A TABLE TO FACILITATE THE FITTING OF CERTAIN LOGISTIC CURVES

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The most useful generalization of the logistic curve is that having the form

$$(1) \qquad y = \frac{k}{1+e^{a+bx+cx^2+gx^3 \cdots}}$$

In practice it will seldom be found necessary to use higher powers of x. This equation may also be written

$$(2) \qquad Y = a + bx + cx^2 + gx^3$$

in which $Y \equiv \log \frac{k-y}{y}$.

If we can evaluate the constant k with reasonable accuracy, the value of Y corresponding to each observed value of y can be computed, and then the values of the coefficients $a, b, c,$ and g, in equation (1) may be obtained by fitting equation (2) as a generalized parabola by the method of least squares.

The normal equations necessary to make this fit will be found to be

$$a\Sigma x^0 + b\Sigma x + c\Sigma x^2 + g\Sigma x^3 = \Sigma Y$$
$$a\Sigma x + b\Sigma x^2 + c\Sigma x^3 + g\Sigma x^4 = \Sigma xY$$
$$a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 + g\Sigma x^5 = \Sigma x^2 Y$$
$$a\Sigma x^3 + b\Sigma x^4 + c\Sigma x^5 + g\Sigma x^6 = \Sigma x^3 Y$$

In the special case where the observations have been made at regular intervals (that is, where the successive values of x are in arithmetic progression) the solution of these normal equations may be greatly simplified. We may then select an arbitrary origin in the middle of the range of observations, so that for every positive value of x there will be a corresponding negative value of equal absolute magnitude. Thus the sums of the odd powers of x will all be zero.

If the number of observations be odd, the middle one will, of course, be chosen for the origin, and the unit of the scale will be the interval between successive values of x . If the number of observations be even, the origin will be midway between the middle pair of observations, and it will be found more convenient to take half the interval as scale unit. In the former case, x will take all integral values between $+n$ and $-n$, while in the latter case x may take only the odd integral values.

If we set the sums of the odd powers of x in the normal equations equal to zero, and solve them simultaneously, we derive the following formulae for the literal coefficients:

$$A = \frac{\Sigma Y \cdot \Sigma X^4 - \Sigma X^2 Y \cdot \Sigma X^2}{\Sigma X^4 \cdot \Sigma X^0 - (\Sigma X^2)^2}, \quad C = \frac{\Sigma X^2 Y \cdot \Sigma X^0 - \Sigma Y \cdot \Sigma X^2}{\Sigma X^4 \cdot \Sigma X^0 - (\Sigma X^2)^2},$$

$$B = \frac{\Sigma XY \cdot \Sigma X^6 - \Sigma X^3 Y \cdot \Sigma X^4}{\Sigma X^6 \cdot \Sigma X^2 - (\Sigma X^4)^2}, \quad G = \frac{\Sigma X^3 Y \cdot \Sigma X^2 - \Sigma XY \cdot \Sigma X^4}{\Sigma X^6 \cdot \Sigma X^2 - (\Sigma X^4)^2}.$$

The use of capital letters indicates that the equation has been referred to the arbitrary origin.

In these formulae the factors involving Y must be computed from the observations, but those in which X alone occurs may be tabulated for all convenient values of n . Since Y does not occur in the denominators at all, these may be tabulated in the same way.

TABLE TO BE USED WHEN THE NUMBER OF OBSERVATIONS IS ODD

n	Σx^0	Σx^2	Σx^4	Σx^6	$\Sigma x^4 \cdot \Sigma x^0 - (\Sigma x^2)^2$	$\Sigma x^6 \cdot \Sigma x^2 - (\Sigma x^4)^2$	$\Sigma x^4 \div \Sigma x^2$
1	3	2	2	2	2	0	1.0
2	5	10	34	130	70	144	3.4
3	7	28	196	1,588	588	6,048	6.0
4	9	60	708	9,780	2,772	85,536	11.8
5	11	110	1,958	41,030	9,438	679,536	17.8
6	13	182	4,550	134,342	26,026	3,747,744	25.0
7	15	280	9,352	369,640	61,880	16,039,296	33.4
8	17	408	17,544	893,928	131,784	56,930,688	43.0
9	19	570	30,666	1,956,810	257,754	174,978,144	53.8
10	21	770	50,666	3,956,810	471,086	479,700,144	65.8
11	23	1,012	79,948	7,499,932	814,660	1,198,248,480	79.0
12	25	1,300	121,420	13,471,900	1,345,500	2,770,653,600	93.4
13	27	1,638	178,542	23,125,518	2,137,590	6,002,352,720	109.0
14	29	2,030	255,374	38,184,590	3,284,946	12,298,837,824	125.8
15	31	2,480	356,624	60,965,840	4,904,944	24,014,605,824	143.8
16	33	2,992	487,696	94,520,272	7,141,904	44,957,265,408	163.0
17	35	3,570	654,738	142,795,410	10,170,930	81,097,765,056	183.4
18	37	4,218	864,690	210,819,858	14,202,006	141,549,364,944	205.0
19	39	4,940	1,125,332	304,911,620	19,484,348	239,891,292,576	227.8
20	41	5,740	1,445,332	432,911,620	26,311,012	395,928,108,576	251.8
21	43	6,622	1,834,294	604.443.862	35,023,758	637,992,775,728	277.0
22	45	7,590	2,302,806	831,203,670	46,018,170	1,005,920,381,664	303.4
23	47	8,648	2,862,488	1,127,275,448	59,749,032	1,554,840,524,160	331.0
24	49	9,800	3,526,040	1,509,481,400	76.735,960	2,359,959,638,400	359.8
25	51	11,050	4,307,290	1,997,762,650	97,569,290	3,522,530,138,400	389.8

TABLE TO BE USED WHEN THE NUMBER OF OBSERVATIONS IS EVEN.

n	ΣX^0	ΣX^2	ΣX^4	ΣX^6	$\Sigma X^4 \cdot \Sigma X^0 - (\Sigma X^2)^2$	$\Sigma X^6 \cdot \Sigma X^2 - (\Sigma X^4)^2$	$\Sigma X^4 \div \Sigma X^2$
1	2	2	2	2	0	0	1.0
3	4	20	164	1,460	256	2,304	8.2
5	6	70	1,414	32,710	3,584	290,304	20.2
7	8	168	6,216	268,008	21,504	6,386,688	37.0
9	10	330	19,338	1,330,890	84,480	65,235,456	58.6
11	12	572	48,620	4,874,012	256,256	424,030,464	85.0
13	14	910	105,742	14,527,630	652,288	2,038,772,736	116.2
15	16	1,360	206,992	37,308,880	1,462,272	7,894,388,736	152.2
17	18	1,938	374,034	85,584,018	2,976,768	25,960,393,728	193.0
19	20	2,660	634,676	179,675,780	5,617,920	75,123,949,824	238.6
21	22	3,542	1,023,638	351,208,022	9,974,272	196,144,058,880	289.0
23	24	4,600	1,583,320	647,279,800	16,839,680	470,584,857,600	344.2
25	26	5,850	2,364,570	1,135,561,050	27,256,320	1,051,840,857,600	404.2
27	28	7,308	3,427,452	1,910,402,028	42,561,792	2,213,790,808,320	469.0
29	30	8,990	4,842,014	3,100,048,670	64,440,320	4,424,337,967,104	538.6
31	32	10,912	6,689,056	4,875,056,032	94,978,048	8,453,141,250,048	613.0
33	34	13,090	9,060,898	7,457,991,970	136,722,432	15,525,242,320,896	692.2
35	36	15,540	12,062,148	11,134,523,220	192,745,728	27,535,076,464,896	776.2
37	38	18,278	15,810,470	16,265,976,038	266,712,576	47,338,548,401,664	865.0
39	40	21,320	20,437,352	23,303,463,560	362,951,680	79,144,486,327,296	958.6
41	42	24,682	26,088,874	32,803,672,042	486,531,584	129,030,886,752,768	1,057.0
43	44	28,380	32,926,476	45,446,398,140	643,340,544	205,615,957,434,624	1,160.2
45	46	32,430	41,127,726	62,053,929,390	840,170,496	320,919,084,186,624	1,268.2
47	48	36,848	50,887,088	83,612,360,048	1,084,805,120	491,452,517,928,960	1,381.0
49	50	41,650	62,416,690	111,294,934,450	1,386,112,000	739,590,829,286,400	1,498.6

Finally, the sign of G is determined by the direction in which the curve approaches the asymptote $y = 0$, and this may readily be told by inspection. But it not infrequently happens that a slight error in one of the observations may be sufficient to give G the wrong sign. In this case the limits between which the observations were taken must be changed, or a new value of k must be tried, or the faulty observation must be adjusted by a smoothing formula. It is obviously important therefore that some means be provided for determining the sign of G before the values of the coefficients are determined.

The condition that G shall be negative is $\frac{\Sigma X^3 Y}{\Sigma X Y} > \frac{\Sigma X^4}{\Sigma X^2}$. The second term in this inequality may be tabulated in the same way. The accompanying tables show the values of the functions

$$\Sigma X^0, \ \Sigma X^2, \ \Sigma X^4, \ \Sigma X^6, \ \Sigma X^4 \cdot \Sigma X^0 - (\Sigma X^2)^2,$$
$$\Sigma X^6 \, \Sigma X^2 - (\Sigma X^4)^2 \text{ and } \Sigma X^4 \div \Sigma X^2$$

for all values of n from 0 to 25 when the number of observations is odd and from 0 to 49 when they are even.

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