draw a curve of errors which, as a rule, will deviate very little from the original. All this, however, holds good only of the curves of presumptive errors. With the actual ones we cannot operate in this way, and the transition from the latter to the former seems in the meantime to depend on the eye's sense of beauty.

V. FUNCTIONAL LAWS OF ERRORS,

§ 17. Laws of errors may be represented in such a way that the frequency of the results of repetitions is stated as a mathematical function of the number, or numbers, expressing the results. This method only differs from that of curves of errors in the circumstance that the curve which represents the errors has been replaced by its mathematical formula; the relationship is so close that it is difficult, when we speak of these two methods, to maintain a strict distinction between them.

In former works on the theory of observations the functional law of errors is the principal instrument. Its source is mathematical speculation; we start from the properties which are considered essential in ideally good observations. From these the formula for the typical functional law of errors is deduced; and then it remains to determine how to make computations with observations in order to obtain the most favourable or most probable results.

Such investigations have been carried through with a high degree of refinement; but it must be regretted that in this way the real state of things is constantly disregarded. The study of the curves of actual errors and the functional forms of laws of actual errors have consequently been too much neglected.

The representation of functional laws of errors, whether laws of actual errors or laws of presumptive errors founded on these, must necessarily begin with a table of the results of repetitions, and be founded on interpolation of this table. We may here be content to study the cases in which the arguments (i. e. the results of the repetitions) proceed by constant differences, and the interpolated function, which gives the frequency of the argument, is considered as the functional law of errors. Here the only difficulty we encounter is that we cannot directly employ the usual Newtonian formula of interpolation, as this supposes that the function is an integral algebraic one, and gives infinite values for infinite arguments, whether positive or negative, whereas here the frequency of these infinite arguments must be $= 0$. We must therefore employ some artifice, and an obvious one is to interpolate, not the frequency itself, y, but its reciprocal, $\frac{1}{y}$. This, however, turns out to be inapplicable; for $\frac{1}{y}$ will often become infinite for finite arguments, and will, at any rate, increase much faster than any integral function of low degree.

But, as we have already said, the interpolation generally succeeds, when we apply it to the logarithm of the frequency, assuming that

$$\operatorname{Log} y = a + bx + cx^2 + \ldots + gx^{2n},$$

where the function on the right side begins with the lowest powers of the argument x, and ends with an even power whose coefficient g must be *negative*. Without this latter condition the computed frequency,

$$y = 10^{a+bx+cx^2+\ldots+gx^{2n}}, \tag{1}$$

would again become infinitely great for $x = \pm\infty$. That the observed frequency is often $= 0$, and its logarithm $-\infty$ like $\frac{1}{y}$, does no harm. Of course we must leave out these frequencies of the interpolation, or replace them by very small finite frequencies, a few of which it may become necessary to select arbitrarily. As a rule it is possible to succeed by this means. In order to represent a given law of actual errors in this way, we must, according to the rule of interpolation, determine the coefficients $a, b, c, \ldots g$, whose number must be at least as large as that of the various results of repetitions with which we have to deal. This determination, of course, is a troublesome business.

Here also we may suppose that the law of presumptive errors is simpler than that of the actual errors. And though this, of course, does not imply that log y can be expressed by a small number of terms containing the lowest powers of x, this supposition, nevertheless, is so obvious that it must, at any rate, be tried before any other.

§ 18. Among these, the simplest case, namely that in which Log y is a function of x of the second degree

$$\operatorname{Log} y = a + bx - cx^2,$$

gives us the typical form for the functional law of errors, and for the curve of errors, or with other constants

$$y = he^{-\frac{1}{2}\left(\frac{x-m}{n}\right)^2} = h10^{-0{\cdot}21715\left(\frac{x-m}{n}\right)^2}, \tag{2}$$

where

$$e = 1 + \frac{1}{1} + \frac{1}{1\cdot2} + \frac{1}{1\cdot2\cdot3} + \ldots = 2{\cdot}71828.$$

The function has therefore no other constants than those which may be interpreted as unit for the frequencies h, and as zero m and unit n for the observed values; the corresponding typical curve of errors has therefore in all essentials a fixed form.

The functional form of the typical law of errors has applications in mathematics which are almost as important as those of the exponential, logarithmic, and trigonometrical functions. In the theory of observations its importance is so great that, though it has been over-estimated by some writers, and though many good observations show presumptive as well as actual laws of errors that are not typical, yet every student must make himself perfectly familiar with its properties.

Expanding the index we get

$$e^{-\frac{1}{2}\left(\frac{x-m}{n}\right)^2} = e^{-\frac{1}{2}\left(\frac{m}{n}\right)^2}\cdot e^{\frac{mx}{n^2}}\cdot e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2}, \tag{3}$$

so that the general function resolves itself into a product of three factors, the first of which is constant, the second an ordinary exponential function, while the third remains a typical functional law of errors. Long usage reduces this form to e^{-x^2}; but this form cannot be recommended. In the majority of its purely mathematical applications $e^{-\pi x^2}$ is preferable, unless (as in the whole theory of observations) the factor $\frac{1}{2}$ in the index is to be preferred on account of the resulting simplification of most of the derived formulæ.

The differential coefficients of $e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2}$ with regard to x are

$$\left.\begin{aligned}
De^{-\frac{1}{2}\left(\frac{x}{n}\right)^2} &= -n^{-2}xe^{-\frac{1}{2}\left(\frac{x}{n}\right)^2}\\
D^2e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2} &= \quad n^{-4}(x^2-n^2)e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2}\\
D^3e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2} &= -n^{-6}(x^3-3n^2x)e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2}\\
D^4e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2} &= \quad n^{-8}(x^4-3n^2\cdot 2x^2+1\cdot 3n^4)e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2}\\
D^5e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2} &= -n^{-10}(x^5-5n^2\cdot 2x^3+3\cdot 5n^4x)e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2}\\
D^6e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2} &= \quad n^{-12}(x^6-5n^2\cdot 3x^4+3\cdot 5n^4\cdot 3x^2-1\cdot 3\cdot 5n^6)e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2}
\end{aligned}\right\} \tag{4}$$

The law of the numerical coefficients (products of odd numbers and binomial numbers) is obvious. The general expression of $D^r e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2}$ can be got from a comparison of the coefficients to $(-m)^r$ of the two identical series for equation (3), one being the Taylor series, the other the product of $e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2}$ and the two exponential series with m^2 and m as arguments. It can also be induced from the differential equation

$$n^2D^{r+2}\varphi + xD^{r+1}\varphi + (r+1)D^r\varphi = 0.$$

Inversely, we obtain for the products of the typical law of errors by powers of x

$$\left.\begin{aligned}
x\varphi &= -n^2D\varphi\\
x^2\varphi &= \quad n^4D^2\varphi + n^2\varphi\\
x^3\varphi &= -n^6D^3\varphi - 3n^4D\varphi\\
x^4\varphi &= \quad n^8D^4\varphi + 6n^6D^2\varphi + 3n^4\varphi\\
x^5\varphi &= -n^{10}D^5\varphi - 10n^8D^3\varphi - 15n^6D\varphi\\
x^6\varphi &= \quad n^{12}D^6\varphi + 15n^{10}D^4\varphi + 45n^8D^2\varphi + 15n^6\varphi\\
\varphi &= e^{-\frac{1}{2}\left(\frac{x}{n}\right)^2},
\end{aligned}\right\} \tag{5}$$

the numerical coefficients being the same as above (4). This proposition can be demonstrated by the identical equation $n^{-2}x^{r+1}\varphi = -D(x^r\varphi)+rx^{r-1}\varphi$.

By means of these formulæ every product of any integral rational function by exponential functions and functional typical laws of errors can be reduced to the form

$$k_0\varphi - \frac{k_1}{\lfloor 1} D\varphi + \frac{k_2}{\lfloor 2} D^2\varphi - \frac{k_3}{\lfloor 3} D^3\varphi + \dots, \qquad (6)$$

where

$$\varphi = e^{-\frac{1}{2}\left(\frac{x-m}{n}\right)^2},$$

and thus they can easily be differentiated and integrated. Every quadrature of this form can be reduced to

$$f_1(x)e^{-\frac{1}{2}\left(\frac{x-m}{n}\right)^2} + f_2(x)\int e^{-\frac{1}{2}\left(\frac{x-m}{n}\right)^2} dx,$$

where $f_1(x)$ and $f_2(x)$ are integral rational functions; thus a very large class of problems can be solved numerically by aid of the following table of the typical or exponential functional law of errors, $\eta = e^{-\frac{1}{2}z^2}$, together with the table of its integral $\int_0^z \eta dz$.

z	$\int_0^z \eta dz$	$\eta = e^{-\frac{1}{2}z^2}$	$\frac{d\eta}{dz}$	$\frac{d^2\eta}{dz^2}$	$\frac{d^3\eta}{dz^3}$	$\frac{d^4\eta}{dz^4}$
0·0	0·00000	1·0000	0·000	−1·00	0·0	3
0·1	0·09983	·9950	− ·100	− ·99	·3	3
0·2	0·19867	·9802	− ·196	− ·94	·6	3
0·3	0·29556	·9560	− ·287	− ·87	·8	2
0·4	0·38968	·9231	− ·369	− ·78	1·0	2
0·5	0·47993	·8825	− ·441	− ·66	1·2	1
0·6	0·56586	·8353	− ·501	− ·53	1·3	1
0·7	0·64680	·7827	− ·548	− ·40	1·4	0
0·8	0·72227	·7261	− ·581	− ·26	1·4	−0
0·9	0·79194	·6670	− ·600	− ·13	1·3	−1
1·0	0·85562	0·6065	−0·607	0·00	1·2	−1
1·1	0·91325	·5461	− ·601	·11	1·1	−2
1·2	0·96488	·4868	− ·584	·21	·9	−2
1·3	1·01067	·4296	− ·558	·30	·7	−2
1·4	1·05089	·3753	− ·525	·36	·5	−2
1·5	1·08585	·3247	− ·487	·41	·4	−2
1·6	1·11585	·2780	− ·445	·43	·2	−2
1·7	1·14161	·2357	− ·401	·45	·0	−1
1·8	1·16325	·1979	− ·356	·44	−·1	−1
1·9	1·18133	·1645	− ·313	·43	−·2	−1
2·0	1·19629	0·1353	−0·271	0·41	−·3	−1
2·1	1·20858	·1103	− ·232	·38	−·3	−0
2·2	1·21846	·0889	− ·196	·34	−·4	−0
2·3	1·22643	·0710	− ·163	·30	−·4	−0

z	$\int_0^z \eta dz$	$\eta = e^{-\frac{1}{2}z^2}$	$\frac{d\eta}{dz}$	$\frac{d^2\eta}{dz^2}$	$\frac{d^3\eta}{dz^3}$	$\frac{d^4\eta}{dz^4}$
2·4	1·23277	0·0561	−0·135	0·27	−0·4	0
2·5	1·23775	·0439	− ·110	·23	− ·4	0
2·6	1·24163	·0340	− ·089	·20	− ·3	0
2·7	1·24462	·0261	− ·071	·16	− ·3	0
2·8	1·24691	·0198	− ·056	·14	− ·3	0
2·9	1·24864	·0149	− ·043	·11	− ·2	0
3·0	1·24993	0·0111	−0·033	0·09	−0·2	0·3
3·1	1·25089	·0082	− ·025	·07	− ·2	0
3·2	1·25159	·0060	− ·019	·06	− ·1	0
3·3	1·25210	·0043	− ·014	·04	− ·1	0
3·4	1·25247	·0031	− ·011	·03	− ·1	0
3·5	1·25273	·0022	− ·008	·02	− ·1	0
3·6	1·25292	·0015	− ·006	·02	− ·1	0
3·7	1·25304	·0011	− ·004	·01	− ·0	0
3·8	1·25313	·0007	− ·003	·01	− ·0	0
3·9	1·25319	·0005	− ·002	·01	− ·0	0
4·0	1·25323	0·0003	−0·001	0·01	$-\frac{1}{60}$	$\frac{1}{18}$
4·1	1·253[illegible]	·0002	− ·001	·00		
4·2	1·25328	·0001	− ·001	·00		
4·3	1·25329	·0001	− ·000	·00		
4·4	1·25330	·0001	− ·000	·00		
4·5	1·25331	·0000	− ·000	·00		
∞	$\sqrt{\frac{1}{2}\pi}$	·0000	− ·000	·00	0	0

Here η, $\frac{d^2\eta}{dz^2}$, $\frac{d^4\eta}{dz^4}$ are, each of them, the same for positive and negative values of z; the other columns of the table change signs with z.

The interpolations are easily worked out by means of Taylor's theorem:

$$\eta_{(z+\zeta)} = \eta + \frac{d\eta}{dz}\cdot\zeta + \frac{1}{2}\frac{d^2\eta}{dz^2}\cdot\zeta^2 + \frac{1}{6}\frac{d^3\eta}{dz^3}\cdot\zeta^3 + \frac{1}{24}\frac{d^4\eta}{dz^4}\cdot\zeta^4 + \cdots \quad (7)$$

and

$$\int_0^{z+\zeta}\eta dz = \int_0^z \eta dz + \eta\cdot\zeta + \frac{1}{2}\frac{d\eta}{dz}\cdot\zeta^2 + \frac{1}{6}\frac{d^2\eta}{dz^2}\cdot\zeta^3 + \frac{1}{24}\frac{d^3\eta}{dz^3}\cdot\zeta^4 + \frac{1}{120}\frac{d^4\eta}{dz^4}\cdot\zeta^5 + \cdots \quad (8)$$

The typical form for the functional law of errors (2) shows that the frequency is always positive, and that it arranges itself symmetrically about the value $x = m$, for which the frequency has its maximum value $y = h$. For $x = m \pm n$ the frequency is $y = h\cdot 0{\cdot}60653$. The corresponding points in the curve of errors are the points of inflexion. The area between the curve of errors and the axis of abscissae, reckoned from the middle to $x = m \pm n$, will be $nh\cdot 0{\cdot}85562$; and as the whole area from one asymptote to the other is $nh\sqrt{2\pi} = nh\cdot 2{\cdot}50663$, only $nh\cdot 0{\cdot}39769$ of it falls outside either of the inflexions, consequently not quite that sixth part (more exactly 16 per ct.) which is the foundation of the rule, given in § 11, as to the limit between the great and small errors.

The above table shows how rapidly the function of the typical law of errors decreases toward zero. In almost all practical applications of the theory of observations $\varepsilon^{-\frac{1}{2}z^2} = 0$, if only $z > 5$. Theoretically this superior assymptotical character of the function is expressed in the important theorem that, for $z = \pm\infty$, not only $\varepsilon^{-\frac{1}{2}z^2}$ itself is $= 0$ but also all its differential coefficients; and that, furthermore, all products of this function by every algebraic integral function and by every exponential function, and all the differential quotients of these products, are equal to zero.

In consequence of this theorem, the integral $\int_{-\infty}^{+\infty}\varepsilon^{-\frac{1}{2}z^2}dz = \sqrt{2\pi}$ can be computed as the sum of equidistant values of $\varepsilon^{-\frac{1}{2}z^2}$ multiplied by the interval of the arguments without any correction. This simple method of computation is not quite correct, the underlying series for conversion of a sum into an integral being only semiconvergent in this case; for very large intervals the error can be easily stated, but as far as intervals of one unit the numbers taken out of our table are not sufficient to show this error.

If the curve of errors is to give relative frequency directly, the total area must be $1 = nh\sqrt{2\pi}$; h consequently ought to be put $= \frac{1}{n\sqrt{2\pi}}$.

Problem 1. Prove that every product of typical laws of errors in the functional form $= h\varepsilon^{-\frac{1}{2}\left(\frac{x-m}{n}\right)^2}$, with the same independent variable x, is, itself a typical law of errors. How do the constants h, m, and n change in such a multiplication?

Problem 2. How small are the frequencies of errors exceeding 2, 3, or 4 times the mean error, on the supposition of the typical law of errors?

Problem 3. To find the values of the definite integrals

$$s_r = \int_{-\infty}^{+\infty} x^r e^{-\frac{1}{2}\left(\frac{x-m}{n}\right)^2} dx.$$

Answer: $s_{2i+1} = 0$ and $s_{2i} = 1 \cdot 3 \cdot 5 \ldots (2i-1)\, n^{2i+1}\sqrt{2\pi}$.

§ 19. Nearly related to the typical or exponential law of errors in functional form are the binomial functions, which are known from the coefficients of the terms of the n^{th} power of a binomial, regarded as a function of the number x of the term.

n	$x =$ 0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1
8	1	8	28	56	70	56	28	8
9	1	9	36	84	126	126	84	36
10	1	10	45	120	210	252	210	120
11	1	11	55	165	330	462	462	330
12	1	12	66	220	495	792	924	792
13	1	13	78	286	715	1287	1716	1716
14	1	14	91	364	1001	2002	3003	3432

For integral values of the argument the binomial function can be computed directly by the formula

$$\left.\begin{aligned} \beta_n(x) &= \frac{1 \cdot 2 \cdot 3 \ldots n}{1 \cdot 2 \cdot 3 \ldots x \cdot 1 \cdot 2 \cdot 3 \ldots (n-x)} = \beta_n(n-x) \\ &= \frac{n(n-1)\ldots(n-x+1)}{1 \cdot 2 \ldots x}. \end{aligned}\right\} \quad (9)$$

When the binomial numbers for n are known, those for $n+1$ are easily found by the formula

$$\beta_{n+1}(x) = \beta_n(x) + \beta_n(x-1). \quad (10)$$

By substitution according to (9) we easily demonstrate the proposition that, for any integral values of n, r, and t

$$\beta_n(t)\beta_{n-t}(r) = \beta_n(r)\cdot\beta_{n-r}(t), \tag{11}$$

which means that, when the trinomial $(a+b+c)^n$ is developed, it is indifferent whether we consider it to be $((a+b)+c)^n$ or $(a+(b+c))^n$.

For fractional values of the argument x, the binomial function $\beta_n(x)$ can be taken in an infinity of different ways, for instance by

$$\beta_0(x) = \frac{\sin \pi x}{\pi x}.$$

This formula results from a direct application of Lagrange's method of interpolation, and leads by (10) to the more general formula

$$\beta_n(x) = \frac{1\cdot 2\ldots n}{(1-x)(2-x)\ldots(n-x)}\,\frac{\sin \pi x}{\pi x}. \tag{12}$$

This species of binomial function may be considered the simplest possible, and has some importance in pure mathematics; but as an expression of frequencies of observed values, or as a law of errors, it is inadmissible because, for $x > n$ or x negative, it gives negative values alternating with positive values periodically.

This, however, may be remedied. As $\beta_0(x)$ has no other values than 0 and 1, when x is integral, we can put for instance

$$\beta_0(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2;$$

by (10) then

$$\left.\begin{aligned}\beta_1(x) &= \left(\frac{1}{x^2}+\frac{1}{(x-1)^2}\right)\frac{\sin^2 \pi x}{\pi^2}\\ \beta_2(x) &= \left(\frac{1}{x^2}+\frac{2}{(x-1)^2}+\frac{1}{(x-2)^2}\right)\frac{\sin^2 \pi x}{\pi^2}\end{aligned}\right\} \tag{13}$$

Here the values of the binomial function are constantly positive or 0. But this form is cumbersome; and although for $x = \infty$ the function and its principal coefficients are $= 0$, this property is lost here, when we multiply by integral algebraic or by exponential functions.

These unfavourable circumstances detract greatly from the merits of the binomial functions as expressions for continuous laws of errors.

When, on the contrary, the observations correspond only to integral values of the argument, the original binomial functions are most valuable means for treating them. That $\beta_n(x) = 0$, if $x > n$ or negative, is then of great importance. But this case must be referred to special investigations.

§ 20. To represent non-typical laws of errors in functional form we have now the choice between at least three different plans:

1) the formula (1) or

$$y = e^{\alpha+\beta x+\gamma x^2+\dots-\varkappa x^{2r}},$$

2) the products of integral algebraic functions by a typical function or (6)

$$y = k_0\varphi - \frac{k_1}{\underline{|1}} D\varphi + \frac{k_2}{\underline{|2}} D^2\varphi - \frac{k_3}{\underline{|3}} D^3\varphi + \dots, \qquad \varphi = e^{-\frac{1}{2}\left(\frac{x-m}{n}\right)^2},$$

3) a sum of several typical functions

$$y = \Sigma_1^r h_i e^{-\frac{1}{2}\left(\frac{x-m_i}{n_i}\right)^2}. \tag{14}$$

This account of the more prominent among the functional forms, which we have at our disposal for the representation of laws of errors, may prove that we certainly possess good instruments, by means of which we can even in more than one form find general series adapted for the representation of laws of errors. We do not want forms for the series, required in theoretical speculations upon laws of errors; nor is the exact representation of the actual frequencies more than reasonably difficult. If anything, we have too many forms and too few means of estimating their value correctly.

As to the important transition from laws of actual errors to those of presumptive errors, the functional form of the law leaves us quite uncertain. The convergency of the series is too irregular, and cannot in the least be foreseen.

We ask in vain for a fixed rule, by which we can select the most important and trustworthy forms with limited numbers of constants, to be used in predictions. And even if we should have decided to use only the typical form by the laws of presumptive errors, we still lack a method by which we can compute its constants. The answer, that the "adjustment" of the law of errors must be made by the "method of least squares", may not be given till we have attained a satisfactory proof of that method; and the attempts that have been made to deduce it by speculations on the functional laws of errors must, I think, all be regarded as failures.

VI. LAWS OF ERRORS

EXPRESSED BY SYMMETRICAL FUNCTIONS.

§ 21. All constants in a functional law of errors, every general property of a curve of errors or, generally, of a law of numerical errors, must be symmetrical functions of the several results of the repetitions, i. e. functions which are not altered by interchanging two or more of the results. For, as all the values found by the repetitions correspond to the same essential circumstances, no interchanging whatever can have any influence on the law of errors. Conversely, any symmetrical function of the values of the