CONCERNING THE LIMITS OF A MEASURE OF SKEWNESS

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In a recent note in the Annals of Mathematical Statistics,* Hotelling and Solomons devised an ingenious method of showing that the measure of skewness s defined by the equation

$$s = \frac{\text{mean} - \text{median}}{\text{standard deviation}}$$

cannot be greater than unity in absolute value. I am venturing to offer another proof of the same fact, which seems to me to be of interest because it employs an important and well-known algebraic inequality.

With Hotelling and Solomons, I shall assume that we are concerned with n readings, or x's, with median zero and mean $\bar{x}$, where $\bar{x}$ of course is $\Sigma x/n$ We may show that the absolute value of s cannot be greater than one by showing that $1/s^2$ is not less than one. Making obvious substitutions, we must then show that

$$\frac{n\Sigma x^2}{(\Sigma x)^2} \geqq 2.$$

Now according to a known theorem if $a, b, \cdots, k$ are n positive numbers, and if m is a number not lying between zero and one, then

$$\frac{a^m + b^m + \cdots + k^m}{n} \geqq \left(\frac{a+b+\cdots+k}{n}\right)^m.$$

*Vol. 3, no. 2, May, 1932, 141-2.

While the proof of this theorem is given in Chrystal, we shall outline a (simplified) proof for the case $m=2$, to make this note self-contained. For any number r we obviously have $(r-1)^2 \geqq 0$. Now let r equal $na/_{a+b+\cdots+k}$, $nb/_{a+b+\cdots+k}$, $\cdots$, $nk/_{a+b+\cdots+k}$ in turn. The first of these gives

$$\frac{n^2a^2}{(a+b+\cdots+k)^2}-\frac{2na}{(a+b+\cdots+k)}+1\geqq 0,$$

while the others give similar inequalities. Summing these inequalities we have

$$\frac{n^2(a^2+b^2+\cdots+k^2)}{(a+b+\cdots k)^2}-2n+n\geqq 0,$$

which is Chrystal's theorem,* for $m=2$. The proof shows that some of the numbers $a, b, \cdots, k$ can be zero; in fact, some can be negative, provided $a+b+\cdots\cdots+k$ is not zero.

Now, suppose we have an odd number of readings, say $n=2s+1$. Since the median reading is zero, there are s non-negative readings, which we shall now call y's, and s non-positive readings, which we shall call z's. We have at once, by the above.

$$\frac{s\Sigma y^2}{(\Sigma y)^2}\geqq 1,$$

$$\frac{s\Sigma z^2}{(\Sigma z)^2}\geqq 1,$$

It follows immediately that

$$s\left(\frac{\Sigma y^2+\Sigma z^2}{(\Sigma y)^2+(\Sigma z)^2}\right)\geqq 1$$

*Chrystal, Algebra, Part II, 2nd ed., 1922, p. 49.

and, since $n = 2s+1$ that

$$n\left(\frac{\Sigma y^2 + \Sigma z^2}{(\Sigma y)^2 + (\Sigma z)^2}\right) > 2.$$

Finally,

$$n\frac{\Sigma x^2}{(\Sigma x)^2} = n\frac{\Sigma y^2 + \Sigma z^2}{(\Sigma y + \Sigma z)^2} = n\frac{\Sigma y^2 + \Sigma z^2}{(\Sigma y)^2 + (\Sigma z)^2 + 2(\Sigma y)(\Sigma z)} > 2,$$

since $2(\Sigma y)(\Sigma z)$ is certainly not positive.

This proof is valid unless all the y's are zero or all the z's are zero. Suppose the latter is the case. Then

$$\frac{n\Sigma x^2}{(\Sigma x)^2} = \frac{n\Sigma y^2}{(\Sigma y)^2} > 2\left(\frac{s\Sigma y^2}{(\Sigma y)^2}\right) > 2.$$

If all the readings are zero our definition of s does not give a definite value.

If n is even, not odd, the proof may be modified by properly defining the median. In this case we can show again that

$$\frac{n\Sigma x^2}{(\Sigma x)^2} \geqq 2,$$

but the possibility of the equality cannot be ruled out.

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