THE SIMULTANEOUS DISTRIBUTION OF MEAN AND STANDARD DEVIATION IN SMALL SAMPLES

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1. Introduction. If samples of n items are selected at random from a normal universe, it is well known that the arithmetic mean $\bar{x}$ and standard deviation s computed from samples are independent in the probability sense and that the simultaneous frequency distribution is

$$F(\bar{x},s)=Cs^{n-2}e^{-\frac{ns^2+n\bar{x}^2}{2\sigma^2}}$$

If, however, the parent population is other than the normal type, there appears to be little known regarding the form of $F(\bar{x},s)$. In the present paper, we propose to determine the simultaneous frequency function of the arithmetic mean and standard deviation in samples of small numbers of items selected at random from a rather arbitrary universe. For convenience, we shall classify frequency distributions according as the range of the independent variable is $(-\infty,\infty)$, $(0,\infty)$ or $(0,a)$, $a>0$. We shall further assume that the total area under the distribution function is unity.

2. The simultaneous distribution of $\bar{x}$ and s in samples of $n=2$. Let $f(x)$, $-\infty<x<\infty$ be the frequency unction of the variable x Let x_1 and x_2 be two independentl erved values of x. write

$$x_1+x_2=2\bar{x}$$
$$x_1^2+x_2^2=2s^2+2\bar{x}^2$$

We seek the function $F(\bar{x}, s)$ such that $F(\bar{x}, s)\,d\bar{x}\,ds$ is, to within infinitesimals of higher order, the probability of the simultaneous occurrence of $\bar{x}$ in $(\bar{x}, \bar{x}+d\bar{x})$ and s in $(s, s+ds)$. For $\bar{x}$ and s assigned, x_1 may have either value $\bar{x}-s$ or $\bar{x}+s$ and x_2 is uniquely determined by $x_2 = 2\bar{x}-x_1$.

Thus

$$F(\bar{x},s)\,d\bar{x}\,ds = f(\bar{x}-s)f(2\bar{x}-x_1)\,dx_1\,dx_2$$

$$+f(\bar{x}+s)f(2\bar{x}-x_1)\,dx_1\,dx_2\,.$$

Since $dx_1\,dx_2 = 2\,d\bar{x}\,ds$ we have

$$(1) \qquad F(\bar{x},s) = 4f(\bar{x}-s)\,f(\bar{x}+s).$$

If $f(x)$ is defined on the interval $(0, \infty)$, we note, for $\bar{x}$ assigned, that $s \leq \bar{x}$. Thus (1) is valid for this type of frequency function but the surface is limited by the x-axis and the line $s=\bar{x}$

If $f(x)$ is defined on the interval $(0, a)$, we note, for $\bar{x}$ assigned on $(0, a/2)$, that $s \leq \bar{x}$; and, for $\bar{x}$ assigned on $(a/2, a)$, that $s \leq a-\bar{x}$. Accordingly, for this kind of frequency function, (1) is valid but the surface is limited by the x-axis and the lines $s=\bar{x}$, $s=a-\bar{x}$.

As simple illustrations, let us find the correlation surface for the mean and standard deviation of samples of two items drawn from distributions of various types.

Example 1. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{x^2}{2\sigma^2}}, \qquad -\infty < x < \infty$$

Then

$$F(\bar{x}, s) = \frac{2}{\sigma^2 \pi} e^{-\frac{s^2 + \bar{x}^2}{\sigma^2}}$$

the well known result.

Example 2. Let

$$f(x) = e^{-x}, \qquad 0 \leq x < \infty.$$

Then

$$F(\bar{x}, s) = 4e^{-2\bar{x}}$$

over the open region of the $\bar{x}s$-plane bounded by the $\bar{x}$-axis and the line $s = \bar{x}$.

Example 3. Let

$$f(x) = \frac{1}{a}, \qquad 0 \leq x \leq a.$$

Then

$$F(\bar{x}, s) = \frac{4}{a^2},$$

over the region of the $\bar{x}s$-plane bounded by the isosceles triangle with sides $s = 0$, $s = \bar{x}$ and $s = a - \bar{x}$. With a uniform distribution proportional to $4/a^2$ over this triangle, it follows incidentally from very elementary geometry that the marginal totals of the distribution of $\bar{x}$ are given by the known values

$$\phi(\bar{x}) = \frac{4}{a^2} \bar{x}, \qquad 0 \leq \bar{x} \leq \frac{a}{2},$$

$$= \frac{4}{a^2}(a - \bar{x}), \qquad \frac{a}{2} \leq \bar{x} \leq a,$$

and that the marginal totals for the distribution of s are given by

$$\psi(s) = \frac{4}{a^2}(a-2s), \qquad 0 \le s \le \frac{a}{2},$$

which is the result given by Rider.[1]

3. The simultaneous distribution of $\bar{x}$ and s in samples of $n=3$. Consider first a frequency function $f(x), -\infty < x < \infty$. We have

$$x_1 + x_2 + x_3 = 3\bar{x},$$

$$x_1^2 + x_2^2 + x_3^2 = 3s^2 + 3\bar{x}^2.$$

Upon eliminating x_3, we have

$$2x_1^2 + 2x_1x_2 + 2x_2^2 - 6\bar{x}x_1 - 6\bar{x}x_2 - 3s^2 + 6\bar{x}^2 = 0.$$

From simple properties of this ellipse, it follows, for assigned $\bar{x}$ and s that x_1 may be chosen arbitrarily from the interval $(\bar{x} - s\sqrt{2}, \bar{x} + s\sqrt{2})$. With x_1 assigned, x_2 must be selected with certainty as either

$$\frac{3\bar{x} - x_1 - \left[6s^2 - 3(x_1 - \bar{x})^2\right]^{\frac{1}{2}}}{2} \quad \text{or}$$

$$\frac{3\bar{x} - x_1 + \left[6s^2 - 3(x_1 - \bar{x})^2\right]^{\frac{1}{2}}}{2}$$

Finally we must have

$$x_3 = 3\bar{x} - x_1 - x_2.$$

[1] P. R. Rider, On the distribution of ratio of mean to standard deviation etc., Biometrika, vol. 21 (1929) pp. 124-141.

Thus

$$F(\bar{x},s)d\bar{x}\,ds = 2\int_{\bar{x}-s\sqrt{2}}^{\bar{x}+s\sqrt{2}} f(x_1)f(x_2)f(x_3)dx_1dx_2dx_3.$$

From

$$x_1 = x_1,$$
$$x_2 = \frac{3\bar{x}-x_1 \pm \left[6s^2-3(x_1-\bar{x})^2\right]^{\frac{1}{2}}}{2},$$
$$x_3 = 3\bar{x}-x_1-x_2$$

we obtain

$$dx_1dx_2dx_3 = \frac{9s}{\left[6s^2-3(x_1-\bar{x})^2\right]^{\frac{1}{2}}}\,dx_1d\bar{x}\,ds$$

$$= 9s\,dx_1d\bar{x}\,ds/R$$

where

$$R \equiv \left[6s^2-3(x_1-\bar{x})^2\right]^{\frac{1}{2}}$$

Thus

(2)

$$F(\bar{x},s) = 18s\int_{\bar{x}-s\sqrt{2}}^{\bar{x}+s\sqrt{2}} \frac{1}{R}\,f(x_1)\,f\left(\frac{3\bar{x}-x_1+R}{2}\right)f\left(\frac{3\bar{x}-x_1-R}{2}\right)dx_1$$

If $f(x)$ is defined on the interval $(0, \infty)$, we note, for $\bar{x}$ assigned, that $0 \le s \le \bar{x}\sqrt{2}$. Thus the surface is limited by the $\bar{x}$ -axis and the line $s = \bar{x}\sqrt{2}$. Moreover, since x_1, x_2, x_3 are non-negative, x_1 may be selected from the interval $(\bar{x} - s\sqrt{2}, x + s\sqrt{2})$ only as long as $s \le \bar{x}\sqrt{2}/2$. If

$$\bar{x}\; 2/\sqrt{2} \le s \le \bar{x}\sqrt{2}.$$

then x_1 may be selected from the intervals

$$\left(0, \frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}\right)$$

and

$$\left(\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}, \bar{x} + s\sqrt{2}\right).$$

Accordingly, for this type of frequency function,

$$F(\bar{x}, s) = 18s \int_{\bar{x} - s\sqrt{2}}^{\bar{x} + s\sqrt{2}} \frac{1}{R} f(x_1) f\left(\frac{3\bar{x} - x_1 + R}{2}\right) f\left(\frac{3\bar{x} - x_1 - R}{2}\right) dx_1,$$

$$0 \le s \le \frac{\bar{x}\sqrt{2}}{2},$$

(2.1)

$$= 18s \left[\int_0^{\frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}}^{\bar{x} + s\sqrt{2}} \right]$$

$$\frac{1}{R} f(x_1) f\left(\frac{3\bar{x} - x_1 + R}{2}\right) f\left(\frac{3\bar{x} - x_1 - R}{2}\right) dx_1,$$

$$\frac{\bar{x}\sqrt{2}}{2} \le s \le \bar{x}\sqrt{2}.$$

If $f(x)$ is defined on the interval (O, a), we note:

for $O \leq \bar{x} \leq a/3$, $\qquad O \leq s \leq \bar{x}\sqrt{2}$;

for $a/3 \leq \bar{x} \leq 2a/3$, $\qquad O \leq s \leq \left[2\bar{x}^2 - 2a\bar{x} + \frac{2a}{3}^2\right]^{\frac{1}{2}}$;

for $2a/3 \leq \bar{x} \leq a$, $\qquad O \leq s \leq (a-\bar{x})\sqrt{2}$.

Thus in this case, the surface is limited by the x-axis, the lines $s = \bar{x}\sqrt{2}$ and $s = (a-\bar{x})\sqrt{2}$ and the hyperbola

$$s = \left[2\bar{x}^2 - 2a\bar{x} + 2a^2/3\right]^{\frac{1}{2}}.$$

(Fig. 1.). Now x, may be selected from the interval $(\bar{x} - s\sqrt{2},\ x + s\sqrt{2})$ as long as $s \leq \bar{x}\sqrt{2}/2$ and $s \leq (a-\bar{x})\sqrt{2}/2$. This holds for that part of the surface over the region bounded by OPa. For that part of the surface over the region bounded by OPU, x, may be selected from the intervals

$$\left(O,\ \frac{3\bar{x} - \left[6s^2 - 3\bar{x}^2\right]^{\frac{1}{2}}}{2}\right)$$

and

$$\left(\frac{3\bar{x} + \left[6s^2 - 3\bar{x}^2\right]^{\frac{1}{2}}}{2},\ \bar{x} + s\sqrt{2}\right).$$

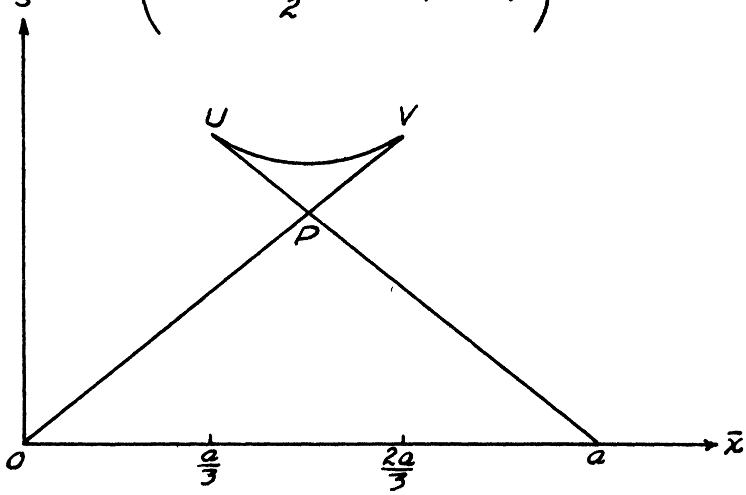

Fig. 1

It is clear that the ranges of arbitrary selection of x_1 for that part of the surface over the region bounded by PVa are

$$\left(\bar{x} - s\sqrt{2},\ \frac{3\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2} \right)$$

and

$$\left(\frac{3\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2},\ a \right).$$

Finally, we find that x_1 may be selected from the intervals

$$\left(0,\ \frac{3\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2}\right),$$

$$\left(\frac{3\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2},\ \frac{3\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2}\right),$$

and

$$\left(\frac{3\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2},\ a\right)$$

for that part of the surface over the region bounded by PUV. If we adopt the notation

$$\phi = \phi(x_1, \bar{x}, s) \equiv \frac{1}{R} f(x_1) f\left(\frac{3\bar{x}-x_1+R}{2}\right) f\left(\frac{3\bar{x}-x_1-R}{2}\right),$$

we have

$$(2.2)\quad F(\bar{x},s) = 18s \int_{\bar{x}-s\sqrt{2}}^{\bar{x}+s\sqrt{2}} \phi\, dx_1,$$

$$= 18s\left[\int_0^{\frac{3\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2}}^{\bar{x}+s\sqrt{2}}\right]\phi\, dx_1,$$

$$= 18s\left[\int_{\bar{x}-s\sqrt{2}}^{\frac{3\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2}}^{a}\right]\phi\, dx_1$$

$$= 18s\left[\int_0^{\frac{3\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2}}^{\frac{3\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2}}^{a}\right]\phi\, dx_1,$$

for the parts of the surface over the regions indicated above.

In order to illustrate the theory, we shall consider a few examples.

Example 1. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

By (2),

$$F(\bar{x},s) = \frac{3\sqrt{3}}{\sigma^3\sqrt{2\pi}}\, s e^{-\frac{3s^2+3\bar{x}^2}{2\sigma^2}}$$

Example 2. Let

$$f(x) = e^{-x}, \qquad 0 \leq x < \infty.$$

By (2.1),

$$F(\bar{x},s) = 6\sqrt{3}\,\pi s e^{-3\bar{x}} \qquad 0 \leq s \leq \frac{\bar{x}\sqrt{2}}{2},$$

$$= 6\sqrt{3}\, s e^{-3\bar{x}}\left[\arcsin \frac{\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} + \arcsin\frac{\bar{x}}{s\sqrt{2}}\right.$$

$$\left. - \arcsin \frac{\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} + \frac{\pi}{2}\right], \quad \frac{\bar{x}\sqrt{2}}{2} \leq s \leq \bar{x}\sqrt{2}.$$

Example 3. Let

$$f(x) = \frac{1}{a}, \qquad 0 \leq x \leq a.$$

By (2.2),

$$F(\bar{x},s) = \frac{6\sqrt{3}\,\pi s}{a^3}, \text{ over } OPa,$$

$$= \frac{6\sqrt{3}s}{a^3}\left[\arc\sin \frac{\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} + \arc\sin\frac{\bar{x}}{s\sqrt{2}}\right.$$

$$\left. - \arc\sin \frac{\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} + \frac{\pi}{2}\right], \text{ over } OPU,$$

$$= \frac{6\sqrt{3}s}{a^3}\left[\arc\sin \frac{\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}}\right.$$

$$+ \arc\sin\frac{a-\bar{x}}{s\sqrt{2}} - \arc\sin\frac{\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}}$$

$$\left. + \frac{\pi}{2}\right], \text{ over } PVa,$$

$$= \frac{6\sqrt{3}s}{a^3}\left[\arc\sin \frac{\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}}\right.$$

$$+ \arc\sin\frac{\bar{x}}{s\sqrt{2}} + \arc\sin\frac{\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}}$$

$$+ \arc\sin\frac{a-\bar{x}}{s\sqrt{2}}$$

$$- \arc\sin\frac{\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}}$$

$$\left. - \arc\sin\frac{\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}}\right], \text{ over } PVU.$$

I have succeeded in obtaining the marginal totals for s from O to $a\sqrt{2}/4$ by integrating $F(\bar{x},s)$ with respect to $\bar{x}$ from the boundary (Fig. 1) $s=\bar{x}\sqrt{2}$ to $s=(a-\bar{x})\sqrt{2}$ and obtain as a result the parabola which is known[1] to give the distribution of s from $s=0$ to $s=a\sqrt{6}/6$.

4. The simultaneous distribution of $\bar{x}$ and s in samples of $n=4$ We shall consider first samples of four items drawn from a universe characterized by a law of frequency $f(x)$, $-\infty < x < \infty$. Then

$$x_1 + x_2 + x_3 + x_4 = 4\bar{x},$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4s^2 + 4\bar{x}^2.$$

The elimination of x_4 yields

$$x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3-4\bar{x}x_1-4\bar{x}x_2-4\bar{x}x_3-2s^2+6\bar{x}^2=0$$

It follows from the properties of this ellipsoid that x_1 may be chosen arbitrarily from the interval $(\bar{x}-s\sqrt{3},\ \bar{x}+s\sqrt{3})$. For x_1 assigned, the region of arbitrary selection of x_2 is determined by the properties of the ellipse and is

$$\left(\frac{4\bar{x}-x_1-2\left[6s^2-2(x_1-\bar{x})^2\right]^{\frac{1}{2}}}{3},\ \frac{4\bar{x}-x_1+2\left[6s^2-2(x_1-\bar{x})^2\right]^{\frac{1}{2}}}{3}\right)$$

Upon solving for x_3 in terms of x_1 and x_2 we have

$$x_3=\frac{4\bar{x}-x_1-x_2\pm\left[8s^2-8\bar{x}^2+8\bar{x}x_1+8\bar{x}x_2-2x_1x_2-3x_1^2-3x_2^2\right]^{\frac{1}{2}}}{2}$$

while x_4 is uniquely determined by $x_4 = 4\bar{x}-x_1-x_2-x_3$.
If we write

$$T\equiv\left[8s^2-8x^2+8\bar{x}x_1+8\bar{x}x_2-2x_1x_2-3x_1^2-3x_2^2\right]^{\frac{1}{2}}$$

and

$$\phi\equiv f(x_1)f(x_2)f\left(\frac{4\bar{x}-x_1-x_2+T}{2}\right)f\left(\frac{4\bar{x}-x_1-x_2-T}{2}\right)$$

[1] H. L. Rietz [Paper to appear presently in Biometrika].

then

$$(3) \quad F(\bar{x},s) = 32s \int_{\bar{x}-s\sqrt{3}}^{\bar{x}+s\sqrt{3}} \int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}} \frac{1}{f}\phi \, dx_2 \, dx_1 .$$

The integration can be carried out in an obvious manner when $f(x)$ is the normal frequency function.

In case $f(x)$ is defined on the interval $(0, \infty)$, we note, for $\bar{x}$ assigned, that $s \leq \bar{x}\sqrt{3}$. Thus the surface is limited by the $\bar{x}$-axis and the line $s = \bar{x}\sqrt{3}$ Moreover, x_1 may be selected from the interval $(\bar{x}-s\sqrt{3}, \bar{x}+s\sqrt{3})$ with x_2 chosen as above only as long as $s \leq \bar{x}\sqrt{3}/3$ If $\bar{x}\sqrt{3}/3 \leq s \leq \bar{x}$, then x_1 may be chosen from either of the two intervals

$$\left(0, \frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}\right)$$

and

$$\left(\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}, \bar{x}+s\sqrt{3}\right)$$

with x_2 chosen as above; or x_1 may be selected from the interval

$$\left(\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}, \frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}\right)$$

with x_2 taken from either

$$\left(0, \frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}\right)$$

or

$$\left(\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}, \frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}\right).$$

when $\bar{x} \leq s \leq \bar{x}\sqrt{3}$ we may have

$$0 \leq x_1 \leq 2\bar{x} - [2s^2-2\bar{x}^2]^{\frac{1}{2}}$$

and

$$2\bar{x}+[2s^2-2\bar{x}^2]^{\frac{1}{2}} \le x_1 \le \frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}$$

with either

$$0 \le x_2 \le \frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}$$

or

$$\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2} \le x_2 \le \frac{4\bar{x}-x_1+2[6s-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}$$

Or we may have

$$\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3} \le x_1 \le \bar{x}+s\sqrt{3}$$

with

$$\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3} \le x_2 \le \frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}$$

Accordingly, for this kind of frequency function,

$$F(\bar{x},s)=32s\int_{\bar{x}-s\sqrt{3}}^{\bar{x}+s\sqrt{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-x)^2]^{\frac{1}{2}}}{3}} \frac{1}{T}\phi\, dx_2\, dx_1,$$

(3.1)

$$0 \le s \le \frac{\bar{x}\sqrt{3}}{3},$$

$$=32s\Bigg[\int_0^{\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}$$

$$+\int_{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\bar{x}+s\sqrt{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}$$

$$+\int_{\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}} \int_{0}^{\frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}$$

$$\left. +\int_{\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}} \int_{\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}} \right] \frac{1}{T}\phi\, dx_2\, dx_1,$$

$$\frac{\bar{x}\sqrt{3}}{3} \leq s \leq \bar{x},$$

$$=32s\left[\int_{0}^{2\bar{x}-[2s^2-2\bar{x}^2]^{\frac{1}{2}}} \int_{0}^{\frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}\right.$$

$$+\int_{0}^{2\bar{x}-[2s^2-2\bar{x}^2]^{\frac{1}{2}}} \int_{\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}$$

$$+\int_{2\bar{x}+[2s^2-2\bar{x}^2]^{\frac{1}{2}}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}} \int_{0}^{\frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}$$

$$+\int_{2\bar{x}+[2s^2-2\bar{x}^2]^{\frac{1}{2}}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}} \int_{\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}$$

$$\left. +\int_{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\bar{x}+s\sqrt{3}} \int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}} \right] \frac{1}{T}\phi\, dx_2\, dx_1,$$

$$\bar{x} \leq s \leq \bar{x}\sqrt{3}$$

By similar reasoning, the writer has determined $F(\bar{x}, s)$ for $n = 4$ and $f(x)$ defined on the interval $(0, a)$. The results, however, are quite lengthy and formal and will not be presented here.

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