THE USE OF THE RELATIVE RESIDUAL IN THE APPLICATION OF THE METHOD OF LEAST SQUARES

By

WALTER A. HENDRICKS
Junior Biologist, Bureau of Animal Industry,
U. S. Department of Agriculture.

The method of least squares offers a precise method of fitting a curve describing the relation between two or more related, measurable variables, but certain criteria must be fulfilled to justify its application. First, the type of equation selected for fitting must be the true mathematical expression of the law governing the relationship of the variables. Secondly, all errors of measurement, made in obtaining the observed values of the variables when the data were collected, must be distributed according to the well-known laws of probability.[1]

This paper is concerned with the latter of these two criteria. The fundamental theory upon which the method of least squares is based can be found in any text-book on the subject and need not be elaborated upon here. However, it may be well to point out a very pertinent, if somewhat elementary, aspect of the theory which facilitates the ready visualization of the fundamental concepts involved.

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[1]Steinmetz, C. P Engineering Mathematics. McGraw-Hill Book Co., New York (1917).

The application of the method of least squares to curve fitting, as ordinarily described in works on the subject, is perfectly analogous to the calculation of the arithmetic mean of a number of measurements made upon a single, constant quantity. This may be easily demonstrated as follows:

Let $Y = f(X)$ describe the relation existing between an independent variable, X, and a dependent variable, Y. If it is desired to find the most probable value of the dependent variable when X has some definite value, X_1, the most direct method of procedure would be to make a number of measurements of Y at this value of X and calculate their arithmetic mean, provided, of course, that the errors of measurement were distributed according to the laws of probability in a normal frequency distribution. According to the elementary theory of statistics, the most probable value of the dependent variable, $f(X_1)$, would be such that the sum of the squares of the deviations of the actual measurements from this value would be a minimum.

If X is conceived to be varying in value so rapidly that it is impossible to make more than one measurement of Y at any value of X, this direct method can not be employed. However, the most probable value of $f(X_1)$ can still be determined. Let $Y_1, Y_2, Y_3, \ldots Y_n$ each represent a measured value of Y at values $X_1, X_2, X_3, \ldots X_n$, respectively, of the independent variable. Since the errors of measurement are assumed to be distributed according to the law of chance, an error of a given magnitude is equally likely to occur at any value of X. In other words, exactly the same errors would be made in obtaining one measurement of each of the quantities $Y_1, Y_2, Y_3, \ldots Y_n$ as if $f(X_1)$ were measured n times. These errors may, therefore, be considered as having been made in measuring a single, constant quantity. Therefore, if $f(X)$ denotes the most probable value of Y at any value of X and Y denotes the corresponding observed value, the most probable values of the dependent variable which can be calculated from any set of data are such that the sum of the squares of the differences, $f(X)-Y$, is a minimum.

It is important to bear in mind that this conception of the distribution of errors of measurement is justified only when an error of a given magnitude is equally likely to occur at any value of X. In actual practice it often happens that this ideal condition is not realized. The magnitude of the errors of measurement is often influenced by the magnitude of the quantity which is being measured. In obtaining the live weights of animals at different ages, for example, it is common practice to use a less delicate balance in making the weighings as the animals become larger, and the magnitude of the errors of measurement increases as the sensitivity of the balance decreases. Other factors which tend to increase the magnitude of the errors may also be in operation. The error, or rather the unreliability, of the weight of a 1,000 pound steer would be greater than that of a 100 pound calf, even though an equally sensitive balance were used in making both weighings, because of a greater content of material in the digestive tract and excretory organs and the increased effect of the movements of the animal.

It is highly probable that in many fields of investigation such disturbing influences are encountered more frequently than the ideal conditions which justify the application of the method of least squares as ordinarily described.

Pearl and Reed recognized the need for modifying the application of the method of least squares to compensate for changes in the probability of the occurrence of an error of any given magnitude and suggested, as stated by Pearl,[1] that it would be more logical in many instances to employ residuals of the type $\frac{f(X)-Y}{Y}$. The use of such residuals was based on the assumption that if the errors of measurement were expressed as percentages of the

[1]Pearl, Raymond. Studies in Human Biology. Williams & Wilkins, Baltimore (1924).

magnitude of the quantities measured, the percentage errors would be distributed at random according to the law of probability. In many practical problems this assumption appears to be justifiable.

The study herein reported was made to determine the extent of the error made when the method of least squares as ordinarily described is applied to data in which the percentage, rather than the absolute errors of measurement are distributed according to the law of chance.

The writer desired a hypothetical set of errors of measurement which, when expressed as percentages of the quantities measured, would come as near as possible to forming a normal frequency distribution.

TABLE I

Ideal Frequency Distribution of 41 Throws of 12 Dice in Which a Throw of 4, 5, or 6 Points Is Considered a Success.

SUCCESSES	FREQUENCY
2	1
3	2
4	5
5	8
6	9
7	8
8	5
9	2
10	1
	—
Total	41

Mills[1] gives the results of fitting a normal frequency curve to Weldon's distribution of 4096 throws of 12 dice, described by Yule,[2] in which a throw of 4, 5, or 6 points was considered a success. If each frequency, calculated from the fitted curve, is divided by 100 and the results rounded off to whole numbers, the frequency distribution given in Table I is obtained.

If hypothetical errors of measurement are substituted for

TABLE II

Ideal Frequency Distribution of 41 Hypothetical Percentage Errors of Measurement.

ERROR (Per cent of quantity measured)	FREQUENCY
+ 8	1
+ 6	2
+ 4	5
+ 2	8
0	9
− 2	8
− 4	5
− 6	2
− 8	1
Total	41

[1]Mills, F. C. Statistical Methods Applied to Economics and Business. Henry Holt & Co., New York (1924).

[2]Yule, G. Udny. Introduction to the Theory of Statistics. Charles Griffin & Co., Ltd., London (1927).

successes in this frequency table, the resulting distribution may be considered to represent a distribution of random errors of measurement which might be made in obtaining a series of 41 measurements of a variable. The most probable error should obviously be zero. If the total range in magnitude of the errors is assumed to be from $+8$ per cent to -8 per cent and the precision of measurement is such that each error differs from the next larger or smaller error by 2 per cent, the distribution of these hypothetical errors of measurement should be as given in Table II.

From the simple equation, $Y = 100\,X^2$, 41 values of Y were calculated, using values of X from 1 to 41, inclusive. Each calculated value of Y was then changed by algebraically subtracting the hypothetical errors of measurement given in Table II. All the percentage errors of each magnitude were arbitrarily distributed as uniformly as possible throughout the data. These altered values of Y will hereafter be termed the "observed" values and the original values, from which they were calculated, the "true" values. The observed values of Y, together with the true values and the assumed errors of measurement from which they were calculated, are given in Table III.

In order to be certain that the errors were actually distributed in such a manner that the probability of the occurrence of a percentage error of any given magnitude was the same at all values of X, the writer employed Pearson's method of square contingency as described by Yule.[1] A 16-cell contingency table was constructed in which the percentage errors were classified according to the values of X at which they occurred. The chi-square test for contingency was applied to this table.

Table IV shows the actual distribution of the percentage errors, together with the corresponding theoretical frequencies. Since there are 4 rows and 4 columns of cells in the table, the number of algebraically independent differences between theoret-

[1]Loc. cit.

TABLE III

Calculation of the Observed Values of Y from the True Values.

X	$100X^2$	ERROR		Y	X	$100X^2$	ERROR		Y
		Per cent	Actual Units				Per cent	Actual Units	
1	100	− 2	− 2	102	22	48400	+ 8	+ 3872	44528
2	400	+ 4	+ 16	384	23	52900	0	0	52900
3	900	0	0	900	24	57600	− 4	− 2304	59904
4	1600	− 4	− 64	1664	25	62500	− 2	− 1250	63750
5	2500	+ 6	+ 150	2350	26	67600	0	0	67600
6	3600	− 6	− 216	3816	27	72900	− 2	− 1458	74358
7	4900	+ 2	+ 98	4802	28	78400	+ 6	+ 4704	73696
8	6400	0	0	6400	29	84100	+ 2	+ 1682	82418
9	8100	− 2	− 162	8262	30	90000	+ 4	+ 3600	86400
10	10000	+ 2	+ 200	9800	31	96100	− 4	− 3844	99944
11	12100	+ 4	+ 484	11616	32	102400	+ 2	+ 2048	100352
12	14400	− 4	− 576	14976	33	108900	0	0	108900
13	16900	− 8	− 1352	18252	34	115600	+ 2	+ 2312	113288
14	19600	+ 2	+ 392	19208	35	122500	− 2	− 2450	124950
15	22500	− 2	− 450	22950	36	129600	0	0	129600
16	25600	0	0	25600	37	136900	+ 4	+ 5476	131424
17	28900	+ 2	+ 578	28322	38	144400	− 6	− 8664	153064
18	32400	+ 4	+ 1296	31104	39	152100	− 2	− 3042	155142
19	36100	− 2	− 722	36822	40	160000	0	0	160000
20	40000	0	0	40000					
21	44100	+ 2	+ 882	42318	41	168100	− 4	− 6724	174824

ical and observed frequencies is (4 - 1)(4 - 1) + 1 or 10. The value of X^2, calculated from the data in Table IV, is 1.3171. The corresponding value of P, which is the probability that as bad, or worse, an agreement between observed and theoretical frequencies could occur from the fluctuations of random sampling is, according to Pearson's Tables,[1] 0.996911 or almost certainty. The percentage errors were, therefore, distributed in such a manner as to be uncorrelated with the values of X at which they were used.

The equation, $Y = AX^2$, was fitted to the hypothetical set of data in Table III by the method of least squares as ordinarily described. If AX^2 represents a calculated value of the dependent variable and Y represents the corresponding observed value, the difference between these two values is $AX^2 - Y$ and the square of the difference is $A^2X^4 - 2AX^2Y + Y^2$. The sum of the squares of all the differences is $A^2\Sigma X^4 - 2A\Sigma X^2Y + \Sigma Y^2$. The value of this expression will be a minimum when its derivative with respect to A is equal to zero. Differentiating and equating to zero yields the following equations for the determination of A:

(1) $$2A\Sigma X^4 - 2\Sigma X^2Y = 0$$

(2) $$A = \frac{\Sigma X^2Y}{\Sigma X^4}$$

The value of A calculated from the data in Table III by means of equation (2) is 100.6250.

If residuals of the type suggested by Pearl and Reed are employed, A is calculated as follows. Let AX^2 represent a calculated value of the dependent variable, as before, and let Y represent the corresponding observed value. Then the difference between the two values, expressed as a fraction of the observed

[1] Pearson, Karl. Tables for Statisticians and Biometricians. Cambridge University Press, London (1924).

TABLE IV

Chi-square test for contingency applied to the distribution of the percentage errors of measurement. The theoretical frequencies for each compartment are given in parentheses.

Value of X	Magnitude of Error (Per cent)				
	0.0 to ±1.9	± 2.0 to ± 3.9	± 4.0 to ± 5.9	± 6.0 and over	Total
1 to 10	2 (2.1951)	4 (3.9024)	2 (2.4390)	2 (1.4634)	10
11 to 20	2 (2.1951)	4 (3.9024)	3 (2.4390)	1 (1.4634)	10
21 to 30	2 (2.1951)	4 (3.9024)	2 (2.4390)	2 (1.4634)	10
31 to 41	3 (2.4146)	4 (4.2926)	3 (2.6829)	1 (1.6097)	11
Total	9	16	10	6	41

$\chi^2 = 1.3171$
$n' = 10$
$P = 0.996911$

value, is $\frac{AX^2-Y}{Y}$ or $\frac{AX^2}{Y}$ - 1. The square of this relative deviation is $\frac{A^2X^4}{Y^2} - \frac{2AX^2}{Y} + 1$ and the sum of the squares of the 41 relative deviations is $A^2\Sigma\frac{X^4}{Y^2} - 2A\Sigma\frac{X^2}{Y} + 41$. This expression will likewise have its minimum value when its derivative with respect to A is equal to zero. Differentiating and equating to zero, as before, leads to the following equations for the determination of A :

(3) $$2A\Sigma\frac{X^4}{Y^2} - 2\Sigma\frac{X^2}{Y} = 0$$

(4) $$A = \frac{\Sigma\frac{X^2}{Y}}{\Sigma\frac{X^4}{Y^2}}$$

Applying equation (4) to the given set of data gives a value of 99.7573 for A. This value of A is closer to the true value 100, than the value which was calculated by means of equation (2) but the improvement was not as great as might be expected.

It occurred to the writer that if the deviations of the calculated, from the observed, values of the dependent variable were expressed as fractions of the calculated values, a more accurate value of A could be obtained.

The relative deviation expressed in this manner is $\frac{AX^2-Y}{AX^2}$ or $1 - \frac{A^{-1}Y}{X^2}$. The square of this deviation is $1 - \frac{2A^{-1}Y}{X^2} + \frac{A^{-2}Y^2}{X^4}$ and the sum of the squares of the 41 relative deviations is $41 - 2A^{-1}\Sigma\frac{Y}{X^2} + A^{-2}\Sigma\frac{Y^2}{X^4}$. Differentiating this expression with respect to A and equating to zero yields the following equations for the determination of A :

(5) $$2A^{-2}\Sigma\frac{Y}{X^2} - 2A^{-3}\Sigma\frac{Y^2}{X^4} = 0$$

(6) $$A = \frac{\Sigma\frac{Y^2}{X^4}}{\Sigma\frac{Y}{X^2}}$$

The value of A, calculated from the data by means of equation (6), is 100.1210 which is nearer to the true value than either of the values calculated by the two preceding methods. However, it is evident that equation (6) failed to give results as precise as one would expect, in view of the method by which the observed values of Y were obtained.

The reason for this discrepancy can be made most apparent by returning to the analogy existing between the application of the method of least squares to curve fitting and the calculation of the arithmetic mean of a number of measurements of a single, constant quantity.

Let $m_1, m_2, m_3, \ldots m_n$ represent measured values of the same constant quantity and let their arithmetic mean be represented by M. If each measurement is divided by the arithmetic mean of all the measurements, the resulting distribution of these relative values will be normal if the original measurements were distributed normally. The arithmetic mean of these relative values will obviously be unity.

Let $\frac{m_1}{M}, \frac{m_2}{M}, \frac{m_3}{M}, \ldots \frac{m_n}{M}$ represent the relative values of the measurements. The arithmetic mean of these values is unity. Therefore, the deviation of any relative value, $\frac{m}{M}$; from the mean is $1 - \frac{m}{M}$.

Let it be assumed that the value of the arithmetic mean of the original measurement, M, is unknown and is represented by Z. Then any measurement, m, expressed as a fraction of Z, is $\frac{m}{Z}$. According to the discussion in the two preceding paragraphs, it might appear that Z must have such a value that the sum of the squares of the deviations, $1 - \frac{m}{Z}$, is a minimum. However, this is not the case. It may be demonstrated that the value of the expression $\Sigma\,(1 - \frac{2m}{Z} + \frac{m^2}{Z^2})$, is a minimum when Z has some other value than the arithmetic mean of the original measurements. The sum of the squares of the residuals may be written, $n - 2Z^{-1}\Sigma\, m + Z^{-2}\Sigma\, m^2$. Differentiating this expression with respect to Z and equating to zero yields the following equations for the determination of Z :

(7) $$2Z^{-2}\Sigma m - 2Z^{-3}\Sigma m^2 = 0$$

(8) $$Z = \frac{\Sigma m^2}{\Sigma m}$$

The value of Z, calculated by means of equation (8), is obviously not the arithmetic mean of the original measurements. The fallacy in the deduction of this equation is readily apparent.

Instead of using residuals of the type, $1 - \frac{m}{Z}$, and differentiating the sum of the squares of the residuals with respect to Z, one should use residuals of the type, $V - \frac{m}{Z}$, in which V represents the arithmetic mean of the relative values, $\frac{m}{Z}$, of the measurements. The sum of the squares of the residuals should be differentiated with respect to V. The square of the residual, $V - \frac{m}{Z}$, is $V^2 - \frac{2m}{Z} + \frac{m^2}{Z^2}$ and the sum of the squares of all the residuals may be written $nV^2 - \frac{2V}{Z}\Sigma m + \frac{1}{Z^2}\Sigma m^2$ Differentiating with respect to V and equating to zero yields the following equations for the determination of V :

(9) $$2nV - \frac{2}{Z}\Sigma m = 0$$

(10) $$V = \frac{\frac{1}{Z}\Sigma m}{n}$$

Since the value of V is known to be unity, equation (10) may be written:

(11) $$n = \frac{1}{Z}\Sigma m$$

from which Z may be readily calculated as follows:

(12) $$Z = \frac{\Sigma m}{n}$$

Equation (12) is obviously nothing more than the simple formula for the calculation of the arithmetic mean of the original measurements, which is sufficient evidence that the reasoning involved in its deduction is sound.

It is now readily apparent why equation (6) did not yield results which were consistent with the data in Table III. The ratio, $\frac{Y}{AX^2}$, is analogous to the ratio, $\frac{m}{Z}$, and residuals of the type, $V - \frac{Y}{AX^2}$, should have been used in fitting the equation instead of residuals of the type, $1 - \frac{Y}{AX^2}$.[1] The square of the residual, $V - \frac{Y}{AX^2}$, is $V^2 - \frac{2VY}{AX^2} + \frac{Y^2}{A^2X^4}$ The sum of the squares of the 41 residuals is $41V^2 - \frac{2V}{A}\Sigma\frac{Y}{X^2} + \frac{1}{A^2}\Sigma\frac{Y^2}{X^4}$ Differentiating this expression with respect to V and equating to zero yields the following equations for the determination of V:

(13) $$82V - \frac{2}{A}\Sigma\frac{Y}{X^2} = 0$$

(14) $$V = \frac{\frac{1}{A}\Sigma\frac{Y}{X^2}}{41}$$

Substituting the known value, unity, for V in equation (14) yields the following equations for the determination of A.

(15) $$41 = \frac{1}{A}\Sigma\frac{Y}{X^2}$$

(16) $$A = \frac{\Sigma\frac{Y}{X^2}}{41}$$

[1]Residuals of the type, $\frac{AX^2}{Y} - 1$, are analogous to those of the type, $\frac{Z}{m} - 1$, which also lead to incorrect results.

Applying equation (16) to the data in Table III gives A a value of 100.0000, which coincides exactly with the true value from which the data were originally calculated. Equation (16) was, therefore, the correct equation to use in interpreting the data given in Table III. Although the use of residuals of the types, $\frac{AX^2}{Y} - 1$ and $1 - \frac{Y}{AX^2}$, gave better approximations to the true values of A than the use of the simple residuals, $AX^2 - Y$, neither of the two gave results which were entirely in accord with the derivation of the data.

Yule[1] suggested that the geometric mean might often prove useful in comparing the frequency distributions of different sets of data, in which the dispersion of the individual measures about their means was influenced by the magnitude of the means. It appeared to the writer that the use of residuals of the type, $\log AX^2 - \log Y$, might give a good approximation to the true value of A in fitting the given equation. It is evident that the ratio, $\frac{AX^2}{Y}$, approaches unity as the residual, $\log AX^2 - \log Y$, approaches zero.

This logarithmic residual may be written, $\log A + 2 \log x - \log Y$, and its square is $(\log A)^2 + 4(\log X)^2 + (\log Y)^2 + 4(\log A)(\log X) - 2(\log A)(\log Y) - 4(\log X)(\log Y)$. The sum of the squares of the 41 residuals is $41(\log A)^2 + 4\Sigma(\log X)^2 + \Sigma(\log Y)^2 + 4(\log A)\Sigma(\log X) - 2(\log A)\Sigma(\log Y) - 4\Sigma(\overline{\log X} . \overline{\log Y})$. Differentiating this expression with respect to $\log A$ and equating to zero yields the following equations for the determination of A:

$$(17) \qquad 82(\log A) + 4\Sigma(\log X) - 2\Sigma(\log Y) = 0$$

$$(18) \qquad \log A = \frac{(\log Y) - 2\Sigma(\log X)}{41}$$

[1]Loc. cit.

The value of $\log A$, calculated from the given set of data by means of equation (18), is 1.9997369, which gives A a value of 99.9394. This value of A comes closer to the true value than those calculated by means of residuals of the types, $1 - \frac{Y}{AX^2}$ and $\frac{AX^2}{Y} - 1$. However, since the use of the geometric mean is not rigorously justified when the distribution of the measures about the arithmetic mean is symmetrical, the use of logarithmic residuals in curve fitting can not give precise results when the errors of measurement are distributed as they were in the given set of data.

In any application of the method of least squares to a practical problem, the procedure of the investigators should be governed by the nature of the data to which it is being applied. In many instances the correct procedure can be deduced by a careful consideration and evaluation of the accuracy of the methods of measurement used in obtaining the data. Unfortunately, however, some sources of error are not always readily apparent at the time the data are collected, and occasionally can not be quantitatively estimated even though they are known to exist. If the nature of the mathematical relationship existing between the dependent and independent variables is known, all that remains is to find the most probable values of the constants in the equation.

A statistical study of the deviations of the observed values of the dependent variable from the corresponding calculated values, obtained after fitting the equation by several different methods, may be of much help in deciding which method of fitting was most consistent with the nature of the data. For example, Table V gives the results of applying the chi-square test for contingency to the distribution of the deviations of the observed values of Y from the calculated values obtained when residuals of the type, $AX^2 - Y$, were used in fitting the equation, $Y = AX^2$, to the data in Table III. The value of P is only 0.005061 and a mere inspection of the table itself shows that large deviations tend to occur more frequently, and small deviations less frequently as

TABLE V

Chi-square test for contingency applied to the distribution of the deviations of the type, AX^2-Y. The theoretical frequencies for each compartment are given in parentheses.

Value of X	Magnitude of Deviation				
	10 to ± 1999	± 2000 to ± 3999	± 4000 to ± 5999	± 6000 and over	Total
1 to 10	10 (7.3171)	0 (1.2197)	0 (0.9756)	0 (0.4878)	10
11 to 20	10 (7.3171)	0 (1.2197)	0 (0.9756)	0 (0.4878)	10
21 to 30	6 (7.3171)	1 (1.2197)	3 (0.9756)	0 (0.4878)	10
31 to 41	4 (8.0488)	4 (1.3415)	1 (1.0732)	2 (0.5366)	11
Total	30	5	4	2	41

$\chi^2 = 23.5989$

$n' = 10$

$P = 0.005061$

the values of X increase. If the true nature of the values of Y in Table III were not known in advance, this distribution of the deviations would be sufficient evidence that the method of fitting the equation was not consistent with the accuracy of the measurements made when the data were collected.

Tables VI, VII, and VIII give, respectively, the distributions of the deviations of the types, $\frac{AX^2}{Y} - 1$, $1 - \frac{Y}{AX^2}$, and $\log AX^2 - \log Y$, when the corresponding residuals were used in fitting the equation.[1] The value of P is high in each case, indicating that, although the use of residuals of these types did not give results which were precisely accurate, nevertheless, they yielded values of A which were well within the limits of the probable error to be expected in any practical investigation.

As a matter of fact, this is a rather fortunate circumstance, since the only method of fitting the equation given above which yielded exactly the correct value of A cannot be applied to fitting an equation containing more than one undetermined constant. The applicability of residuals of the types, $1 - \frac{Y}{f(X)}$ and $\log f(X) - \log Y$ is also somewhat limited. However, any equation which can be fitted by the method of least squares at all can still be fitted when residuals of the type, $\frac{f(X)}{Y} - 1$, are employed.

SUMMARY AND CONCLUSIONS

The method of least squares can be a more valuable tool in statistical work when the fundamental theory upon which the method is based is taken into consideration. The use of residuals of the type, $f(X) - Y$, is probably justified in fewer practical

[1]The distribution of the deviations obtained when the equation was fitted to the data by means of equation (16) is identical with the distribution of the errors given in Table IV.

problems than the use of residuals of some other form. The type of residual to be employed should be governed by the nature of the data to which the method of least squares is being applied.

The use of relative residuals of the type suggested by Pearl and Reed may be of much value in many instances but will not give results which are precisely accurate, even though the distribution of the percentage errors of measurement is strictly normal. The results can be improved by expressing the deviations of the observed from the calculated values of the dependent variable as fractions of the calculated, rather than the observed, value.[1]

The use of logarithmic residuals may give more accurate results than the use of residuals of the type suggested by Pearl and Reed, even though the distribution of the percentage errors of measurement is normal.

The chi-square test for contingency may be of much help in selecting the type of residual most consistent with the errors of measurement made in obtaining the data when sufficient information regarding the accuracy of the measurements is not available.

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[1]Residuals of this type have been used by Hendricks, Lee, and Titus at the U. S. Animal Husbandry Experiment Farm, Beltsville, Maryland, in the fitting of growth curves.

Hendricks, W. A., A. R. Lee, and H. W. Titus. Early growth of White Leghorns, Poultry Sci. 8 (6); pp. 315-327 (1929).

Titus, H. W., and W. A. Hendricks. The Early Growth of Chickens as a Function of Feed Consumption Rather Than of Time. Conference Papers of the Fourth World's Poultry Congress, Section B (Nutrition and Rearing): pp. 285-293 (1930).

The use of such residuals leads to results which appear to give a better description of the data than when simple residuals of the type, $f(X) - Y$, are employed.

TABLE VI

Chi-square test for contingency applied to the distribution of the deviations of the type, $\frac{AX^2}{Y} - 1$. The theoretical frequencies for each compartment are given in parentheses.

Value of X	Magnitude of Deviation				
	0.000 to ± 0.019	± 0.020 to ± 0.039	± 0.040 to ± 0.059	± 0.060 and over	Total
1 to 10	4 (4.1463)	3 (3.1707)	2 (1.7073)	1 (0.9756)	10
11 to 20	4 (4.1463)	4 (3.1707)	1 (1.7073)	1 (0.9756)	10
21 to 31	4 (4.1463)	3 (3.1707)	1 (1.7073)	2 (0.9756)	10
31 to 41	5 (4.5610)	3 (3.4878)	3 (1.8780)	0 (1.0732)	11
Total	17	13	7	4	41

$\chi^2 = 3.8182$

$n' = 10$

$P = 0.921027$

TABLE VII

Chi-square test for contingency applied to the distribution of the deviations of the type, $1 - \frac{Y}{AX^z}$. The theoretical frequencies for each compartment are given in parentheses.

Value of X	Magnitude of Deviation				
	0.000 to ±0.019	± 0.020 to ± 0.039	± 0.040 to ±0.059	± 0.060 and over	Total
1 to 10	3 (3.9024)	4 (3.4146)	2 (1.7073)	1 (0.9756)	10
11 to 20	4 (3.9024)	3 (3.4146)	2 (1.7073)	1 (0.9756)	10
21 to 30	4 (3.9024)	3 (3.4146)	1 (1.7073)	2 (0.9756)	10
31 to 41	5 (4.2927)	4 (3.7561)	2 (1.8780)	0 (1.0732)	11
Total	16	14	7	4	41

$\chi^2 = 3.0984$
$n' = 10$
$P = 0.959091$

TABLE VIII

Chi-square test for contingency applied to the distribution of the deviations of the type, log AX^2- log Y . The theoretical frequencies for each compartment are given in parentheses.

Value of X	Magnitude of Deviation				
	0.000 to ±0.009	± 0.010 to ±0.019	± 0.020 to ±0.029	± 0.030 and over	Total
1 to 10	6 (6.0976)	2 (2.4390)	2 (0.9756)	0 (0.4878)	10
11 to 20	6 (6.0976)	3 (2.4390)	0 (0.9756)	1 (0.4878)	10
21 to 30	6 (6.0976)	2 (2.4390)	1 (0.9756)	1 (0.4878)	10
31 to 41	7 (6.7073)	3 (2.6829)	1 (1.0732)	0 (0.5366)	11
Total	25	10	4	2	41

$\chi^2 = 4.4989$
$n' = 10$
$P = 0.872945$