We sbbstitute only in these the k_i for the s_i, and λ' or λ for μ.

It will be seen that the constants m and n, and the special typical law of errors to which they belong, are generally superfluous. This superfluity in our transformation may be useful in special cases for reasons of convergency, but in general it must be considered a source of vagueness, and the constants must be fixed arbitrarily.

It is easiest and most natural to put

$$m = \lambda_1 \text{ and } n^2 = \lambda_2.$$

In this case we get $k_1 = 0$, $k_2 = 0$, $k_3 = k_0\lambda_3$, $k_4 = k_0\lambda_4$, $k_5 = k_0\lambda_5$, and further

$$k_6 = k_0(\lambda_6 + 10\lambda_3^2)$$
$$k_7 = k_0(\lambda_7 + 35\lambda_4\lambda_3)$$
$$k_8 = k_0(\lambda_8 + 56\lambda_5\lambda_3 + 35\lambda_4^2)$$
$$\cdots\cdots\cdots\cdots$$

The law of the coefficients is explained by writing the right side of equation (30)

$$k_0 e^{m\tau + \frac{n^2}{2}\tau^2 + \log\left(k_0 + \frac{k_3}{\underline{|3}}\tau^3 + \frac{k_4}{\underline{|4}}\tau^4 + \ldots\right) - \log k_0}$$

Expressed by half-invariants in this manner the explicit form of equation (6) is

$$\varphi(x) = \frac{s_0}{\sqrt{2\pi\lambda_2}} e^{-\frac{1}{2}\frac{(x-\lambda_1)^2}{\lambda_2}} \left\{ \begin{array}{l} 1 + \frac{\lambda_3}{6\lambda_2^3}((x-\lambda_1)^3 - 3\lambda_2(x-\lambda_1)) + \\ + \frac{\lambda_4}{24\lambda_2^4}((x-\lambda_1)^4 - 6\lambda_2(x-\lambda_1)^2 + 3\lambda_2^2) + \\ + \frac{\lambda_5}{120\lambda_2^5}((x-\lambda_1)^5 - 10\lambda_2(x-\lambda_1)^3 + 15\lambda_2^2(x-\lambda_1)) + \dots \end{array} \right\} \quad (31)$$

VIII. LAWS OF ERRORS OF FUNCTIONS OF OBSERVATIONS.

§ 26. There is nothing inconsistent with our definitions in speaking of laws of errors relating to any group of quantities which, though not obtained by repeated observations, have the like property, namely, that repeated estimations of a single thing give rise, owing to errors of one kind or other, to multiple and slightly differing results which are *prima facie* equally valid. The various forms of laws of actual errors are indeed only summary expressions for such multiplicity; and the transition to the law of presumptive errors requires, besides this, only that the multiplicity is caused by fixed but unknown circumstances, and that the values must be mutually independent in that sense that none of the circumstances have connected some repetitions to others in a manner which cannot be common to all. Compare § 24, Example 6.

It is, consequently, not difficult to define the law of errors for a function of *one* single observation. Provided only that the function is univocal, we can from each of the observed values $o_1, o_2 \dots o_n$ determine the corresponding value of the function, and

$$f(o_1),\ f(o_2),\ \dots\ f(o_n)$$

will then be the series of repetitions in the law of errors of the function, and can be treated quite like observations.

With respect, however, to those forms of laws of errors which make use of the idea of frequency (probability) we must make one little reservation. Even though o_i and o_k are different, we can have $f(o_i) = f(o_k)$, and in this case the frequencies must evidently be added together. Here, however, we need only just mention this, and remark that the laws of errors when expressed by half-invariants or other symmetrical functions are not influenced by it.

Otherwise the frequency is the same for $f(o_i)$ as for o_i, and therefore also the probability. The ordinates of the curves of errors are not changed by observations with discontinuous values; but the abscissa o_i is replaced by $f(o_i)$, and likewise the argument in the functional law of errors. In continuous functions, on the other hand, it is the areas between corresponding ordinates which must remain unchanged.