APPROXIMATION AND GRADUATION ACCORDING TO THE PRINCIPLE OF LEAST SQUARES BY ORTHOGONAL POLYNOMIALS*

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PREFACE.

In the present paper the mathematical theory of approximation by orthogonal polynomials is given in its entirety and accompanied by nearly all of the necessary demonstrations. This has necessitated some mathematical preliminaries.

Statisticians less interested in mathematics will find it sufficient to read § 12 on the recapitulation of the operations, the beginning of § 9 concerning the computation of the mean binomial moments and the mean orthogonal moments, § 11 dealing with the method of addition of differences, and the examples of § 13. In paragraphs § 14 and § 15 dealing with correlation and in § 17 treating graduation, one may observe the end-formulae and skip the rest. With the aid of these sections and the tables, one may readily employ the methods and attain results with a very small amount of labor.

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§ 1. *Introduction.* It has been shown that in any case of approximating a function $F(x)$ it is advantageous to develop that function in a series of orthogonal functions. It was *Fourier* who first used such an expansion in his treatment of trigonometric series. The first expansion in orthogonal polynomials was performed by *Legendre.* In Legendre's polynomials the variable x is a continuous one, it takes on every value between -1 and $+1$. Orthogonal polynomials with respect to a discontinuous variable, where x assumes only the N values $x_0, x_1, \cdots, x_{N-1}$ have been deduced by *Chebisheff*[1] who has treated the particular case of two orthogonal polynomials with respect to equidistant variables.

Since then several authors have investigated this subject. *Poin-*

[1]Chebisheff. Sur les fractions continues, Journal de Mathématiques pures and appliquées 1858, T. III (Oeuvres Tome I. 203).
Sur l'interpolation par la méthode des moindres carrés. Mem. Acad. Imp. de St. Pétersbourg, 1859 (Oeuvres Tome I. p. 473).
Sur l'interpolation des valeurs équidistantes. 1875 (Oeuvres Tome II. p. 219).

caré[2] and *Quiquet*[3] considered orthogonal polynomials of a discontinuous variable in the case of non-equidistant values. *Gram*[4] employed these polynomials for $x = -n, \cdots, -1, 0, 1, \cdots, n$. *Jordan*[5] considered the general case of equidistant orthogonal polynomials for equidistant values of x between a and b, the interval of two consecutive values of x being h ; moreover he treated the particular case of polynomials relative to $x = 0, 1, 2, \cdots, (n-1)$ which Chebisheff has examined in another way. *Essher* in his first publication[6] used such polynomials with respect to $x = -\frac{1}{2}(N-1) \cdots, \frac{1}{2}(N-1)$, and in his second[7] for $x = -n, \cdots, 0, 1, \cdots, n$. *Lorentz*[8] also introduced orthogonal polynomials for $x = -n, \cdots, 0, 1, \cdots, n$ and for $x = -2n+1, \cdots, -1, 1, 3, \cdots, 2n-1$, the interval in this latter case being obviously equal to two.

In later publications I showed new methods for using orthogonal polynomials for approximation[9] and graduation[10] which permit the results to be reached very rapidly. In the present paper the general case of orthogonal polynomials for equidistant values of the variable is to be discussed; the formulae given are valid for all orthogonal polynomials of equidistant values such as the polynomials of Gram, Essher and Lorentz. These are also discussed in this paper. At the end some very useful tables are appended.

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[2] ***Poincairé.*** Calcul des probabilités, Paris, 1896. p. 251.

[3] ***A. Quiquet.*** Sur une methode d'interpolation exposée par Henri Poincare. Proc. of the fifth International Congress of Mathematicians. Cambridge. 1913. p. 385.

[4] *J. Gram.* Ueber partielle Ausgleichung mittelst Orthogonalfunktionen, Bull. de l'Association des Actuaires Suisses, 1915.

[5] *Ch. Jordan.* Sur une série de Polynomes dont chaque somme partielle represente la meilleure approximation d'un degré donné suivant la méthode des moindres carrés. Proc. of the London Math. Soc. 1921.

[6] *F. Essher.* Ueber die Sterblichkeit in Schweden, Lund 1920

[7] *F. Essher.* On some methods of Interpolation. Scandia, 1930.

[8] *P. Lorentz.* Der Trend, Vierteljahreshefte zur Konjunkturforschung, Berlin 1928. Zweite Auflage, 1931.

[9] *K. Jordan,* Berechnung der Trendlinie auf Grund der Theorie der kleinsten Quadrate, Mitt. der Ungarischen Landeskommission für Wirtschaftsstatistik und Konjuncturenforschung 1930.

Ch. Jordan, Sur la détermination de la tendance séculare des grandeurs statistiques par la méthode des moindres carrés. Journal de la Société Hongroise de Statistique, 1929.

[10] *Ch. Jordan,* Statistique Mathématique, p. 291, Paris 1927, Gauthier Villars.

§ 2. *Some formulae of the Calculus of Finite Differences.* Since the functions considered in this paper correspond to $x = a$, $a+h$, $a+2h$, $a+3h \cdots$, the increment h being constant, the Calculus of Finite Differences will be found very useful.

By the first difference of $F(x)$ we mean $F(x+h) - F(x)$ which is denoted by $\Delta F(x)$, so that $\Delta F(x) = F(x+h) - F(x)$.

The n-th difference of $F(x)$ will be defined as

$$\Delta^n F(x) = \Delta\left[\Delta^{n-1} F(x)\right].$$

We shall term $F(x)$ the *indefinite sum of* $f(x)$ and denote it by $\Sigma f(x)$ if $\Delta F(x) = f(x)$. It follows that $\Sigma f(x) = F(x) + C$, where C is an arbitrary constant or a periodic function of periodicity h.

The n-th sum of $F(x)$ will be defined by

$$\Sigma^n f(x) = \Sigma\left[\Sigma^{n-1} f(x)\right].$$

This contains an arbitrary polynomial of the $(n-1)th$ degree, considering only the polynomial and neglecting the arbitrary periodic function.

It would be more precise to add the increment h to the above notation for the difference Δ and the indefinite sum Σ. Thus we could use $\underset{h}{\Delta}$ and $\underset{h}{\Sigma}$, respectively; but since in our formulae the increment will generally equal h we shall omit this index except in cases where doubt might otherwise arise.

By the *definite sum of* $f(x)$ *between* a *and* b, the following sum is understood (b being equal to $a + nb$ and n an integer)

$$f(a) + f(a+h) + f(a+2h) + \cdots + f(b-h)$$

and is denoted by

$$\sum_{x=a}^{b} f(x).$$

It can be shown that if $F(x)$ is the definite sum of $f(x)$, the above sum is equal to the difference of the values of $F(x)$ taken at the limits, so that we have

$$\sum_{x=a}^{b} f(x) = f(a) + f(a+h) + \cdots + f(b-h) = F(b) - F(a).$$

According to our definition it is evident that the value of the function $f(x)$ at the upper limit, i.e. $f(b)$, is not included in the sum between a and b . This terminology, although rather unusual, will be adhered to throughout this paper.

We shall have occasion to employ the following formulae of the calculus of finite differences.

Formula of differencing by parts, or of a product:

$$(1) \quad \Delta\left[u(x)\cdot v(x)\right] = u(x)\cdot\Delta v(x) + v(x+h)\cdot\Delta u(x).$$

Formula of the n-th difference of a product:

$$(2) \quad \Delta^{n}\left[u(x)\cdot v(x)\right] = \sum_{s=0}^{n+1} \binom{n}{s} \Delta^{n-s} u(x+sh)\cdot\Delta^{s} v(x).$$

Formula of the summation by parts, or of the sum of a product:

$$\Sigma\left[u(x)\cdot v(x)\right] = u(x)\cdot\Sigma v(x) - \Sigma\left[\Delta u(x)\cdot\Sigma v(x+h)\right].$$

Formula of the sum of a product, $u(x)$ being a polynomial of the n-th degree:

$$(3) \quad \begin{aligned} \Sigma\left[u(x)\cdot v(x)\right] &= u(x)\cdot\Sigma v(x) - \Delta u(x)\cdot\Sigma^{2} v(x+h) \\ &+ \Delta^{2} u(x)\cdot\Sigma^{3} v(x+2h) - \cdots + (-1)^{n}\Delta^{n} u(x)\cdot\Sigma^{n+1} v(x+nh). \end{aligned}$$

In the second member of this equation $\Sigma\, v(x)$ contains one arbitrary constant, $\Sigma^{2} v(x+h)$ contains, besides this, one more, and so on. Ultimately the second member will contain $n+1$ arbitrary constants — but since the first member contains but one constant it is evident that after simplification all of the terms of the arbitrary polynomial must necessarily vanish except the single constant term arising from $\Sigma^{n+1} v(x+nh)$.

Generalized binomial coefficients. We shall denote the generalized binomial coefficient of the n-th degree by

$$\binom{x}{n}_{h} = \frac{x(x-h)\cdot(x-2h)\cdots(x-nh+h)}{1\cdot 2\cdot 3\cdots n},$$

where the index h is associated with the decrement. If $h=1$ the index will be omitted, and the expression above will be equal to the ordinary binomial coefficient, $\binom{x}{n}$.

Let us mention the following well-known formulae:

$$\Delta^s \binom{x}{n}_h = h^s \binom{x}{n-s}$$

$$\Sigma \binom{x}{n}_h = \frac{1}{h} \binom{x}{n+1}.$$

Expansion of a function in a series of generalized binomial coefficients:

$$f(x) = f(a) + \binom{x-a}{1}_h \frac{\Delta f(a)}{h} + \cdots + \binom{x-a}{n} \frac{\Delta^n f(a)}{h^n}. \tag{4}$$

The generalized binomial coefficient can be expressed as an ordinary one by merely changing the variable. Thus if we place $(x-a)/h = \xi$ we have that

$$\binom{x}{n}_h = h \binom{\xi}{n}$$

and consequently if we write $F(\xi) = f(a+h\,\xi)$ it follows that $\Delta^s F(0) = \underset{h}{\Delta^s} f(a)$, and formula (4) may be written

$$\begin{aligned} F(\xi) &= F(0) + \binom{\xi}{1} \cdot \Delta F(0) + \binom{\xi}{2} \cdot \Delta^2 F(0) + \\ &\quad \cdots + \binom{\xi}{n} \cdot \Delta^n F(0). \end{aligned} \tag{4'}$$

Formulae (4) and (4′) are two different forms of Newton's series. The great importance of Newton's formula to the statistician is not yet sufficiently recognized by the latter, since he nearly always develops his functions in power series in spite of the fact that he is generally primarily concerned with the differences and the sums of his function. Now if a function be expanded in a Newton series as in (4) above, its m-th difference and its sum can be obtained immediately by means of the following formulae:

(5)
$$\Delta^m f(x) = \Delta^m f(a) + \binom{x-a}{1}_h \frac{\Delta^{m+1} f(a)}{h} + \cdots + \binom{x-a}{n-m}_h \frac{\Delta^n f(a)}{h^{n-m}};$$

(6)
$$\Sigma f(x) = \binom{x-a}{1}_h \frac{f(a)}{h} + \binom{x-a}{2}_h \frac{\Delta f(a)}{h^2} + \cdots + \binom{x-a}{n+1}_h \frac{\Delta^n f(a)}{h^{n+1}}$$

These operations would be very complicated if $f(x)$ were expanded into a power series. Although it is true that a power series would be more advantageous for determining either the derivatives or the integral of $f(x)$, we may remark that the statistician hardly ever needs these quantities. And for nearly all other operations, Newton's formula is at least as convenient as an expansion in a power series.

To illustrate the last remark — if $f(x)$ corresponding to a given value of x is needed, then in the case of a power series it is necessary to compute the values of $x, x^2, x^3, \cdots$ and these are obtained most readily by means of successive multiplication. In the case of a Newton series it is necessary to calculate $(x-a), (x-a)(x-a-h), (x-a)(x-a-h)(x-a-2h), \cdots\cdots$ and these should also be obtained by successive multiplication. The formula for changing the origin and that for changing the interval of observation are given in the following paragraph and are as simple as those arising in the case of power series. If a statistician expands a function into a power series and needing the differences of the function for $x = a$ calculates them separately, he doubles his work since these differences are precisely the coefficients in Newton's formula. In statistical research Newton's series should always be preferred to the power series.

§ 3. *Changing the origin.* In mathematical statistics it frequently occurs that it is necessary to change the origin of a set of observations. For instance, if $f(a)$ is the value of some quantity corresponding to the first of January of the year 1901

$$a = \text{January } 1, 1901,$$

h represents the interval of a year, and it follows that $x = a + \xi h$ represents January 1, $(1901 + \xi)$ where ξ is an integer. If we know the *Newton* expansion of $f(x)$ in generalized binomial coefficients $\binom{x-a}{\nu}_h$ that is

$$f(x) = f(a) + \binom{x-a}{1}_h \frac{\Delta f(a)}{h} + \dots + \binom{x-a}{n}_h \frac{\Delta^n f(a)}{h^n},$$

and desire the values of $f(x)$ counted from the first of July of 1901, then denoting

$$c = \text{July } 1, 1901$$

we need $f(x)$ expanded into a series of generalized binomial coefficients $\binom{x-c}{\nu}_h$, that is

$$(\alpha) \qquad f(x) = f(c) + \binom{x-c}{1}_h \frac{\Delta f(c)}{h} + \dots + \binom{x-c}{n}_h \frac{\Delta^n f(c)}{h^n}.$$

The coefficients of this expansion must be so determined that $x = c + \xi h$ will correspond to July 1, $(1901 + \xi)$. These are easily obtained by putting $x = c$ in the first equation of this paragraph,

$$f(c) = f(a) + \binom{c-a}{1}_h \frac{\Delta f(a)}{h} + \dots + \binom{c-a}{n}_h \frac{\Delta^n f(a)}{h^n}$$

and also placing $x = c$ in the s-th difference of the same equation, so that we have

$$(7) \quad \Delta^s f(c) = \Delta^s f(a) + \binom{c-a}{1}_h \frac{\Delta^{s+1} f(a)}{h} + \dots + \binom{c-a}{n-s}_h \frac{\Delta^n f(a)}{h^{n-s}}.$$

Substituting these values into equation (α) yields the required expansion.

Remark. If $c-a=mh$ where m is an integer, from the above equations it follows that

$$f(c)=f(a+mh) \text{ and } \Delta^s f(c)=\Delta^s f(a+mh).$$

An example of this is given in § 13.

§ 4. *Changing the interval* h. Sometimes a changing of the interval is needed in Newton's formula; this occurs for instance in statistics when a function $f(x)$, giving the value of some quantity corresponding to the middle of the year x, is known for several consecutive years $(h=1)$ by its Newton expansion (4), and it is necessary to obtain the values of the quantity corresponding to the first of each month. It would of course be possible to calculate these values by placing $x=1/12, 2/12, 3/12, \cdots$ in Newton's formula, but it is more advantageous to change the interval $(h=1)$ in formula (4) and deduce another one in which both the increment of the differences and the decrement of the generalized binomial coefficients are $k=1/12$. The formula thus obtained leads, by the method of summation of the differences (§ 11), more rapidly to the results. So, starting from formula

$$(8) \qquad f(x)=\sum_{m=0}^{n+1} \frac{1}{h^m}\binom{x-a}{m}_h \mathop{\Delta}_h^m f(a)$$

we have to deduce

$$(9) \qquad f(x)=\sum_{m=0}^{n+1} \frac{1}{k^m}\binom{x-a}{m}_k \mathop{\Delta}_k^m f(a).$$

To obtain this, it is sufficient to know that for $x=a$ the differences of the generalized binomial coefficients $\binom{x-a}{s}_h$ in a system of differences in which the increment is k may be written

$$(10) \qquad \left[\underset{k}{\Delta}^{s} \binom{x-a}{m}_h \right]_{x=a} = A^s_m .$$

To obtain these numbers, A^s_m, let us write the following identity,

$$\underset{k}{\Delta}^{s} \binom{x-a}{m}_h = \underset{k}{\Delta}^{s} \left[\binom{x-a}{m-1}_h \cdot \frac{x-a-mh+h}{m} \right]$$

and then deduce the s-th difference. By formula (2) we have

$$\underset{k}{\Delta}^{s} \binom{x-a}{m}_h = \frac{x-a-mh+h}{m} \underset{k}{\Delta}^{s} \binom{x-a}{m-1}_h + \frac{sk}{m} \underset{k}{\Delta}^{s-1} \binom{x-a}{m-1}_h .$$

Placing in this equation $x=a$, we obtain

$$(11) \qquad A^s_m = \frac{sk-(m-1)h}{m} A^s_{m-1} + \frac{sk}{m} A^{s-1}_{m-1} .$$

The complete solution of this *Equation of Partial Differences* with the interval k would be very complicated, but one may readily solve it for some particular values of s and then deduce successively the other values of A^s_m starting from the initial values which follow immediately from (10). These are that

$$A^1_1 = k,$$

$$A^0_m = 0, \text{ except that } A^0_0 = 1,$$

and $\quad A^{m+\nu}_m = 0$

Equation (11) can be solved *first* for $s=m$ yielding

$$A^m_m = kA^{m-1}_{m-1} = k^{m-1} A^1_1 = k^m ,$$

and *secondly* for $s=1$ from which we obtain

$$A'_m = \frac{k-(m-1)h}{m} A'_{m-1}$$

$$A'_{m-1} = \frac{k-(m-2)h}{m} A'_{m-2}$$

. . . .

so that by multiplying we easily obtain

$$A'_m = \binom{k}{m}_h ,$$

and *thirdly* for $s=m-1$ we may express successively the values of

$$A_m^{m-1}, A_{m-1}^{m-2}, \ldots\ldots, A'_2$$

and then multiply each $A_{m-\nu}^{m-\nu-1}$ by $k^{\nu}(m-\nu)/m$ and adding the products obtain the result

$$A_m^{m-1} = (m-1)k^{m-2}\cdot\binom{k}{2}_h .$$

The other values of A_m^s are obtained as indicated above. The following table contains the numbers necessary for binomial coefficients up to the fifth degree.

$A'_1 = k$ $\quad A'_2 = \binom{k}{2}_h$ $\quad A_2^2 = k^2$

$A'_3 = \binom{k}{3}_h$ $\quad A_3^2 = 2k\binom{k}{2}_h$ $\quad A_3^3 = k^3$

$A'_4\binom{k}{4}_h$ $\quad A_4^2 = \frac{k}{6}(7k-11h)\cdot\binom{k}{2}_h$ $\quad A_4^3 = 3k^2\binom{k}{2}_h$ $\quad A_4^4 = k^4$

$A'_5 = \binom{k}{5}_h$ $\quad A_5^2 = \frac{k}{2}(3k-5h)\cdot\binom{k}{3}_h$ $\quad A_5^3 = \frac{k^2}{2}(5k-7h)\binom{k}{2}_h$

$A_5^4 = 4k^3\binom{k}{2}_h$ $\quad A_5^5 = k^5$

The numbers A_m^s being known, we can immediately express $\binom{x-a}{m}_h$ in a *Newton* series, with the increment equal to k,

$$\binom{x-a}{m}_h = \binom{x-a}{1}_k \frac{A_m^1}{k} + \binom{x-a}{2}_k \frac{A_m^2}{k^2} + \binom{x-a}{3}_k \frac{A_m^3}{k^3} + \cdots .$$

It follows from (8) that

$$f(x) = \sum_{m=0}^{n+1} \Delta_h^m f(a) \cdot \frac{1}{h^m} \sum_{\nu=0}^{m+1} \binom{x-a}{\nu}_k \cdot \frac{A_m^\nu}{k^\nu}$$

and consequently the s-th difference of $f(x)$ for $x=a$ will be

$$\Delta_k^s f(a) = \sum_{m=s}^{n+1} \frac{1}{h^m} \cdot A_m^s \Delta_h^m f(a) .$$

When these values are placed in equation (9) the problem is solved. An example is given in § 13.

§ 5. *The problem of approximation.* The number of the given values will always be denoted by N in this paper. The values $y_0, y_1, y_2, \cdots, y_{N-1}$ correspond to $x=a, a+h, a+2h, \cdots, b-h$ where $b=a+Nh$. A parabola of the n-th degree, $y=f_n(x)$ is to be determined according to the principle of least squares, that is, so that the sum of the squares of the deviations $\left[f_n(x)-y\right]$ for $x=0,1,2,\cdots,N-1$ shall be a minimum. Hence the parameters in $f_n(x)$ must be so determined that the expression

$$S = \sum_{x=a}^{b} \left[f_n(x)-y\right]^2 \tag{12}$$

shall be a minimum.

To solve the problem in the ordinary way would require the solution of $n+1$ determinants of the n-th order. This would be very laborious, as those who have employed *Gauss's* method to solve this problem know. It is far more convenient to first expand the function $f(x)$ into a series of *orthogonal polynomials.*

Let $U_\nu = U_\nu(x)$ be such a polynomial — it is termed orthogonal if it satisfies the following relation

$$\sum_{x=a}^{b} U_\nu U_\mu = 0 \tag{13}$$

for all values of ν different from μ. The expansion of $f_n(x)$ can be written as follows

$$f_n(x) = a_0 U_0 + a_1 U_1 + a_2 U_2 + \cdots\cdots + a_n U_n \tag{14}$$

where the a_ν are constant parameters which must be evaluated according to the principle of least squares.

To render expression (12) a minimum it is necessary that the first derivative of S with respect to a_ν should vanish for all values of ν. This will produce $n+1$ equations which determine the $n+1$ parameters, namely

$$\sum_{x=a}^{b} U_\nu \left[a_0 U_0 + a_1 U_1 + a_2 U_2 + \cdots\cdots + a_n U_n - y \right] = 0$$

As a consequence of relation (13) these equations are so simplified that we may write

$$a_\nu \sum_{x=a}^{b} U_\nu^2 - \sum_{x=a}^{b} U_\nu \cdot y = 0. \tag{15}$$

The second condition of a minimum, namely that the expression

$$\sum_{\nu=0}^{n+1} \sum_{\mu=0}^{n+1} \frac{\partial^2 S}{\partial a_\nu \cdot \partial a_\mu} \, da_\nu \, da_\mu$$

shall be a *positive definite form* for all values of da_ν and da_μ, is also satisfied since in consequence of (13) this quantity is equal to

$$\sum_{x=a}^{b} U_\nu^2 (da_\nu)^2 > 0.$$

From (15) it may be concluded that the coefficients a_ν are independent of the degree n of the parabola of approximation. Consequently, if the coefficients $a_0, a_1, \cdots, a_n$ corresponding to a parabola of degree n have been calculated, then to obtain a further parabola of degree $n+1$ it is sufficient to determine only one further coefficient, a_{n+1} — the others will remain unchanged. This is of great importance. If the series (14) is limited at any term, the remaining expression will always satisfy condition (12).

§ 6. *Deduction of the polynomial* U_ν. Instead of starting from relation (13) we shall employ the following equivalent formula,

$$\sum_{x=a}^{b} F_{m-1}(x) \cdot U_m(x) = 0,$$

where $F_{m-1}(x)$ is an arbitrary polynomial of degree $m-1$. If we were to expand $F_{m-1}(x)$ into a series of U_ν polynomials we would return back to condition (13) again.

Applying formula (3), of the sum of a product, to the above expression yields

$$\Sigma\left[F_{m-1} \cdot U_m\right] = F_{m-1} \Sigma^1 U_m(x) - \Delta F_{m-1} \cdot \Sigma^2 U_m(x+h)$$
$$+ \Delta^2 F_{m-1} \cdot \Sigma^3 U_m(x+2h) - \cdots + (-1)^{m-1} \cdot \Delta^{m-1} F_{m-1} \Sigma^m U_m(x+mh-h).$$

Now, $\Sigma U_m(x)$ contains an arbitrary constant to which may be assigned such a value that $\Sigma U_m(a)=0$. But $\Sigma^2 U_m(x+h)$ contains an additional constant which can be chosen so that $\Sigma^2 U_m(a+h)=0$. Continuing after this fashion we may dispose of all these arbitrary constants in such a way that the expression for the definite sum will vanish for the lower limit $x=a$, that is

$$\Sigma \left[F_{m-1} \cdot U_m\right]_{x=a} = 0 .$$

But in order that the definite sum may be equal to zero it is necessary for the above expression to vanish also for the upper limit, $x=b$. But since $F(b)$ is arbitrary for all values of s it follows each expression obtained for $s=0,1,2,\cdots,m-1$ must vanish separately for $x=b$. From this we conclude that $(x-a)$ and $(x-b)$ must both be multiplying factors of $U(x)$. Considering for the moment only the first of these factors we may therefore write

$$U(x)=(x-a)\omega(x).$$

Applying to this expression the formula for the sum of a product, (3), we have

$$\Sigma^2 U_m(x) = \sum_{\nu=0}^{m+1} (-1)^\nu \frac{1}{h^{\nu+1}} \binom{x-a+\nu h}{\nu+2}_h \Delta^\nu \omega(x).$$

By successive summation we should find that $(x-a)(x-a-h)\cdots(x-a-mh+h)$ is a multiplying factor of $\Sigma^m U_m(x)$ and that we can assign the following form to this expression

$$\Sigma^m U_m(x) = \binom{x-a}{m}_h \psi(x) . \tag{16}$$

As $(x-b)$ must also be a multiplying factor of $\Sigma U_m(x)$, the same reasoning leads to the expression of $\Sigma^m U_m(x)$

$$\Sigma^m U_m(x) = C\binom{x-a}{m}_h \binom{x-b}{m}_h .$$

As this sum must be of degree $2m$, it follows that C is an arbitrary constant and we conclude that the general formula of the orthogonal polynomials with respect to $x = a, a+h, \cdots, b-h$, is the following

$$U_m(x) = C\Delta^m \left[\binom{x-a}{m}_h \binom{x-b}{m}_h\right] . \tag{17}$$

Starting from this expression, there are two different ways of deducing the expansion of $U_m(x)$ in *Newton*-series, as has been shown in the paper [5]. First, we can utilize formula (2), giving the m-th difference of a product, and obtain

$$U_m(x) = Ch^m \sum_{s=0}^{m+1} \binom{m}{s}\binom{x-a+sh}{s}_h \binom{x-b}{m-s}_h \tag{18}$$

Secondly, we can develop $\Sigma^m U_m$ into a *Newton*-series of generalized binomial coefficients $\binom{x-b}{s}_h$. According to formula (5), we have then that

$$\Sigma^m U_m(x) = C \sum_{s=0}^{2m+1} \frac{1}{h^s} \binom{x-b}{s}_h \Delta^s \left[\binom{x-a}{m}_h \binom{x-b}{m}_h\right]_{x=b} . \tag{19}$$

The s-th difference in this formula can be written according to (2) in the following manner

$$\left[h^s \sum_{\nu=0}^{s+1} \binom{s}{\nu} \binom{x-a+\nu h}{m-s+\nu}_h \binom{x-b}{m-\nu}_h\right] = \binom{s}{m}\binom{x-a+mh}{2m-s}_h h^s,$$

so it follows that

$$(20) \qquad \Sigma^m U_m(x) = C \sum_{s=m}^{2m+1} \binom{s}{m}\binom{b-a+mh}{2m-s}_h \binom{x-b}{s}_h ;$$

and finally putting $s=m+\nu$ into this expression, and determining the m-th difference, we see that

$$(21) \qquad U_m = Ch^m \sum_{\nu=0}^{m+1} \binom{x-b}{\nu}_h \binom{m+\nu}{m}\binom{b-a+mh}{m-\nu}_h .$$

Let us note that $U_0 = C$.

As $\Sigma^m U_m(x)$ is symmetric with respect to a and b, we can get two other formulae for $U_m(x)$ from (18) and (21) changing a into b and inversely. For instance, remarking that $b-a=Nh$, and that

$$\binom{a-b+mh}{m-\nu}_h = \binom{-N+m}{m-\nu} h^{m-\nu} = (-1)^{m-\nu}\binom{N-\nu-1}{m-\nu} h^{m-\nu}$$

it follows from (21) that

$$(21') \quad U_m(x) = Ch^{2m} \sum_{\nu=0}^{m+1} (-1)^{m-\nu} h^{-\nu} \binom{m+\nu}{m}\binom{N-\nu-1}{m-\nu}\binom{x-a}{\nu}_h .$$

The constant C is arbitrary, whatever value may be chosen for it. The orthogonal polynomials introduced by different investigators differ only in the value attributed to C. As these polynomials are closely related to *Legendre's* polynomials it seems most advisable to choose C in such a manner that for $h=0$ the limit of the polynomial U_m shall be equal to *Legendre's* polynomial P_m. For this purpose, we must put into (19) and (21) $a=-1$ and, $b=1$ and, $C=1.2.3\ldots m/2^m h^m$; then, deducing the limit for $h=0$, we obtain two known formulae for *Legendre's* polynomials.[11]

The choice of C is only important if we want to compute numerical values of $U_m(x)$ corresponding to any value of x; in this case C should be chosen so that the calculation of $U_m(x)$ shall be as short as possible. As we shall see later Essher in his first paper and also Gram proceeded in this manner. The above-given value that I chose for C is in this respect also very acceptable; and whenever numerical values of a_m are needed, we will adopt this value. It will be shown that our problem can be solved in a general way by leaving the constant C arbitrary.

§ 7. *Determination of the coefficients* a_m. These are given by formula (15); but first it is necessary to know the value of

$$\sum_{x=a}^{b} U_m^2$$

To determine this quantity we shall apply formula (3) of the sum of a product to the following indefinite sum

$$\begin{aligned}\Sigma\left[U_m U_m\right] = {} & U_m(x)\Sigma U_m(x) - \Delta U_m(x)\Sigma^2 U_m(x+h) \\ (22) \qquad & + \Delta^2 U_m(x)\Sigma^3 U_m(x+2h) - \cdots \\ & + (-1)^m \Delta^m U_m(x)\Sigma^{m+1} U_m(x+mh).\end{aligned}$$

[11] I proceeded in this way in the paper [5] *loc. cit.;* formulae (19) and (21) of the present paper are identical with (9) and (13) of the first paper, only the notation is somewhat different.

Let us now determine the quantities $\Sigma^{\mu} U_m(x+\mu h-h)$; they are easily obtained, if $\mu < m+1$, by deducing the $(m-\mu)$-th difference of $\Sigma^m U_m(x+\mu h-h)$. Starting from formula (20), after having replaced x by $x+\mu h-h$, it follows that

$$
(23) \qquad \begin{aligned} &\Delta^{m-\mu}\left[\Sigma^m U_m\right] = \Sigma^{\mu} U_m(x+\mu h-h) = \\ &\qquad C h^{\mu} \sum_{s=m}^{2m+1} \binom{x+\mu h-h-b}{s-m+\mu} \binom{s}{m} \binom{b-a+mh}{2m-s}. \end{aligned}
$$

At the upper limit, $x=b$ the first generalized binomial coefficient figuring in this formula can be expressed by an ordinary one,

$$
\binom{\mu h-h}{s-m+\mu}_h = \binom{\mu-1}{s-m+\mu} h^{s-m+\mu}
$$

but since $s \geqq m$, it follows that this expression is equal to zero; indeed an ordinary binomial coefficient $\binom{r}{t}$ is equal to zero, if r and t are integers, and if $r < t$.

$\Sigma^m U_m(x+\mu h-h)$ being symmetrical with respect to a and b, its $(m-\mu)$-th difference $\Sigma^{\mu} U_m(x+\mu h-h)$ will be symmetrical too; but this quantity, as we have seen, is equal to zero for $x=b$, therefore it must also be equal to zero for $x=a$.

We conclude that at both limits $x=a$ and $x=b$ all the terms of (22) will vanish in which $\mu < m+1$, — that is all terms except the last.

To evaluate this last term, we shall determine first the indefinite sum of $\Sigma^m U_m$. From (20) we easily get, after putting $x+mh$ instead of x,

$$\Sigma^{m+1} U_m(x+mh) = \frac{C}{h} \sum_{s=m}^{2m+1} \binom{x+mh-b}{s+1}_h \binom{s}{m} \binom{b-a+mh}{2m-s}_h .$$

Since $m < s+1$, this expression will vanish at the upper limit $x=b$ and its value at the lower limit, $x=a$, will be

$$\Sigma^{m+1} U_m(a+mh) = \frac{C}{h} \sum_{s=m}^{2m+1} \binom{s}{m} \binom{a-b+mh}{s+1}_h \binom{b-a+mh}{2m-s}_h .$$

Noting that $b-a=Nh$, we can express the two generalized binomial coefficients of this formula by ordinary ones; indeed they are equal to

$$\binom{-N+m}{s+1}\binom{N+m}{2m-s} h^{2m+1} = (-1)^{s+1} \binom{N-m+s}{s+1}\binom{N+m}{2m-s} h^{2m+1}$$

$$= (-1)^{s+1} \binom{N+m}{2m+1}\binom{2m+1}{s+1} h^{2m+1},$$

so that we have

$$\Sigma^{m+1} U_m(a+mh) = C h^{2m} \binom{N+m}{2m+1} \sum_{s=m}^{2m+1} (-1)^{s+1} \binom{s}{m}\binom{2m+1}{s+1}.$$

According to a known formula of *Combinatory Analysis* (*Netto*, p. 250.) the last sum of this equation is equal to $(-1)^{m+1}$. So it follows that

$$\Sigma^{m+1} U_m(a+mh) = (-1)^{m+1} C h^{2m} \binom{N+m}{2m+1}.$$

Moreover from (21) we deduce that

$$\Delta^m U_m(x) = C h^{2m} \binom{2m}{m}.$$

Now we have determined the values, at the limits, of all the quantities figuring in equation (22); hence the definite sum will be

$$(24) \qquad \sum_{x=a}^{b} U_m^2 = C^2 h^{4m} \binom{2m}{m} \binom{N+m}{2m+1}.$$

It is useful to remark that this expression is independent of the origin of the variable x. Let us note also that $\Sigma U_0^2 = C^2 N$.

To determine the coefficient a_m from equation (15) it is still necessary to determine $\Sigma y \cdot U_m(x)$. For this purpose, let us start from equation (21'), that is from the *Newton*-series of U_m, with respect to the generalized binomial coefficient $\binom{x-a}{s}_h$. We have

$$(25) \quad \sum_{x=a}^{b} U_m y = C h^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m} \binom{-N+m}{m-\nu} h^{-\nu} \sum_{x=a}^{b} \binom{x-a}{\nu}_h y.$$

In the analysis of continuous variables the quantity $\int_a^b x^s y\,dx$ is known as the moment of y of the s-th degree (we will say the s-th power-moment). We have seen that in analysing equidistant discontinuous variables it is not advantageous to operate with powers, but that it is far better to express the quantities in binomial coefficients. Indeed if an expression were given in power-series, we could not consider it to be a full solution, as it would still be advantageous to transform it into a binomial series. Therefore we have no use for power-moments and shall introduce binomial moments. The generalized *binomial moment* of degree ν will be denoted by $\mathcal{B}_\nu$ and defined by the following equation

$$(26) \qquad \sum_{x=a}^{b} \binom{x-a}{\nu}_h y = \mathcal{B}_\nu .$$

This can be easily expressed by ordinary binomial coefficients, for if we introduce a new variable $\xi = (x-a)/h$, we have

$$(27) \qquad \mathcal{B}_\nu = h^\nu \sum_{\xi=0}^{N} \binom{\xi}{\nu} y .$$

As will be shown later in (§ 9) there is a far better method for rapidly computing the binomial moments than there is in the case of power-moments.

Several statisticians have remarked that it is not advisable to introduce moments of high order into calculations. In fact, if N is large, these numbers will increase rapidly with the order of the moment, will become very large, and their coefficients in the formulae will necessarily become very small. It is very difficult to operate with such numbers, the causes of errors being many.

To obviate this inconvenience, I have introduced the *mean binomial moment;* that of the ν-*th* degree will be denoted by $\mathcal{T}_\nu$ and defined by

$$\mathcal{T}_\nu = \sum_{x=a}^{b} \binom{x-a}{\nu}_h y \Big/ \sum_{x=a}^{b} \binom{x-a}{\nu}_h .$$

The sum figuring in the denominator is according to § 2 equal to

$$\frac{1}{h} \binom{b-a}{\nu+1}_h = \binom{N}{\nu+1} h^\nu ,$$

so that T_ν will be

$$(28) \qquad T_\nu = \sum_{x=a}^{b} \binom{x-a}{\nu}_h y \Big/ \binom{N}{\nu+1} h^\nu .$$

If we introduce the variable ξ mentioned above, we obtain

$$T_\nu = \sum_{\xi=0}^{N} \binom{\xi}{\nu} y \Big/ \binom{N}{\nu+1} .$$

Consequently the mean binomial moments are independent of the origin of the variable x and of the interval h. The binomial moment increases rapidly with N and also with ν, though less rapidly than the ordinary power-moments. However, the mean binomial moment remains of the order of magnitude of y, whatever N or ν may be. For instance if $y=k$, it follows that $T_\nu=k$ for any value of ν.

Substituting in (25) the value of T_ν obtained in (28), we have

$$\sum_{x=a}^{b} U_m(x) y = Ch^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m} \binom{-N+m}{m-\nu} \binom{N}{\nu+1} T_\nu .$$

To this expression we can give the following form,

$$(-1)^m Ch^{2m} (m+1) \binom{N}{m+1} \sum_{\nu=0}^{m+1} (-1)^\nu \binom{m+\nu}{m} \binom{m}{\nu} \frac{T_\nu}{\nu+1} .$$

To simplify we shall write

$$(29) \qquad \beta_{m\nu} = (-1)^{m+\nu} \binom{m+\nu}{m} \binom{m}{\nu} \frac{1}{\nu+1}$$

These numbers are integers, and as they are very useful are presented in the following table, which presents all the numbers necessary for parabolas up to the tenth degree.

Table for $\beta_{m\nu}$

$m \backslash \nu$	0	1	2	3	4	5	6	7	8	9	10
1	-1	1									
2	1	-3	2								
3	-1	6	-10	5							
4	1	-10	30	-35	14						
5	-1	15	-70	140	-126	42					
6	1	-21	140	-420	630	-462	132				
7	-1	28	-252	1050	-2310	2772	-1716	426			
8	1	-36	420	-2310	6930	-12012	12012	-6435	1430		
9	-1	45	-660	4620	-18018	42042	-60060	51480	-24310	4862	
10	1	-55	990	-8580	42042	-126126	240240	-291720	218780	-92378	16796

The following relation can be used for checking these numbers:

$$\beta_{m0} + \beta_{m1} + \beta_{m2} + \cdots + \beta_{mm} = 0.$$

Moreover, let us put

$$(30) \qquad \sum_{\nu=0}^{m+1} \beta_{m\nu} T_\nu = \Theta_m \quad [12]$$

If we already know the mean binomial moments, the values of Θ_m may readily be computed with the aid of the table above.

Finally we obtain

$$(31) \qquad \sum_{x=a}^{b} U_m(x) y = C h^{2m}(m+1)! \binom{N}{m+1} \Theta_m .$$

As this expression could be termed the orthogonal moment of degree m of y, we could consider Θ_m as a certain *mean ortho-*

---

[12] $\beta_{m\nu}$ and Θ_m of this paper correspond respectively to $(-1)^m \beta_{m\nu}$ and $(-1)^m \Theta_m$ of the paper *loc. cit.* [9].

gonal moment of y of degree m. These quantities, as we shall see, are very important; they are independent of the origin, of the interval, and of the constant C. Thus, they are valid for all orthogonal polynomials.

In particular $\Theta_0 = T_0 = B_0/N$ is the arithmetic mean of the given quantities y.

Finally, equations (15), (24) and (31) yield, after simplification, the formula for a_m, that is,

$$a_m = \frac{(2m+1)\Theta_m}{Ch^{2m}\binom{N+m}{m}} . \tag{32}$$

The coefficient a_m is independent of the origin of x. In particular if $m=0$, we have $a_0 = \Theta_0/C$.

Now the equation of the approximating parabola is known, in the form of its expansion into a series of U_m polynomials (13), and our problem is solved. But if it is necessary to compute a table of the values of the parabola $f_n(x)$ corresponding to $x=a$, $a+h$, $a+2h$, $\cdots$, the corresponding values of $U_m(x)$ must first be calculated by formula (21′). Although this seems easy enough, yet if N is large, the computation is a tedious one even with the aid of Table IV which presents the values of the binomial coefficients, $\binom{\xi}{s}$ for integral values of ξ. If it should be necessary to compute $\binom{\xi}{s}$ for non-integer values of ξ, the calculation must be made in the ordinary way, that is, by multiplication. At all events the calculation would not be shorter if the $U_m(x)$ were expanded into power-series. The labor will be decreased, however, if tables giving the values of $U_m(x)$ corresponding to $x=a$, $a+h$, $a+2h$, $\cdots$, are available. I adopted this procedure in my paper published in 1921 and later *Essher,* and also *Lorentz,* did the same[18].

[18]See *loc. cit.* [5], [9], [10], [7] and [8].

It will be shown, however, that these tables are superfluous, as by a transformation of formula (14) into a *Newton*-series, we can get the required values still more rapidly by the method of *addition of differences;* and if an interpolation is necessary for any values whatsoever of x , *Newton's* formula will give it in the shortest way.

Moreover, by this method we shall be independent of the value of the constant C , that is of the orthogonal polynomial chosen.

We will give in § 10 formulae leading directly to *Newton's* expansion, so that the computer will have nothing to do with the orthogonal polynomials themselves. He will only have to compute the binomial moments B_ν , and then deduce the mean orthogonal moments Θ_m, which will give, with appropriate coefficients, the solution in *Newton's* series.

§ 8. *Determination of the measure of obtained approximation.* The approximation is generally measured by the *mean-square-deviation* σ_n^2 , that is by the mean of the squares of deviations between the parabola and the given values y . It is expressed by

$$\sigma_n^2 = \frac{1}{N} \sum_{x=a}^{b} \left[f_n(x) - y \right]^2 .$$

If in this formula we put in place of $f_n(x)$ its expansion (14) in orthogonal polynomials, we shall obtain in consequence of the condition, (13) of orthogonality,

$$\sigma_n^2 = \frac{1}{N} \sum_{x=a}^{b} y^2 + \frac{1}{N} \sum_{m=0}^{n+1} \left[a_m^2 \sum_{x=a}^{b} U_m^2 - 2a_m \sum_{x=a}^{b} U_m \, y \right] ,$$

and on account of (13)

$$\sigma_n^2 = \frac{1}{N} \sum_{x=a}^{b} y^2 - \frac{1}{N} \sum_{m=0}^{n+1} \left[a_m^2 \sum_{x=a}^{b} U_m^2 \right] . \tag{32'}$$

We can still simplify this result, since from formulae (24) and (32) it follows that

$$\frac{1}{N} a_m^2 \sum_{x=a}^{b} U_m^2 = \frac{(2m+1)\binom{N-1}{m}\theta_m^2}{\binom{N+m}{m}} \tag{33}$$

And if to abbreviate we write

$$C_m = (2m+1)\binom{N-1}{m} \Big/ \binom{N+m}{m} \tag{34}$$

and note that $c_0 = 1$, then the mean-square-deviation will be

$$\sigma_n^2 = \frac{1}{N}\sum_{x=a}^{b} y^2 - \theta_0^2 - C_1\theta_1^2 - C_2\theta_2^2 - \cdots - C_n\theta_n^2 . \tag{35}$$

The coefficients C_m can be easily calculated by (34). using Table IV for the binomial coefficients. But we shall see that C_m is equal to the absolute value of a certain quantity, which we have denoted by C_{m0} and which is given in Table III for values of N up to 100.

As the mean orthogonal moments are already known, therefore to obtain σ_n^2 it is sufficient to compute Σy^2.

Remark. All quantities figuring in (35) are independent of the origin, of the interval and of the constant C ; consequently this formula is valid for all systems of orthogonal polynomials.

Sometimes it is necessary to know σ_{ns}^2 , the mean-square of deviations between two parabolas, one of degree n , the other of degree s , (where $n > s$), both approximating to the same values of y ; that is

$$\sigma_{ns}^2 = \frac{1}{N}\sum_{x=a}^{b} \left[f_n(x) - f_s(x) \right]^2 .$$

If in this formula we place the values of $f_n(x)$ and $f_s(x)$ expressed in series (13) of orthogonal polynomials, we have

$$f_n(x) - f_s(x) = a_{s+1} U_{s+1} + a_{s+2} U_{s+2} + \cdots + a_n U_n$$

and in consequence of (14)

$$\sigma_{sn}^2 = \frac{1}{N} \sum_{m=s+1}^{n+1} a_m^2 \sum_{x=a}^{b} U_m^2$$

then, using formulae (33) and (34) we find that

$$\sigma_{ns}^2 = C_{s+1} \Theta_{s+1}^2 + \cdots + C_n \Theta_n^2 .$$

Having obtained the equation of an approximating parabola of degree n, it may develop that the resultant approximation is not close enough, the mean-square-deviation being too large. We may then pass on to a parabola of degree $n+1$ by determining only the one additional coefficient a_{n+1}, —the others do not change. For this purpose we must compute T_{n+1} and then Θ_{n+1} ; the coefficient will be given by (32). The new mean-square-deviation will be

$$\sigma_{n+1}^2 = \sigma_n^2 - C_{n+1} \Theta_{n+1}^2 .$$

Remark. If the binomial moments $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ were given, and we proceeded to determine a polynomial of degree n such that its first $n+1$ binomial moments should equal respectively to the values given above, thus employing the *principle of moments,* we should reach the same result as though we had imposed the principle of *least squares.* In the case of polynomials both principles lead to the same function.

§ 9. *Determination of the binomial moments. Chetverikoff* has given a very good method for their determination, which dispenses with all multiplication. We have seen that

$$B_\nu = \sum_{x=a}^{b} \binom{x-a}{\nu}_h y = h^\nu \sum_{\xi=0}^{N} \binom{\xi}{\nu} y \,.$$

Chetverikoff's method produces the last sum, that is the ordinary binomial moment $(h=1)$; to obtain the generalized binomial moment B_ν this must still be multiplied by h^ν; but it is needless to carry out this multiplication, since we need only the mean-binomial-moment T_ν, which is given in both cases by

$$T_\nu = \sum_{\xi=0}^{N} \binom{\xi}{\nu} y \Big/ \binom{N}{\nu+1} .$$

The method consists in the following: Let us denote by $y(\xi)$ the value of y corresponding to ξ ; in the first column of a table the values of $y(\xi)$ are written in the reverse order of magnitude of ξ, that is

$$y(N-1),\ y(N-2),\ \dots\dots\ y(1),\ y(0).$$

Into the first line of all the columns we write the same number, $y(N-1)$. Into the ν-th line of column μ we put the sum of the two numbers figuring in line $\nu-1$ of column μ, and in line ν of column $\mu-1$.

In column μ we stop at the line $N-\mu+2$; the number figuring there will be $\Sigma\binom{\xi}{\mu-2} y$; to obtain the mean binomial-moment of degree $\mu-2$ this must be divided by $\binom{N}{\mu-1}$. If we want the mean-binomial-moments $T_0, T_1, T_2, \cdots, T_n$, we must compute $n+1$ columns. An example is given in § 13.

An elementary demonstration of this method is given in the papers loc. cit. [9] and [10]. We will give here a more direct one. Let us denote by $\varphi(\nu,\mu)$ the number written into the ν-th line of column μ. The rule of computation will be

$$\varphi(\nu,\mu-1)+\varphi(\nu-1,\mu)=\varphi(\nu,\mu).$$

This is an equation of *partial differences* of the first order which may be written as follows:

$$(\alpha) \qquad \varphi(\nu+1,\mu+1)-\varphi(\nu+1,\mu)-\varphi(\nu,\mu+1)=0.$$

Let us solve it utilizing *Laplace's* method of *generating functions*.

We will call $u=u(t,\mu)$ the generating function of $\varphi(\nu,\mu)$ with respect to ν, if in the expansion of u in powers of t the coefficient of t^ν is equal to $\varphi(\nu,\mu)$, where μ is a parameter of u. Since $\varphi(0,\mu)=0$, then we have,

$$u=\varphi(1,\mu)t+\varphi(2,\mu)t^2+\cdots+$$
$$\varphi(\nu,\mu)t^\nu+\varphi(\nu+1,\mu)t^{\nu+1}+\cdots$$

From this we easily deduce the generating function of $\varphi(\nu+1,\mu)$. If we divide both members by t, the coefficient of t^ν in the second member will be $\varphi(\nu+1,\mu)$. Hence $u(t,\mu)/t$ is the generating function sought. Since μ is a parameter, it follows from the preceding that the generating function of $\varphi(\nu,\mu+1)$ will be $u(t,\mu+1)$, and that of $\varphi(\nu+1,\mu+1)$ will be $u(t,\mu+1)/t$. If in equation (α) above we substitute the corresponding generating functions for the function Φ, we obtain

$$(\beta) \qquad (1-t)u(t,\mu+1) - u(t,\mu) = 0.$$

In this way we have reduced the partial difference-equation (α) of the function Φ to an ordinary difference-equation of its generating function u. The solution expanded into powers of t, will give the function Φ itself, which is sought.

Equation (β) is a homogeneous linear equation with constant coefficients, t being only a parameter from this point of view, and can be solved immediately, yielding

$$(\gamma) \qquad u = \omega(t)/(1-t)^{\mu}$$

where $\omega(t)$ is an arbitrary function of t; and may be determined by the initial values, that is, by the values put into the first column. Placing $\mu = 1$ into equation (γ) we obtain the generating function of these values. Hence,

$$\omega(t)/(1-t) = y(N-1)t + y(N-2)t^2 + \cdots + y(0)t^N.$$

Finally we have

$$u = \left[y(N-1)t + \cdots + y(N-z)t^z + \cdots + y(0)t^N\right](1-t)^{-\mu+1}.$$

Since

$$(1-t)^{-\mu+1} = \sum_{s=0}^{\infty} \binom{\mu-2+s}{\mu-2} t^s,$$

we may obtain the coefficient of t^{ν} in the expansion of u.

By letting $s = \nu - z$ we have therefore that

$$\Phi(\nu,\mu) = \sum_{z=1}^{N+1} \binom{\mu - 2 + \nu - z}{\mu - 2} y(N - z).$$

If we place in this expression $\nu = N - \mu + 2$ and $N - z = \xi$, we see that

$$\Phi(N - \mu + 2, \mu) = \sum_{\xi=0}^{N} \binom{\xi}{\mu - 2} y(\xi).$$

We conclude, therefore, that the number figuring in line $N - \mu + 2$ of column μ will be the ordinary binomial moment of degree $\mu - 2$. It was this that was to be demonstrated.

§ 10. *Transformation of the orthogonal series into Newton's expansion.* Since the approximating parabola and the mean square deviation are independent of the constant of the orthogonal polynomial used, it is natural to transform equation (13) so that it shall also be independent of this constant. This can be done by a transformation into *Newton's* series.

We have seen that the coefficients of *Newton's* series (4) are

$$f_n(a), \Delta f_n(a), \Delta^2 f_n(a), \dots, \Delta^n f_n(a).$$

To obtain this series, therefore, it is sufficient to determine these quantities, starting from the orthogonal expansion.

Deriving from formula (13) the s-th difference of $f_n(x)$, we have

$$\Delta^s f_n(x) = \sum_{m=s}^{n+1} a_m \Delta^s U_m(x). \tag{36}$$

To obtain $\Delta^s U_m$, we start from formula (21′). Since $\binom{x-a}{\nu-s}_h$ is a multiplying factor of the s-th difference of U_m, we conclude that for $x=a$, all terms obtained from (21′) will vanish except the term in which $\nu=s$. Hence we have for $s=0,1,2,\cdots,m$

$$\Delta^s U_m(a)=Ch^{2m}\binom{m+s}{m}.$$

Placing this value, and that of a_m taken from (32), into equation (36) we get

$$\Delta^s f_n(a)=\sum_{m=s}^{n+1}(-1)^{m-s}(2m+1)\binom{m+s}{m}\frac{\binom{N-s-1}{m-s}\Theta_m}{\binom{N+m}{m}},$$

where Θ_m is the mean orthogonal moment of § 7. To abbreviate we shall write

$$(37)\qquad C_{ms}=(-1)^{m-s}(2m+1)\binom{m+s}{m}\frac{\binom{N-s-1}{m-s}}{\binom{N+m}{m}}.$$

Finally we find that

$$(38)\qquad \Delta^s f_n(a)=C_{ss}\Theta_s+C_{s+1,\,s}\Theta_{s+1}+\cdots+C_{ns}\Theta_n.$$

For instance, we have for $s=0$,

$$f_n(a)=C_{00}\Theta_0+C_{10}\Theta_1+C_{20}\Theta_2+\cdots+C_{n0}\Theta_n$$

where $C_{00}=1$. Let us remark again, that the second member of equation (38) is independent of the origin, of the interval, and of the constant C of the orthogonal polynomial chosen.

The importance of the numbers C_{ms} was first recognized by *W. Kviatovszky*, who calculated a table for these numbers. This table has not been published. The author's Table III is more extensive, giving these numbers with more decimals for N up to 100, and for parabolas up to the seventh degree.

Having obtained, by the above method, the *Newton* expansion of the approximating parabola, it may happen that the expansion corresponding to a parabola of degree $n+1$ is desired, and this requires the calculation of Θ_{n+1}. The coefficients of the new binomial expansion of $f_{n+1}(x)$ are easily deduced from those of $f_n(x)$ previously obtained, since

$$\Delta^s f_{n+1}(a) = \Delta^s f_n(a) + C_{n+1,s}\,\Theta_{n+1}.$$

The work previously done is therefore not lost.

§ 11. *Method of the addition of differences.* Knowing the coefficients of *Newton's* formula $f_n(a), \Delta f_n(a), \cdots \Delta^n f_n(a)$, we can proceed to calculate a table of the values of $f_n(x)$ corresponding to $x = a,\ a+h,\ a+2h, \cdots, b-h$ by adding the differences. This method has been used by *H. Henning* in his remarkable paper on the Trend-lines.[14] It proceeds as follows. The function $f_n(x)$ being of degree n, it is evident that $\Delta^n f_n(x) = \Delta^n f_n(a)$ is a constant. Into the first line of the first column of a table we shall write the number $\Delta^{n-1} f_n(a)$. Into the other lines of the first column we put the number of the preceding line of the same column, increased by $\Delta^n f_n(a)$. We stop in this column at line $N-n+1$. According to *Newton's* formula, we have

$$\Delta^{n-1} f_n(x+h) = \Delta^{n-1} f_n(x) + \Delta^n f_n(x)$$

[14] *H. Henning,* Die Analyse von Wirschaftskurven, Vierteljahreshefte zur Konjunkturforschung. Berlin 1927.

or

$$\Delta^{n-1} f_n(x) = \Delta^{n-1} f_n(a) + \xi \Delta^n f_n(a)$$

where $\xi = (x-a)/h$; hence the first column will contain the values of the $(n-1)$-th differences of $f_n(x)$. It is advisable, before continuing, to check the last number in this column by the formula above, putting therein $\xi = N - n$.

Into the first line of the μ-th column we write the number $\Delta^{n-\mu} f_n(a)$, and into the ν-th line of the same column the sum of the numbers figuring in line $\nu-1$ of column $\mu-1$ and column μ. The computation will be stopped in this column at the line $N-n+\mu$. The μ-th column will contain the values of the $(n-\mu)$-th differences of $f_n(x)$, which follows from *Newton's* formula

$$\Delta^{n-\mu} f_n(x+h) = \Delta^{n-\mu} f_n(x) + \Delta^{n-\mu+1} f_n(x)$$

or

$$\Delta^{n-\mu} f_n(x) = \Delta^{n-\mu} f_n(a) + \binom{\xi}{1} \Delta^{n-\mu+1} f_n(a) + \tag{39}$$
$$\cdots + \binom{\xi}{\mu} \Delta^n f_n(a).$$

Before going on, the last number of the column μ should be checked by the preceding formula, putting into it $\xi = N - n + \mu - 1$.

We continue in the same manner, — the last column to be computed is the n-th and will contain the values of $f_n(a)$, $f_n(a+h)$, $\ldots$, $f_n(b-h)$. This last number can be checked by formula (39), by making the substitution $\mu = n$ and $\xi = N-1$. An example is given in § 13.

§ 12. *Recapitulation of the operations.* To solve the problem of approximation of § 5, it is necessary first to compute the mean-binomial-moments $T_0, T_1, \cdots, T_n$. This is done by drawing up

Chetverikoff's table (§ 9) and dividing the number in the line $N-\mu+2$ of column μ by $\binom{N}{\mu-1}$. We obtain in this way $T_{\mu-2}$: this must be repeated in every column.

Then the numbers $\beta_{m\nu}$ are taken from Table I., and the mean-orthogonal-moments $\Theta_0, \Theta_1, \cdots \Theta_n$ are calculated by

$$(30) \quad \Theta_0 = T_0, \quad \Theta_m = \beta_{m0} T_0 + \beta_{m1} T_1 + \cdots + \beta_{mm} T_m .$$

Then $\Sigma y^2/N$ is computed. The numbers $C_m = |C_{m0}|$ are taken from Table III, or if this table fails, calculated by formula (34) and by Table IV which gives the binomial coefficients. The mean-square-deviation is calculated by formula

$$(35) \qquad \sigma_n^2 = \Sigma y^2/N - \Theta_0^2 - C_1 \Theta_1^2 + \cdots - C_n \Theta_n^2 .$$

If this quantity is conveniently small, the approximation is considered close enough; if not, we proceed to calculate Θ_{n+1}, and σ_{n+1}^2 and so on until a sufficiently small mean-square-deviation is reached.

Now we proceed to deduce the *Newton's* expansion of the required function $f_n(x)$. The numbers C_{ms} are taken from Table III, or if this table fails, calculated by formula (37) with the aid of Table IV.

The constants of *Newton's* formula are given by (38)

$$f_n(a) = \Theta_0 + C_{10} \Theta_1 + C_{20} \Theta_2 + \cdots + C_{n0} \Theta_n$$

$$\Delta f_n(a) = C_{11} \Theta_1 + C_{21} \Theta_2 + \cdots + C_{n1} \Theta_n$$

$$\Delta^2 f_n(a) = C_{22}\,\theta_2 + C_{32}\,\theta_3 + \cdots + C_{n2}\,\theta_n$$

— — — — — — — — — — — — — —

$$\Delta^s f_n(a) = C_{ss}\,\theta_s + C_{s+1,\,s}\,\theta_{s+1} + \cdots + C_{ns}\,\theta_n$$

$$\Delta^n f_n(a) = C_{nn}\,\theta_n$$

Now the equation of the parabola is known in the form of its *Newton*-series

$$f_n(x) = f_n(a) + \binom{\xi}{1}\Delta f_n(a) + \binom{\xi}{2}\Delta^2 f_n(a) + \cdots + \binom{\xi}{n}\Delta^n f_n(a) \tag{4}$$

where $\xi = (x-a)/h$. This will be considered as the desired solution. As has been said, it is quite useless to expand the parabola in powers of x.

If it is necessary to deduce a parabola of degree $n+1$ starting from equation (4); the new coefficients will be

$$f_{n+1}(a) = f_n(a) + C_{n+1,\,0}\,\theta_{n+1}$$

$$\Delta^s f_{n+1}(a) = \Delta^s f_n(a) + C_{n+1,\,s}\,\theta_{n+1}.$$

Generally a table of the values corresponding to the parabola and to $x = a, a+h, \cdots, b-h$, is needed; this will be computed by the method of addition of differences (§ 11). The last column will contain the required values.

§ 13. *Example 1.* Let us choose an example given by *Lorentz*[15] in which six values of y are given, and where $N=6$,

[15]loc. cit. [8], p. 21.

$h = 2$ and $a = -5$. The approximating parabolas of degrees 1, 2, 3, 4, and 5 are required.

The values of y corresponding to $x = -5, -3, -1, 1, 3, 5$ are written in the first column of the table below in reversed order of magnitude of x . The other numbers of the table are obtained by *Chetverikoff's* method (§ 9).

12293	12293	12293	12293	12293	12293	12293
10875	23168	35461	47754	60047	72340	
10058	33226	68687	116441	176488		
10018	43244	111931	228372			
8530	51774	163705				
7880	59654					

The required *mean-binomial-moments* will be

$T_0 = 59654/6 = 9942,3333$ * $\quad T_3 = 176488/15 = 11765,8667$

$T_1 = 163705/15 = 10913,6667 \quad T_4 = 72340/6 = 12056,6667$

$T_2 = 228372/20 = 11418,6 \quad T_5 = 12293$

The *mean-orthogonal-moments* are

$$\Theta_0 = T_0 = 9942,3333$$
$$\Theta_1 = T_1 - T_0 = 971,3333$$
$$\Theta_2 = 2T_2 - 3T_1 + T_0 = 38,3333$$
$$\Theta_3 = 5T_3 - 10T_2 + 6T_1 - T_0 = 183$$
$$\Theta_4 = 14T_4 - 35T_3 + 30T_2 - 10T_1 + T_0 = 351,6667$$
$$\Theta_5 = 42T_5 - 126T_4 + 140T_3 - 70T_2 + 15T_1 - T_0 = -1152$$

The squares of the mean-orthogonal-moments are

$\Theta_0^2 = 98849985,48 \qquad \Theta_3^2 = 33489$

$\Theta_1^2 = 943488,38 \qquad \Theta_4^2 = 123669,45$

$\Theta_2^2 = 1484,82 \qquad \Theta_5^2 = 1327104$

$$\Sigma y^2/6 = 100960410,3$$

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*Editor's note. In this country we usually write $\frac{69654}{6} = 9942.3333$ instead of $9942,3333$. For the purposes of this paper I prefer to use the continental notation appearing in Professor Jordan's manuscript. I agree that the following typical product of three factors $\frac{1}{k} \cdot 341,77395 \cdot \binom{x+5}{1}$ appearing on page 300 is less liable to be confused than if written $\frac{1}{k} \cdot 341.77395 \cdot \binom{x+5}{1}$.

From Table III we get:

$$C_{10} = -2,14285714 \qquad C_{40} = 0,21428571$$
$$C_{20} = 1,78571429 \qquad C_{50} = -0,02380952$$
$$C_{30} = -0,83333333$$

Now the mean-square-deviations for the parabolas of degree 0, 1, 2, 3, 4, 5 will be:

$$\sigma_0^2 = \Sigma y^2/6 - \theta_0^2 = 2110424,82$$
$$\sigma_1^2 = \sigma_0^2 - C_1\theta_1^2 = 88664,05$$
$$\sigma_2^2 = \sigma_1^2 - C_2\theta_2^2 = 86012,58$$
$$\sigma_3^2 = \sigma_2^2 - C_3\theta_3^2 = 58105,08$$
$$\sigma_4^2 = \sigma_3^2 - C_4\theta_4^2 = 31604,49$$
$$\sigma_5^2 = \sigma_4^2 - C_5\theta_5^2 = 6,80$$

As the parabola of degree 5 passes through the given six points, therefore σ_5^2 should be exactly equal to zero.

Let us now proceed to the determination of *Newton's* formula corresponding to the five parabolas required. It is to be observed that the amount of work required to solve this problem, is independent of the number of observations. First the following numbers must be taken from Table III:

$$C_{11} = 0,85714286 \qquad C_{22} = 1,07142857 \qquad C_{43} = -3$$
$$C_{21} = -2,14285714 \qquad C_{32} = -2,5 \qquad C_{53} = 1,3333333$$
$$C_{31} = 2 \qquad C_{42} = 1,92857143 \qquad C_{44} = 3$$
$$C_{41} = -0,85714286 \qquad C_{52} = -0,5 \qquad C_{54} = -3$$
$$C_{51} = 0,14285714 \qquad C_{33} = 1,66666667 \qquad C_{55} = 6$$

Formulae (38) of the preceding paragraph give:

$$f_1(a) = \Theta_0 + C_{10}\,\Theta_1 = 7860{,}90480$$

$$f_2(a) = f_1(a) + C_{20}\,\Theta_2 = 7929{,}71427$$

$$f_3(a) = f_2(a) + C_{30}\,\Theta_3 = 7777{,}21427$$

$$f_4(a) = f_3(a) + C_{40}\,\Theta_4 = 7852{,}57142$$

$$f_5(a) = f_4(a) + C_{50}\,\Theta_5 = 7779{,}99998$$

$$\Delta f_1(a) = C_{11}\,\Theta_1 = 832{,}57140$$

$$\Delta f_2(a) = \Delta f_1(a) + C_{21}\,\Theta_2 = 750{,}00005$$

$$\Delta f_3(a) = \Delta f_2(a) + C_{31}\,\Theta_3 = 1116{,}00005$$

$$\Delta f_4(a) = \Delta f_3(a) + C_{41}\,\Theta_4 = 814{,}57114$$

$$\Delta f_5(a) = \Delta f_4(a) + C_{51}\,\Theta_5 = 650{,}00002$$

$$\Delta^2 f_2(a) = C_{22}\,\Theta_2 = 41{,}28568$$

$$\Delta^2 f_3(a) = \Delta^2 f_2(a) + C_{32}\,\Theta_3 = -416{,}21432$$

$$\Delta^2 f_4(a) = \Delta^2 f_3(a) + C_{42}\,\Theta_4 = 262{,}00003$$

$$\Delta^2 f_5(a) = \Delta^2 f_4(a) + C_{52}\,\Theta_5 = 838{,}00003$$

$$\Delta^3 f_3(a) = C_{33}\,\Theta_3 = 305{,}00000$$

$$\Delta^3 f_4(a) = \Delta^3 f_3(a) + C_{43}\,O_4 = -705{,}00010$$

$$\Delta^3 f_5(a) = \Delta^3 f_4(a) + C_{53}\,\Theta_5 = -2286{,}00010$$

$$\Delta^4 f_4(a) = C_{44}\,\Theta_4 = 1055{,}00010$$

$$\Delta^4 f_5(a) = \Delta^4 f_4(a) + C_{54}\,\Theta_5 = 4511{,}00010$$

$$\Delta^5 f_5(a) = C_{55}\,\Theta_5 = -6912$$

Before writing the formulae of the parabolas, let us introduce $(x-a)/h=\xi$ and put $f(a+\xi h)=F(\xi)$, we have then

$$F_1(\xi)=7860{,}905+832{,}571\binom{\xi}{1}$$

$$F_2(\xi)=7929{,}714+750\binom{\xi}{1}+41{,}286\binom{\xi}{2}$$

$$F_3(\xi)=7777{,}214+1116\binom{\xi}{1}-416{,}214\binom{\xi}{2}+305\binom{\xi}{3}$$

$$F_4(\xi)=7852{,}571+814{,}571\binom{\xi}{1}+262{,}\binom{\xi}{2}-705\binom{\xi}{3}+1055\binom{\xi}{4}$$

$$F_5(\xi)=7780+650\binom{\xi}{1}+838\binom{\xi}{2}-2286\binom{\xi}{3}+4511\binom{\xi}{4}$$
$$-6912\binom{\xi}{5}$$

These results are exact to three decimal places.

The method of the addition of differences can be applied immediately to these equation.

Example 2. The values corresponding to the approximating parabola of the fifth degree and corresponding to the given values of x in the preceding example are to be determined.

Starting from the last equation above, we will determine $F_5(\xi)$ for $\xi=0,1,2,3,4,5$. Noting that $\Delta^5F(0)=-6912$ the table below is obtained by using the method described in § 11. The last column contains the required values of $F_5(\xi)$, which, as in this case the parabola passes through the given points, should be exactly equal to the given values y in Example 1.

4511	-2286	838	650	7880
-2401	2225	-1448	1488	8530
	-176	777	40	10018
		601	817	10050
			1418	10875
				12293

The results are exact to three decimal places.

Remark on the number of decimals to which the calculations are to be carried out. If for instance a parabola of the fifth degree is to be determined approximating the values of y, which are given with a precision of half a unit, then $\Delta^5f(a)$ should be calculated to seven decimals if the number of the given values is near 50, or to eight decimals if it is near one hundred; $\Delta^4 f(a)$ should be calculated to six or seven respectively; $\Delta^3 f(a)$ to five or six, and so on. The corresponding orthogonal moments and the numbers C_{ms} must be of course calculated to the same number of decimals.

Example 3. Changing the origin. The given values y in Example 1 correspond to

$$\text{July } 1, 1901 + \xi$$

for $\xi = 0, 1, 2, \cdots 5$, where ξ is expressed in years. These values have been approximated by $F_5(\xi)$, obtained in the last equation of Example (1), to abbreviate we shall write it $F(\xi)$. In this case, July 1,1901 corresponds to $a = 0$. It is required to develop the function $F(\xi)$ in a series of binomial coefficients $\binom{x-c}{\nu}$, where $c = -\frac{1}{2}$ corresponds to January 1, 1901. Equation (7) will give for $h = 1$ and $c - a = -\frac{1}{2}$ with an exactitude of three decimals

$$\Delta F(-\tfrac{1}{2}) = -6912$$

$$\Delta^4 F(-\tfrac{1}{2}) = 4511 + \tfrac{1}{2} \cdot 6912 = 7967$$

$$\Delta^3 F(-\tfrac{1}{2}) = -2286 - \tfrac{1}{2} \cdot 4511 - \tfrac{3}{8} \cdot 6912 = -7133{,}5$$

$$\Delta^2 F(-\tfrac{1}{2}) = 838 + \tfrac{1}{2} \cdot 2286 + \tfrac{3}{8} \cdot 4511 + \tfrac{5}{16} \cdot 6912 = 5832{,}625$$

$$\Delta F(-\tfrac{1}{2}) = 650 - \tfrac{1}{2} \cdot 838 - \tfrac{3}{8} \cdot 2286 - \tfrac{5}{16} \cdot 4511$$

$$- \tfrac{35}{128} \cdot 6912 = -3925{,}938$$

$$F(-\tfrac{1}{2}) = 7780 - \tfrac{1}{2} \cdot 650 + \tfrac{3}{8} \cdot 838 + \tfrac{5}{16} \cdot 2286$$

$$+ \tfrac{35}{128} \cdot 4511 + \tfrac{63}{256} \cdot 6912 = 11418{,}102$$

Finally the required formula will be

$$F(\xi)=11418{,}102-3925{,}938\binom{\xi+\frac{1}{2}}{1}+5832{,}625\binom{\xi+\frac{1}{2}}{2}$$

$$-7133{,}5\binom{\xi+\frac{1}{2}}{3}+7967\binom{\xi+\frac{1}{2}}{4}-6912\binom{\xi+\frac{1}{2}}{5}.$$

This equation was checked by calculating $F(0)$ which was necessarily equal to 7780.

Example 4. Changing the interval. Let us suppose the following function given

$$f(x)=7780+\frac{650}{2}\binom{x+5}{1}_2+\frac{838}{2^2}\binom{x+5}{2}_2-\frac{2286}{2^3}\binom{x+5}{3}_2$$

$$+\frac{4511}{2^4}\binom{x+5}{4}_2-\frac{6912}{2^5}\binom{x+5}{5}_2.$$

This is a *Newton*-expansion in which the decrement of the generalized binomial coefficient and the increment of the differences are both $h=2$. It is required to deduce another *Newton*-expansion such that the mentioned decrement and increment should both be $k=1/3$.

For this purpose it is necessary to calculate $\Delta^s f(x)$ for $x=-5$ and $s=1,2,3,4,5$. First we must determine the numbers A_m^s/h^m introduced in § 4.

For $h=2$, and $k=1/3$ we have

$A_1^1/h=1/6$ $\qquad$ $A_2^1/h^2=-5/72$

$A_3^1/h^3=55/1296$ $\qquad$ $A_4^1/h^4=-935/31104$

$A_5^1/h^5=21505/933120$

$A_2^2/h^2 = 1/36$ $\qquad$ $A_3^2/h^3 = -5/216$

$A_4^2/h^4 = 295/15552$ $\qquad$ $A_5^2/h^5 = -55/3456$

$A_3^3/h^3 = 1/216$ $\qquad$ $A_4^3/h^4 = -5/864$

$A_5^3/h^5 = 185/31104$ $\qquad$ $A_4^4/h^4 = 1/1296$

$A_5^4/h^5 = -5/3888$ $\qquad$ $A_5^5/h^5 = 1/7776$

Now we can proceed to calculate the differences $\Delta^s f(a)$ given by the formula in § 4. We have

$$\Delta f(a) = \frac{1}{6}\cdot 650 - \frac{5}{72}\cdot 838 - \frac{55}{1296}\cdot 2286 - \frac{935}{31104}\cdot 4511$$

$$-\frac{21505}{933120}\cdot 6912 = -341,77395$$

$$\Delta^2 f(a) = \frac{1}{36}\cdot 838 + \frac{5}{216}\cdot 2286 + \frac{295}{15552}\cdot 4511$$

$$+\frac{55}{3456}\cdot 6912 = 271,76190$$

$$\Delta^3 f(a) = \frac{-1}{216}\cdot 2286 - \frac{5}{864}\cdot 4511 - \frac{185}{31104}\cdot 6912 = -77,79977$$

$$\Delta^4 f(a) = \frac{1}{1296}\cdot 4511 + \frac{5}{3888}\cdot 6912 = 12,36960$$

$$\Delta^5 f(a) = -\frac{6912}{7776} = -0,88889$$

Hence

$$f(x) = 7780 - \frac{1}{k}\cdot 341,77395\binom{x+5}{1} + \frac{1}{k^2}\cdot 271,76190\binom{x+5}{2}$$

$$-\frac{1}{k^3}(77{,}79977)\binom{x+5}{3}+\frac{1}{k^4}(12{,}3696)\binom{x+5}{4}$$

$$-\frac{1}{k^5}\cdot(0{,}88889)\binom{x+5}{5}$$

where $k=1/3$. If we want to apply the method of the addition of differences it is preferable to change the variable by introducing $\xi=(x-a)/k$ and writing $F(\xi)=f(a+\xi k)$. We have

$$F(\xi)=7780-341{,}77395\binom{\xi}{1}+271{,}7619\binom{\xi}{2}$$

$$-77{,}79977\binom{\xi}{3}+12{,}3696\binom{\xi}{4}-0{,}88889\binom{\xi}{5}.$$

Of course it would have been better to change the variable before beginning to calculate the numbers A_m^s , but we wanted to show the method in its generality. It is advisable to check the above equation by putting into it $\xi=6$: the result must be $F(6)=f(1)=8430$. The checking has given this number with a precision of 5 decimal places.

Starting from $\Delta^5F(\xi)=-0{,}88889$, let us determine the first values of $F(\xi)$ for $\xi=0,1,2,3,\cdots 6$, by the method of the addition of differences

12,36960	- 77,79977	271,76190	- 341,77395	7780
11,48071	- 65,43017	193,96213	- 70,01205	7438,22605
10,59182	- 53,94946	128,53196	123,95008	7368,21400
	- 43,35764	74,58250	252,48204	7492,16408
		31,22486	327,06454	7744,64612
			358,28940	8071,71066
				8430,00006

§ 14. *First problem of correlation.* It is required to determine the coefficient of correlation r_{nm} between the deviations $f_n(x)-y$ of a parabola of degree n approximating to the values y (§ 3) and of the deviations $f'_m(x) - y'$ of a parabola of degree m approximating to other values y' corresponding to the same x quantities. Let us suppose that $n \geqq m$.

The definition of this coefficient is the following:

$$(40) \quad r_{nm} = \frac{1}{N\sigma_n \sigma'_m} \sum_{x=a}^{b} \left[f_n(x) - y\right]\left[f'_m(x) - y'\right]$$

where σ_n^2 is the mean-square-deviation of $f_n(x)$ and y (§ 8); and σ'^2_m the mean-square-deviation of $f_n(x)$ and y'.

Let us put into (40) the values of $f_n(x)$ and $f'_m(x)$ expanded into series of orthogonal polynomials (14); then the sum in the second member of (40) will be

$$(41) \quad \Sigma yy' - \Sigma y(a'_0 U_0 + a'_1 U_1 + \cdots + a'_m U_m)$$

$$-\Sigma y'(a_0 U_0 + a_1 U_1 + \cdots + a_n U_n)$$

$$+ a_0 a'_0 \Sigma U_0^2 + a_1 a'_1 \Sigma U_1^2 + \cdots + a_m a'_m \Sigma U_m^2.$$

This expression may be simplified; indeed, starting from equation (15) we have, after multiplication by a'_s

$$a'_s \Sigma y U_s = a_s a'_s \Sigma U_s^2.$$

Putting into this equation successively $s = 0, 1, 2, \cdots m$, and adding the results, we obtain the values of the second sum of (41). To have the third, we start from the equation analogous to (15)

$$\Sigma y' U_s = a'_s \Sigma U_s^2$$

and multiply both members by a_s ; putting successively $s = 0, 1, 2, \cdots, n$ and adding the results, we obtain an expression for the third sum of (41). Finally this formula will be

$$\Sigma yy' - a_o a_o' \Sigma U_o^2 - a_1 a_1' \Sigma U_1^2 - \cdots - a_n a_n' \Sigma U_n^2.$$

We have still to determine the quantity $a_s a_s' \Sigma U_s^2 / N$; but according to (24) and (32) this expression is equal to $C_m \Theta_m \Theta_m'$, where Θ_m' is the mean-orthogonal-moment of degree m of y'. We conclude that,

$$(42) \quad r_{nm} = \frac{1}{\sigma_n \sigma_m'} \left[\frac{1}{N} \Sigma yy' - \Theta_o \Theta_o' - C_1 \Theta_1 \Theta_1' - \cdots - C_n \Theta_n \Theta_n' \right].$$

Sometimes, n being given, the coefficients of correlation r_{nm} are sought for all values of $m = 0, 1, 2, \cdots; n$.- In this case we will first divide the quantity within the brackets in equation (42) by σ_n and then divide the quotient successively by the values of σ_m' for $m = 0, 1, 2, \cdots, n$.

All the quantities figuring in equation (42) are known from the previous determination of the approximating functions $f_n(x)$ and $f_m'(x)$, except $\Sigma yy'/N$. To obtain the desired coefficient of correlation, it is necessary to compute this additional last quantity.

Formula (42) also shows the importance of the mean-orthogonal moments of the given quantities y and y'; it is independent of the origin, of the interval, and of the constant of the orthogonal polynomial chosen.

The most important particular case of r_{nm} is r_{oo} ; this being the coefficient of correlation of the deviations of y and of y' from their respective averages. Thus r_{oo} shows the simultaneity of the variation from the respective averages. Another important particular case is r_{11} , which gives the correlation between $f_1(x)-y$

and $f_1'(x) - y_1'$ — thus measuring the simultaneity of periodical deviations from the respective linear trend-lines.

If n and m are large, the approximating parabolas will follow the principal secular and periodical variations, will therefore have nearly the same maxima and minima as the values y and y'. In this event the remaining deviations will be mainly due to chance, and thus the coefficient of correlation loses its importance.

§ 15. *Second problem of correlation.* Given the functions $f_n(x)$ and $f_\nu(x)$, of degree n and ν respectively, approximating the values of a quantity y. Let us denote by ξ the deviations of the two corresponding parabolas:

$$\xi = f_n(x) - f_\nu(x).$$

Moreover, let us likewise have given the functions $f_m'(x)$ and $f_\mu'(x)$ of degree m and μ respectively approximating the values of another quantity y', and denote by η the deviations between the corresponding two parabolas.

It is required to find the coefficient of correlation $r_{n\nu,\, m\mu}$ between the deviations ξ and η. According to the definition of th coefficient of correlation we have

$$(43) \quad r_{n\nu,\, m\mu} = \frac{1}{N\sigma_{n\nu}\sigma_{m\mu}} \sum_{x=a}^{b} \left[f_n(x) - f_\nu(x)\right]\left[f_m'(x) - f_\mu'(x)\right]$$

where $\sigma_{n\nu}^2$ denotes the mean-square-deviation of ξ, and $\sigma_{m\mu}'^2$ the mean-square-deviation of η (§ 8). Both are known from the determination of the approximating parabolas.

Let us suppose that $n \geqq m > \nu \geqq \mu$. Substituting, for $f_n(x)$, $f_\nu(x)$, $f_m'(x)$ and $f_\mu'(x)$ their expansions in orthogonal polynomials (14), the sum in the second member of (43) will be

$$\Sigma(a_{\nu+1}U_{\nu+1}+\cdots+a_nU_n)(a'_{\mu+1}U_{\mu+1}+\cdots+a'_mU_m).$$

This yields, in consequence of the orthogonality of the polynomial U_s,

$$a_{\nu+1}a'_{\nu+1}\Sigma U_{\nu+1}^2+\cdots+a_ma'_m\Sigma U_m^2.$$

As we have seen in the preceding paragraph,

$$\frac{1}{N}a_sa'_s\Sigma U_s^2=C_s\Theta_s\Theta'_s.$$

Hence

$$(44)\quad r_{n\nu,m\mu}=\frac{1}{\sigma_{n\nu}\sigma_{m\mu}}\left[C_{\nu+1}\Theta_{\nu+1}\Theta'_{\nu+1}+\cdots+C_m\Theta_m\Theta'_m\right].$$

All the quantities of the second member of this equation are known from the previous parabolic approximation, so that the calculating of the coefficient (44) is easy. We note that it is a simple function of the mean-orthogonal moments, of the mean-square deviations and of the number C_s. (formula 34 or table III).

If m and μ are given and this coefficient must be computed for several values of n and ν, we first divide the quantity within the brackets of formula (44) by $\sigma_{m\mu}$ and then divide the quotient successively by the different values of $\sigma_{n\nu}$.

In the second problem also, the most important particular case is that of $r_{n0,m0}$, especially if n and m are large, in which case the deviations $f_n(x)-y$ and $f'_m(x)-y'$ could be considered as negligible. In this case the coefficient of correlation (44) of the trend-lines is much more important than the coefficient of correlation (42) of the trend-deviations.

Example on correlation. *A. Sipos* has determined trend-lines up to the third degree of Hungarian imports and exports in 1882-1913.[16] The mean orthogonal moments for imports were

$$\Theta_0 = 1254{,}25938 \qquad \Theta_2 = 70{,}63941$$
$$\Theta_1 = 206{,}02671 \qquad \Theta_3 = 21{,}27341$$

The mean-square-deviations corresponding to the parabolas of imports of degree 0, 1, 2, 3 were

$$\sigma_0 = 383{,}777 \qquad \sigma_2 = 83{,}552$$
$$\sigma_1 = 166{,}317 \qquad \sigma_3 = 69{,}330$$

The equation of the third degree parabola of approximation was

$$f_3(x) = 864{,}12484 + 20{,}38562\binom{x}{1} - 2{,}82064\binom{x}{2} + 0{,}45504\binom{x}{3}$$

where $x = 0$ corresponds to 1882 and $f_3(x)$ is given in million gold crowns.

The corresponding values for exports were

$$\Theta'_0 = 1234{,}4 \qquad \Theta'_2 = 37{,}80645$$
$$\Theta'_1 = 192{,}675 \qquad \Theta'_3 = 5{,}58515$$

$$\sigma'_0 = 37{,}587 \qquad \sigma'_2 = 58{,}480$$
$$\sigma'_1 = 96{,}662 \qquad \sigma'_3 = 57{,}184$$

[16] *A. Sipos,* Praktische Anwendung der Trendberechnungsmethode von Jordan. Mitteilungen der Ungarischen Landeskommission für Wirtschaftsstatistik und Konjunkturforschung. Budapest 1930.

The equation of the third degree parabola of exports was

$$f_3'(x) = 821{,}24103 + 15{,}09955\binom{x}{1} + 0{,}28944\binom{x}{2} + 0{,}11947\binom{x}{3}.$$

First problem of correlation. Let us determine the more important particular cases of the coefficient of correlation r_{nm}, between the deviations $y - f_n(x)$ of import and the deviations $y' - f_m'(x)$ of export.

The coefficient of correlation between the deviations of the given quantities and their respective averages is

$$r_{00} = \frac{1}{\sigma_0 \sigma_0'} \left[\frac{1}{N} \Sigma y y' - \Theta_0 \Theta_0' \right],$$

and since $\frac{1}{N} \Sigma y y' = 1672457{,}80$ we have

$$r_{00} = 0{,}9586$$

This correlation is a very strong one.

The coefficient of correlation between the deviations of the given quantities and their respective linear trend-lines is

$$r_{11} = \frac{1}{\sigma_1 \sigma_1'} \left[\frac{1}{N} \Sigma y y' - \Theta_0 \Theta_0' - C_1 \Theta_1 \Theta_1' \right] = 0{,}7669.$$

(The number C_1 was taken from table III. $C_1 = 2{,}81818 \cdots$)
This correlation is still strong enough.

The coefficient of correlation between the deviations of the given quantities and their respective third degree **trend-lines is**

$$r_{33} = \frac{1}{\sigma_3 \sigma_3'} \left[\frac{1}{N} \Sigma y y' - \Theta_0 \Theta_0' - C_1 \Theta_1 \Theta_1' - C_2 \Theta_2 \Theta_2' - C_3 \Theta_3 \Theta_3' \right]$$
$$= 0{,}1739.$$

This correlation is already very small, the deviations are mainly due to chance. From Table III we had $C_2 = 4,14438503$ and $C_3 = 4,80748663$.

Second problem of correlation. The coefficient of correlation $r_{n\nu, m\mu}$ between the deviations of two trend-lines of import of degree n and ν respectively and the deviations of two trend-lines of export of degree m and μ respectively, is to be determined. We will only consider the most important particular case, the correlation between the third degree trend-lines and the respective averages, that is

$$r_{30,30} = \frac{1}{\sigma_{30}\sigma'_{30}}\left[C_1\theta_1\theta'_1 + C_2\theta_2\theta'_2 + C_3\theta_3\theta'_3\right]$$

where

$$\sigma_{30}^2 = C_1\theta_1^2 + C_2\theta_2^2 + C_3\theta_3^2 = 142659,2203$$

and

$$\sigma_{30}'^2 = C_1\theta_1'^2 + C_2\theta_2'^2 + C_3\theta_3'^2 = 110695,9458.$$

And therefore $\sigma_{30} = 377,70256$ and $\sigma'_{30} = 332,71000.$

The quantities in the brackets have already been calculated in the first problem, so that we have

$$r_{30,30} = 0,9828.$$

The obtained value is very near to that of r_{00} calculated above, proving that the third degree trend-lines well represent the given quantities. This would not be true in the case of the second degree trend-lines; in fact we should obtain $r_{20,20} = 0{,}9865$ widely different from r_{20} obtained before.

§ 16. *Some mathematical properties of the orthogonal polynomials. Symmetry of the polynomial* U_m. If we substitute in formula (18) $a+b-h-x$ for x, we get

$$U_m(a+b-h-x) = Ch^m \Sigma \binom{m}{s}\binom{b-h+sh-x}{s}\binom{a-h-x}{m-s}$$

$$= Ch^m \Sigma \binom{m}{s}\binom{x-b}{s}\binom{x-a+mh-sh}{m-s}$$

and putting $s = m-\mu$, it follows that

$$(45) \qquad U_m(a+b-h-x) = (-1)^m U_m(x).$$

This formula shows the symmetry of the polynomial.

Let us consider the particular case,

$$x-a = \frac{1}{2}(b-a-h) = kh.$$

We have, then,

$$b-a = (2k+1)h, \qquad N = 2k+1$$

$$x-a = kh \qquad x-b = -(k+1)h.$$

From (45) it follows that

$$U_m(a+kh) = (-1)^m U_m(a+kh).$$

Hence we have

$$U_{2m+1}(a+kh)=0.$$

From equation (45) we easily obtain

$$\Delta U_m(x)=(-1)^{m+1}\Delta U_m(a+b-2h-x)$$

and

$$\Delta^s U_m(x)=(-1)^{m+s}\Delta^s U_m(a+b-sh-x).$$

Function-equation of $U_m(x)$. This can be deduced in the following way: let us develop xU_m into a series of orthogonal polynomials; we find that

$$xU_m(x)=A_{m-1}U_{m-1}+A_mU_m+A_{m+1}U_{m+1} \tag{46}$$

as, in consequence of the orthogonality of these polynomials, the other terms vanish; indeed, if μ is different from $m-1$, m and $m+1$, it follows that

$$\sum_{x=a}^{b} xU_mU_\mu=0.$$

Since equation (46) holds for every value of x, and $U_m(x)$ is known for three particular values of x, we can determine the coefficients A_s.

We know these values, since equation (21) gives for $x=b$

$$U_m(b)=C(m)h^m\binom{\frac{b-a}{h}+m}{m}$$ [17]

[17]As C can be dependent of m, we will write $C(m)$ in the following formulae, instead of C.

and for $x=a$, after changing a into b and inversely

$$U_m(a)=C(m)(-1)^m h^m \binom{b-a-h}{m}.$$

Moreover, in consequence of the above-mentioned symmetry of the polynomials, we have

$$U_m(b-h)=(-1)^m U_m(a).$$

The two last equations give by using formula (46)

$$(47)\quad aU_m(a)=A_{m-1}U_{m-1}(a)+A_m U_m(a)+A_{m+1}U_{m+1}(a)$$

and

$$(48)\quad \begin{aligned}(b-h)U_m(b-h)=A_{m-1}U_{m-1}(b-h)+A_m U_m(b-h)\\ +A_{m+1}U_{m+1}(b-h).\end{aligned}$$

Multiplying both sides of equation (47) by $(-1)^m$ and adding it to (48), we find that

$$(b+a-h)U_m(b-h)=2A_m U_m(b-h)$$

and

$$A_m=\tfrac{1}{2}(b+a-h).$$

Multiplying both sides of (47) by $(-1)^m$ and subtracting it from (48) yields

$$(b-a-h)U_m(b-h)=2A_{m-1}U_{m-1}(b-h)+2A_{m+1}U_{m+1}(b-h).$$

Since

$$U_m(b-h)=C(m)h^m\binom{b-a-h}{m}$$

we find that

$$\tag{49} \begin{aligned} \tfrac{1}{2}(b-a-h)C(m)h\,\frac{b-a-mh}{m} &= A_{m-1}\,C(m-1) \\ &+ A_{m+1}C(m+1)h^2\,\frac{(b-a-mh)(b-a-mh-h)}{m(m+1)}\,. \end{aligned}$$

As, in consequence of the symmetry of the polynomials, we have

$$U_m(b)=(-1)^mU_m(a-h)=C(m)h^m\binom{b-a+mh}{m}$$

hence we can deduce two other equations analogous to (47) and (48), *i.e.*,

$$b\,U_m(b)=A_{m-1}U_{m-1}(b)+A_m'U_m(b)+A_{m+1}U_{m+1}(b)$$

and

$$(a-h)U_m(a-h) = A_{m-1}U_{m-1}(a-h) + A_m U_m(a-h) + A_{m+1}U_{m+1}(a-h).$$

After multiplying the second equation by $(-1)^m$ and subtracting the result from the first, we obtain

$$\tfrac{1}{2}(b-a+h)\,U_m(b) = A_{m-1}\,U_{m-1}(b) + A_{m+1}U_{m+1}(b)$$

or

$$(50)\qquad \tfrac{1}{2}(b-a+h)C(m)h\,\frac{b-a+mh}{m} = A_{m-1}\,C(m-1)$$

$$+\,A_{m+1}\,C(m+1)\,h^2\,\frac{(b-a+mh)(b-a+mh+h)}{m(m+1)}\;.$$

Finally subtracting (50) from (49), and simplifying, we have

$$A_{m+1} = \frac{(m+1)^2}{2(2m+1)}\;\frac{C(m)}{hC(m+1)}\;.$$

We deduce A_{m-1} from (49) by substituting therein the above value for A_{m+1}, yielding

$$A_{m-1} = \frac{(b-a)^2 m^2 h^2}{2(2m+1)}\;\frac{hC(m)}{C(m-1)}\;.$$

Now the function-equation (46) is known.

We might have proceeded in another way. Having obtained A_m by using the equations (47) and (48), we could determine, for instance A_{m+1} in the usual way by multiplying both members of equation (46) by U_{m+1}, and summing x from a to b,

$$\sum_{x=a}^{b} U_{m+1} U_m x = A_{m+1} \sum_{x=a}^{b} U_{m+1}^2 .$$

Since ΣU_{m+1}^2 is already known from (24), we need only determine the first member and this may be done by applying formula (3) to the quantity

$$\Sigma U_m (x U_m).$$

Difference-equation of $U_m(x)$. We will start from

$$\Delta U_{m+1} = C\Delta^{m+2}\left[\binom{x-a}{m+1}_h \binom{x-b}{m+1}_h\right].$$

According to equation (1) we have

$$\Delta\left[\binom{x-a}{m+1}_h \binom{x-b}{m+1}_h\right] = h\binom{x-a}{m}_h \binom{x-b}{m+1}_h + h\binom{x-b}{m}_h \binom{x-a+h}{m+1}_h .$$

Therefore

$$\Delta U_{m+1} = \frac{hC}{m+1}\Delta^{m+1}\left\{(2x-a-b-mh+h)\cdot\left[\binom{x-a}{m+1}_h \binom{x-b}{m+1}_h\right]\right\} .$$

Applying to this expression formula (2), giving the $m{+}1\,th$ difference of a product, it follows that

$$(51)\quad \Delta U_{m+1} = \frac{h}{m+1}\left[(2x-a-b+mh+3h)\Delta U_m + 2h(m+1)U_m\right].$$

Now let us deduce a second formula for ΔU_{m+1}. We can write this quantity in the following manner

$$\Delta U_{m+1} = C\Delta^{m+2}\left[\binom{x-a}{m+1}_h\binom{x-b}{m+1}_h\right] =$$

$$\frac{C}{(m+1)^2}\,\Delta^{m+2}\left\{\left[x(x-h)+x(h-a-b-2mh)\right.\right.$$

$$\left.\left.+(a+mh)(b+mh)\right]\cdot\left[\binom{x-a}{m}_h\binom{x-b}{m}_h\right]\right\}.$$

Again using formula (2) to deduce the $(m+2)$-th difference of the preceding product, we have after simplification

$$\Delta U_{m+1} = \frac{1}{(m+1)^2}\left[(x-a+2h)(x-b+2h)\Delta^2 U_m\right.$$

$$\left.+(m+2)h(2x+3h-a-b)\Delta U_m + (m+1)(m+2)h^2 U_m\right].$$

Finally, taking into account (51), this equation results in

$$(x-a+2h)(x-b+2h)\Delta^2 U_m + \left[2x-a-b+3h-m(m+1)h\right]h\Delta U_m$$

(52)

$$-m(m+1)h^2 U_m = 0.$$

This is the required difference-equation; it is a linear equation of the second order and can be solved by *Boole's* method.[18] If we put $\xi=(x-a)/h$ the solution will be

$$(53)\quad U_m = Ch^{2m}\sum_{\nu=0}^{m+1}(-1)^{\nu}\binom{m+\nu}{\nu}\binom{m+N}{m-\nu}\binom{\xi+\nu}{\nu}.$$

This expression of U_m differs from those we obtained in paragraph 6.

The roots of $U_m(x)=0$. *L. Fejér*[19] has demonstrated the following theorems concerning these roots:

The roots of $U_m(x)=0$ are all real and single, they are all situated in the interval

$$a,\ b-h$$

Whatever ξ may be, in the interval $a+\xi h,\ a+\xi h+h$ there is at most one root of

$$U_m(x)=0.$$

Fejér showed moreover that if $g_m(x)$ is a polynomial of degree m, and if in its *Newton* expansion the coefficient of $\binom{x}{m}$

[18] *Boole,* Calculus of Finite Differences, 1860, p. 176.
[19] See the Appendix in loc. cit. [5]

is equal to unity, the polynomial which minimizes the following expression

$$\sum_{x=a}^{b} \left[q_m(x) \right]^2$$

is the orthogonal polynomial $U_m(x)$, with the constant C suitably chosen.* Indeed the first conditions of a minimum are that

$$\sum_{x=a}^{b} \binom{x}{\nu} q_m(x) = 0 \quad \text{for} \quad 0 \leqq \nu < m$$

and these are identical with the conditions of orthogonality (14). The second condition of the minimum is always satisfied in these cases, as has been shown in § 5.

§ 17. *Graduation by orthogonal polynomials.* Let us consider an odd number of consecutive values of x, say $2k+1$, and the corresponding values of y. A smoothed value of y is wanted for the central term, viz. for $x=a+kh$. This will be obtained by determining, according to the principle of least squares, a parabola of degree m, so that the sum of the squares of deviations between the parabola and the points x, y shall be a minimum.

The equation of the parabola expressed in orthogonal polynomials (13) will be

$$f_n(x) = a_0 + a_1 U_1(x) + \cdots + a_n U_n(x)$$

and the smoothed value required is given by $f_n(a+kh)$.

In consequence of the symmetry of the polynomials, formula

*See Essher's first polynomial in § 21.

(45), we have $U_{2m+1}(a+kh)=0$, so that the equation of the parabola will give

$$f_n(a+kh)=a_0+a_2U_2(a+kh)+a_4U_4(a+kh)+\cdots$$

From this formula we see that it is useless to consider parabolas of odd degree, as for instance a parabola of the second degree will give the same smoothed value for y as would a parabola of the third degree. Therefore we will consider only parabolas of even degree.

The values of $U_{2m}(a+kh)$ are given by our formulae (18), (21), etc., but we can obtain a much simpler formula for them, starting from the *Function-Equation* of $U_{2m}(x)$ given by formula (46), which, since $A_m=a+kh$, we can write in the following manner

$$\frac{(2k+1)^2-m^2}{2(2m+1)}\,\frac{U_{m-1}}{C(m-1)}+(a+kh-x)\,\frac{U(m)}{h^3C(m)}$$

$$+\frac{(m+1)^2}{2(2m+1)}\,\frac{U_{m+1}}{h^4C(m+1)}=0\,.$$

This holds for every value of x. For $x=a+kh$ the term in U_{2m} will vanish and we have

$$U_{m+1}(a+kh)=-\frac{2k+1-m}{m+1}\,\frac{2k+1+m}{m+1}\,\frac{C(m+1)}{C(m-1)}\,h^4U_{m-1}(a+kh).$$

This equation can be solved by putting into it successively $m=1,2,3$. It follows that

$$U_2(a+kh) = -k(k+1)C(2)h^4$$

$$U_4(a+kh) = \binom{k}{2}\binom{k+2}{2}C(4)h^8$$

and so on

$$U_{2m}(a+kh) = (-1)^m h^{4m} C(2m)\binom{k}{m}\binom{k+m}{m}.$$

For a_{2m} we have found in § 7

$$a_{2m} = \frac{(4m+1)\,\Theta_{2m}}{C(2m)h^{4m}\binom{2k+1+2m}{2m}} .$$

Hence if to abbreviate we write

$$(54) \qquad S_{2m} = (-1)^m(4m+1)\binom{2m}{m}\frac{\binom{k+m}{2m}}{\binom{2k+1+2m}{2m}}$$

we have

$$a_{2m}U_{2m} = S_{2m}\,\Theta_{2m}$$

and finally the required smoothed value of y is given by

$$(55) \qquad f_{2m}(a+kh) = \Theta_0 + S_2\Theta_2 + S_4\Theta_4 + \cdots + S_{2m}\Theta_{2m} .$$

This formula is also independent of the interval and of the constant of the orthogonal polynomial used.

The values of S_{2m} necessary up to parabolas of the tenth degree and up to 29 ordinates are given in Table II, so the calculation of the graduated value is very simple. All we need do is to compute the mean binomial moments by *Chetverikoff's* method (§ 9) and calculate the corresponding mean orthogonal moments Θ_{2m} by formula (30). This must of course be repeated for every value which is to be graduated.

Example 7. Nine point graduation, employing second and fourth degree parabolas. The given values are

x	y	x	y
0	2502	5	2904
1	2548	6	3064
2	2597	7	3188
3	2675	8	3309
4	2770		

The mean binomial moments were calculated by the method of § 9, and were found to be

$T_0 = 2839{,}66667$ $\qquad$ $T_3 = 3182{,}21429$

$T_1 = 3014{,}97222$ $\qquad$ $T_4 = 3225{,}87302$

$T_2 = 3117{,}16667$

Hence the mean orthogonal moments are

$\Theta_0 = T_0 = 2839{,}66667$ $\qquad$ $\Theta_4 = 14\,T_4 - 35\,T_3 + 30\,T_2 - 10\,T_1$

$\Theta_2 = 2\,T_2 - 3\,T_1 + T_0 = 29{,}08335$ $\qquad$ $+ T_0 = -10{,}33148$

From Table II. we take $S_2 = -1{,}81818182$ and $S_4 = 1{,}13286713.$

TABLE II. GRADUATION.

N	S_2	S_4	S_6	S_8	S_{10}
3	- 1				
5	- 1,4285 7143	0,4285 7142 9			
7	- 1,6666 6667	0,8181 8181 8	- 0,1515 1515 2		
9	- 1,8181 8182	1,1328 6713	- 0,3636 3636 4	0,0489 5104 90	
11	- 1,9230 7694	1,3846 1538	- 0,5882 3529 4	0,1417 0040 5	- 0,0150 0364 37
13	- 2	1,5882 3529	- 0,8049 5356 0	0,2631 5789 5	- 0,0508 8167 99
15	- 2,0588 2353	1,7554 1796	- 1,0061 9195	0,4004 5766 6	- 0,1068 5152 8
17	- 2,1052 6316	1,8947 3684	- 1,1899 3135	0,5446 2242 6	- 0,1794 0503 4
19	- 2,1428 5714	2,0124 2236	- 1,3565 2174	0,6898 5507 2	- 0,2644 6776 6
21	- 2,1739 1304	2,1130 4348	- 1,5072 4638	0,8325 8370 8	- 0,3583 1116 7
23	- 2,2	2,2	- 1,6436 7816	0,9707 0819 4	- 0,4578 4204 7
25	- 2,2222 2222	2,2758 6207	- 1,7673 9587	1,1030 7749	- 0,5606 2291 4
27	- 2,2413 7931	2,3426 0289	- 1,8798 6652	1,2291 4349	- 0,6647 9271 2
29	- 2,2580 6451	2,4017 5953	- 1,9824 0469	1,3487 3583	- 0,7689 6251 1

Therefore the smoothed value with a parabola of the second degree will be

$$f_2(4) = \Theta_0 + S_2\Theta_2 = 2786,78785$$

In the paper loc. cit. [9] (p. 31) these values have been approximated by a parabola of the third degree, the value obtained for $f_3(4)$ was (p. 33) $2786,78763$ according to what has been said, this value should be equal to the obtained results for $f_2(4)$

Finally the smoothed value corresponding to a parabola of the fourth degree is

$$f_4(4) = f_2(4) + S_4\Theta_4 = 2775,0837$$

BIBLIOGRAPHICAL AND HISTORICAL NOTES.[20]

§ 18. It was *Chebisheff* who first introduced orthogonal polynomials with respect to a discontinuous variable. He[21] especially treated the case of non-equidistant variables, from the mathematical point of view, in a very interesting manner, but his results were necessarily complicated. As we consider here only equidistant variables, this paper will not be discussed.

But *Chebisheff* also investigated the case of equidistant polynomials in two of his papers. In the first "Sur l'Interpolation par la Méthode des Moindres Carrés"[22] he denotes the orthogonal polynomial of degree m by $\varphi_m(x)$; the variable x taking values differing by unity, from $-\frac{1}{2}(N-1)$ to $\frac{1}{2}(N-1)$ inclusive. His polynomials can be obtained for instance from our formula (21′) by putting therein $h=1$, $a=-\frac{1}{2}(N-1)$ and $C(m!)^2$. Formula (24) will give $\Sigma\varphi_m^2$ by putting into it $h=1$ and $C=(m!)^2$, where $m!$ stands for $1,2,3,\cdots m$.

[20]The numbers of the formulae quoted refer to the present paper.
[21]Sur les Fractions Continues. 1855. Oeuvres, T. I. p. 201.
[22]Oeuvres 1859. Oeuvres, Tome I. p. 474.

In the second paper "Interpolation des valeurs equidistantes" in which he introduces such polynomials,[23] he denotes the polynomial of degree m by $\phi_m(x)$ and the variable x takes the values 1, 2, 3, ... $(n-1)$. These polynomials can also be obtained from formula (21′) and $\Sigma\phi_m^2$ from (24), by putting in them $h=1$, $a=1$, $N=n-1$ and $C=(m!)^2$.

§ 19. *J. P. Gram* utilized orthogonal polynomials for graduation according to the principle of least squares.[24] He denotes the polynomial of degree m by $\psi_m(x)$, the variable x taking the values

$$-\tfrac{1}{2}(N-1), \ldots, -1, 0, 1, \ldots, \tfrac{1}{2}(N-1)$$

where N is an odd number. Then writing

$$\psi_m(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_m x^m$$

he determines the coefficients by formulae

$$\sum_{x=-\frac{1}{2}(N-1)}^{\frac{1}{2}(N+1)} x^s \psi_m(x) = 0 \quad \text{for} \quad 0 \leqq s < m.$$

There are m such equations and $m+1$ unknown coefficients α_s one of them therefore will be arbitrary. *Gram* disposes of the first coefficient which is different from zero in such a manner that all the values of $\psi_m(x)$ corresponding to the considered values of x shall be integers and as small as possible.

For instance in the case of $\psi_1(x)$, it follows that $\alpha_0 = 0$, and in order to have $\psi_1(x) = x$, he puts $\alpha_1 = 1$

Practically he uses only polynomials of the first, second and

[23] 1875. Oeuvres, T. II. p. 270. 219.

[24] Ueber partielle Ausgleichung mittelst Orthogonalfunktionen, Bulletin de l'Association des Actuaires Suisses, 1915.

third degrees; and gives tables for $\psi_2(x)$ and $\psi_3(x)$ available for $N = 7, 9, 11, \cdots, 21$ and containing the corresponding values of

$$\Sigma\psi_1^2, \Sigma\psi_2^2, \Sigma\psi_3^2.$$

The values of $\psi_2(x)$ can be obtained, for instance, from our formula (21′), if we put therein $h=1$, $a=-\frac{1}{2}(N-1)$, and either $C=-1$, if $(N-1)(N-2)$ is not divisible by three, or $C=-\frac{1}{3}$ if it is so divisible.

The values of $\psi_3(x)$ are also obtained from the same formula by putting into it $h=1$, $a=-\frac{1}{2}(N-1)$ and either $C=-1/4$, if $(N-2)(N-3)$ is not divisible by five, or $C=-1/20$, if it is so divisible. $C=1/2$ corresponds to $\psi_1(x)=x$.

Formula (24) would give the values of $\Sigma\psi_1^2, \Sigma\psi_2^2$ and $\Sigma\psi_3^2$ of the table, if we were to substitute $h=1$, and for C the respective values above-mentioned.

In order to give an example, *Gram* calculates the smoothed value corresponding to an eleven-point parabola of the second degree (p. 12). His calculation is short enough, but our general method is still shorter, as may be judged from the following determination of the smoothed central value of his example.

In the first column we have written the values of y corresponding to a reversed order of magnitude of x; the other columns have been obtained by simple addition, as indicated in § 9.

194	194	194	194
179	373	567	761
212	585	1152	1913
124	709	1861	3774
780	1489	3350	7124
000	1489	4839	11963
504	1993	6832	18795
244	2237	9069	27864
000	2237	11306	39170
582	2819	14125	...
000	2819	...	...

Hence

$$T_0 = \Theta_0 = 2819/11 = 256,272727\ldots$$

$$T_1 = 14125/55 = 256,81818\ldots$$

$$T_2 = 39170/165 = 237,393939\ldots$$

and

$$\Theta_2 = 2T_2 - 3T_3 + T_0 = -39,3939\ldots$$

To have the smoothed value, we take from Table II.,

$$S_2 = -1,92307694$$

and it follows that the required smoothed value is

$$f_2(a+kh) = \Theta_0 + S_2\Theta_2 = 332,0303$$

agreeing with *Gram's* result.

Although the calculation has been executed to nine figures, it is a very short one.

§ 20. *Ch. Jordan* "Sur une série de polynomes dont chaque somme partielle représente la meilleure approximation d'un degré donné suivant la méthode des moindres carrés."[25] (1921) In this paper the author treats the mathematical theory of orthogonal polynomials for equidistant values in the general case. Many of their mathematical properties are demonstrated: formulae, difference-equations, function-equations of these polynomials are given and some interesting propositions concerning their roots are demonstrated. Two particular polynomials were introduced. In the

[25]Proceedings of the London Mathematical Society, Vol. XX., p. 297.

first, denoted by $Q_m(x)$, x takes the values $a, a+h, a+2h, \cdots, b-h$, where $b=a+Nh$; and N is an integer. The constant C was chosen as

$$C = m!/2^m h^m.$$

This was done, as has been mentioned, so that for $a=-1$, $b=1$ and $h=0$ the limit of the polynomial $Q_m(x)$ should be identical with *Legendre's* polynomial.

The second particular case denoted by $q_m(x)$ was obtained from $Q_m(x)$ by putting $h=1$, $a=0$ and $b=N$. There are tables in this paper giving the values of $q_m(x)$ for $m=1,2,3,4,5$, for $N=m+1, m+2, \cdots, 20$ and for the values of $x=0,1,2,\cdots, N-1$. Moreover there is a table giving Σq_m^2 for $m=1,2,3,4,5$ and for $N=m+1, m+2, \cdots, 20$.

The problem of approximation was solved in this paper by formula (14) of the present work in the following manner:—the coefficients a_m were calculated from formula (13) above: $\Sigma y q_m(x)$ was computed first by multiplying every value of y by the corresponding values of $q_m(x)$ taken from the tables mentioned and then the products were added. The quantity Σq_m^2 was taken likewise from the tables.

The coefficients a_m being known, the mean square deviation was calculated by formula (32′). And finally, the required values of $f_n(x)$ were obtained by using formula (14), and by taking the necessary values of $q_m(x)$ from the tables.

In a *second* paper "Berechnung der Trendlinie auf Grund der Theorie der kleinsten Quadrate"[20] the determination of the coefficients a_m has been much simplified. These were obtained by multiplying the binomial moments by certain numbers, and were easily calculated with the aid of a table of binomial coefficients $\binom{N}{\nu}$. These tables exhibit the values given for N up to 55 and

[20] Mitteilungen der Ungarischen **Landeskommission für** Wirtschaftsstatistik und Konjunkturforschung. Budapest, 1930.

for V up to ten, and are sufficient for parabolas up to the tenth degree. Although the calculation of ΣU_m^2 was not necessary for the determination of a_m, nevertheless it had to be evaluated for the determination of the mean square deviation. For this purpose a very simple formula was given for ΣQ_m^2 (p. 45).

In this paper the method of approximation by orthogonal polynomials has been freed from the tables giving the values of these polynomials corresponding to the given x values. These tables would be very voluminous if we wanted them extended up to one hundred observations and to parabolas up to the tenth degree.

This has been attained by giving formulae which permit us to pass easily from the orthogonal expansion of the approximating parabola to its *Newton* series, which gives directly the required values by means of the method of addition of the differences and by calculating the coefficients a_m without the evaluation of $\Sigma y\, Q_m(x)$.

The third paper "Sur la détermination de la tendance séculaire des grandeurs statistiques par la méthode des moindres carrés."[27] (1930) published somewhat later than the second, differs from it inasmuch as it introduces the mean orthogonal moments, giving (p. 585) a table for their calculation for parabolas up to the tenth degree,—the present table I. The coefficients a_m, the quantities $\frac{a_m^2}{N}\Sigma Q_m^2$ figuring in the formula of the mean square deviation, and those of $\frac{a_m a_{m'}}{N}\Sigma Q_m^2$ figuring in that of the different coefficients of correlation, are expressed by these moments. The calculation of ΣQ_m^2 became needless, which is very fortunate since

[27]Journal de la Sociáté Hongroise de Statistique, 1929, T. VII. p. 567.

these numbers are very large if N is large, and it would therefore be difficult to operate with such numbers. The table of the binomial coefficients has been extended sufficiently for one hundred observations (N up to 105).

Finally, in the *present* paper the orthogonal polynomials are left in the general form, the arbitrary constant remains entirely in the background, the coefficients a_m are no longer calculated, the mean orthogonal moments alone are used, and by the aid of these quantities the Newton expansion of the approximating parabola and also the mean square deviation and the coefficients of correlation are directly obtained. All unnecessary matter has been cleared away.

Table II. giving numbers S_{2m} useful for graduation, and Table III. rendering it easier to establish *Newton's* formula, are new. Table IV. of the binomial coefficients has been extended by a few lines (up to 110) in order to suffice for parabolic approximation of the tenth degree by the new formulae.

§ 21. *M. F. Essher* "Ueber die Sterblichkeit in Schweden 1886-1914"[28] denotes in this paper the orthogonal polynomial of degree m by $P_m(x)$; x taking the values of $-\frac{1}{2}(N-1),\ldots,\frac{1}{2}(N-1)$ Since the coefficient of x^m is taken as equal to unity, therefore the constant C of the polynomial $U_m(x)$ is, according to formula (21'), $C=m!/\binom{2m}{m}$. Putting this value into (21'), and writing $h=1, a=-\frac{1}{2}(N-1)$, it will give the values of *Essher's* polynomial corresponding to a given x. Formula (24) gives for the above value of C and $h=1$

$$\sum_{x=a}^{b} P_m^2 = \frac{(m!)^2}{\binom{2m}{m}}\binom{N+m}{2m+1}$$

In a *second* paper "On Graduation according to the Method of Least Squares by Means of Certain Polynomials"[29], *Essher* has

[28] Medelanden fran Lunds Astronomiska Observatorium, Lund 1920.
[29] Försäkringsaktiebolaget Skandia 1855-1930, Stockholm, 1930. p. 107.

employed other orthogonal polynomials, denoting them by $X_m(x)$ the variable x taking the values of 1,2,3,...., N. Adopting *Lorenz's* point of view (§ 22), he chose the constant C in such a manner that ΣX_m^2 should be equal to N. From our formula (24) we conclude that in this case

$$C = \sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}}\;.$$

In this way the expression ΣX_m^2 becomes very simple, but the polynomials themselves become complicated.

Putting $h=1$, $a=1$, and the above value of C into formula (21′) we have

$$X_m(x) = \sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}}\;\sum_{\nu=0}^{m+1}\binom{x-N-1}{\nu}\binom{m+\nu}{m}\binom{N+m}{m-\nu}.$$

The coefficient a_m in an expression by *Essher's* polynomials is expressed very simply by our method. Indeed from (32) we get, if we put in this equation $h=1$ and the value or C above,

$$a_m = \Theta_m \sqrt{C_m}$$

where Θ_m is the mean orthogonal moment of degree m and C_m the number given by formula (34), or $C_m = |C_{m0}|$ taken from Table III.

As a comparison let us determine the graduated value corresponding to Age 52, *Essher's* example 10. (p. 116)

The first column below contains the given values corresponding to x in an inverted order of magnitude; the other columns are obtained by the method of § 9. The graduation for the central value will be obtained by an eleven-point parabola of the second degree.

674	674	674	674
873	1547	2221	2895
1005	2552	4773	7668
1216	3768	8541	16209
1331	5099	13640	29849
1239	6338	19978	49827
1640	7978	27956	77783
1385	9363	37319	115102
1366	10729	48048	163150
1315	12044	60092	
851	12895		

Let us remark, that these numbers figure in *Esshcr's* table (p. 117). We shall have

$$T_0 = \Theta_0 = 12895/11 = 1172,272727\ldots$$
$$T_1 = 60092/55 = 1092,581818\ldots$$
$$T_2 = 163150/165 = 988,7878\ldots$$

Hence

$$\Theta_2 = 2T_2 - 3T_1 + T_0 = -127,896969\ldots$$

From Table II. we take $S_2 = -1,92307694$; and finally the required graduated value will be

$$\Theta_0 + S_2\Theta_2 = 1418,22$$

agreeing with *Esshcr's* result.

§ 22. *P. Lorcntz* in the first edition of his paper "Der Trend"[30] introduced orthogonal polynomials, distinguishing two cases according as the number of observations was either even or odd. He denoted polynomials of degree m by $X_m(x)$ and chose them so that ΣX_m^2 should be equal to N.

If m is odd the variable takes the values

$$-\tfrac{1}{2}(N-1),\ldots,-1,0,1,\ldots\tfrac{1}{2}(N-1)$$

[30]Vierteljahreshefte zur Konjunkturforschung. Sonderheft 9, Berlin, 1928.

and the value of C in $U_m(x)$ corresponding to the above condition, taken from (24), is

$$C = \sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}}$$

Hence X_m is given by formula (21′) by putting in it the above value of C and placing $h=1$, $a=-\frac{1}{2}(N-1)$, so that we have

$$X_m(x) = \sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m}\binom{N+m}{m-\nu}\binom{x-\frac{N-1}{2}}{\nu}.$$

If m is even, the variable in $X_m(x)$ takes the values

$$-(N-1), -(N-3), \cdots -1, 1, 3, \cdots, (N-1)$$

and the value of C corresponding to the condition $\Sigma X_m^2 = N$ will be obtained from (24) by putting $h=2$, so that

$$C = \frac{1}{2^{2m}} \sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}}$$

The polynomial is given by (21′) by placing in it this value of C and $h=2$, $a=-(N-1)$. Hence it follows that

$$X_m(x) = \sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}} \sum_{\nu=0}^{m+1} \frac{1}{2^\nu}\binom{m+\nu}{m}\binom{N+m}{m-\nu}\binom{x-N-1}{\nu}_2$$

Whether N be odd or even, the coefficient of a_m is given by our formula (32), the same formula appearing in *Essher's* expansion, *i.e.*,

$$a_m = \theta_m \sqrt{C_m} .$$

Here θ_m is the mean orthogonal moment of degree m, and C_m is given by formula (34), or since $C_m = |C_{m0}|$, by Table III.

The paper contains five decimal tables giving $X_m(x)$ corresponding to the necessary integer values of x up to $m = 5$ and N up to 60. There are also other tables useful for the transformation of the orthogonal series into a power series, and also a table enabling one to change the interval of one year into that of one month.

The second edition of the paper does not differ in principle from the first. The polynomials remain the same, but the tables for $X_m(x)$ have been extended for m up to six, and for N up to eighty.

The only advantage that *Lorentz's* method possesses over ours is that when applying *Chetverikoff's* method to the determination of binomial moments the calculation in his system is a little easier, since the numbers to be added contain one or two figures less. But as this operation is generally made by calculating machines, this is but a slight advantage, and this is largely compensated for in the subsequent operations.

As an example, let us determinate the coefficients corresponding to *Lorentz's* polynomials in the orthogonal expansion of the example in § 13. There, the mean orthogonal moments found were

$$\theta_0 = 9942,3333 \qquad \theta_3 = 183$$
$$\theta_1 = 971,3333 \qquad \theta_4 = 351,666$$
$$\theta_2 = 38,5333 \qquad \theta_5 = -1152$$

We have seen that $C_o = 1$; the other numbers $C_m = |C_{mo}|$ are taken from Table II., their square roots being

$$\sqrt{C_1} = 1,463\ 850\ 109 \qquad \sqrt{C_2} = 1,336\ \ 306\ 210$$

$$\sqrt{C_3} = 0,912\ \ 870\ 929 \qquad \sqrt{C_4} = 0,462\ \ 910\ \ 050$$

$$\sqrt{C_5} = 0,\ 154\ \ 303\ 350$$

Finally, the required coefficients a_m are given by our formula

$$a_m = \theta_m \sqrt{C_m}$$

$$a_0 = 9942,33333 \qquad a_1 = 1421,88641$$

$$a_2 = 51,\ 49233 \qquad a_4 = 162,\ 79003$$

$$a_5 = -177,\ 57459$$

in accordance with *Lorentz's* results.

The corresponding mean square deviations would be given by

$$\sigma_m^2 = \frac{1}{N}\ \Sigma y^2 - a_o^2 - a_1^2 - \cdots - a_m^2\ .$$

Charles Jordan

TABLE III

These numbers are given to ten figures, although generally fewer will be sufficient, especially if as is generally the case the mean orthogonal moments are all of the same order of magnitude. In this event, a fixed number of *decimals*, properly chosen will suffice.

The numbers C_{ms} given are checked by the relation:

$$\sum_{s=0}^{m+1} C_{ms} = 2m+1$$

Remark.

$$\lim_{N=\infty} C_{m0} = 2m+1 \quad \text{and} \quad \lim_{N=\infty} C_{ms} = 0 \quad \text{if} \quad s \neq 0.$$

N	c_{10}	c_{11}	c_{20}	c_{21}
3	-1,5	1,5	0,5	-1,5
4	-1,8	1,2	1	-2
5	-2	1	1,428 571 429	-2,142 857 143
6	-2,142 857 143	0,857 142 857 1	1,785 714 286	-2,142 857 143
7	-2,25	0,75	2,083 333 333	-2,083 333 333
8	-2,333 333 333	0,666 666 666 7	2,333 333 333	-2
9	-2,4	0,6	2,545 454 545	-1,909 090 909
10	-2,454 545 455	0,545 454 545 5	2,727 272 727	-1,818 181 818
11	-2,5	0,5	2,884 615 385	-1,730 769 231
12	-2,538 461 538	0,461 538 461 5	3,021 978 021	-1,648 351 648
13	-2,571 428 571	0,428 571 428 6	3,142 857 143	-1,571 428 571
14	-2,6	0,4	3,25	-1,5
15	-2,625	0,375	3,345 588 235	-1,433 823 529
16	-2,647 058 823	0,352 941 176 4	3,431 372 549	-1,372 549 020
17	-2,666 666 667	0,333 333 333 3	3,508 771 930	-1,315 789 474
18	-2,684 210 526	0,315 789 473 6	3,578 947 368	-1,263 157 894
19	-2,7	0,3	3,642 857 143	-1,214 285 714
20	-2,714 285 714	0,285 714 285 7	3,701 298 702	-1,168 831 169
21	-2,727 272 727	0,272 727 272 7	3,754 940 711	-1,126 482 213
22	-2,739 130 436	0,260 869 565 2	3,804 347 826	-1,086 956 522
23	-2,75	0,25	3,85	-1,05
24	-2,76	0,24	3,892 307 692	-1,015 384 615
25	-2,769 230 769	0,230 769 230 8	3,931 623 933	-0,982 905 982 9
26	-2,777 777 778	0,222 222 222 2	3,968 253 969	-0,952 380 952 4
27	-2,785 714 286	0,214 285 714 2	4,002 463 055	-0,923 645 320 5
28	-2,793 103 447	0,206 896 551 6	4,034 482 758	-0,896 551 723 9
29	-2,8	0,2	4,064 516 128	-0,870 967 741 8
30	-2,806 451 614	0,193 548 387 1	4,092 741 935	-0,846 774 193 4
31	-2,8125	0,1875	4,119 318 182	-0,823 863 636 4
32	-2,818 181 818	0,181 818 181 8	4,144 385 026	-0,802 139 037 4
33	-2,823 529 411	0,176 470 588 2	4,168 067 225	-0,781 512 605 1
34	-2,828 571 429	0,171 428 571 4	4,190 476 192	-0,761 904 761 9
35	-2,833 333 333	0,166 666 666 7	4,211 711 712	-0,743 243 243 3
36	-2,837 837 838	0,162 162 162 2	4,231 863 442	-0,725 462 304 4
37	-2,842 105 263	0,157 894 736 8	4,251 012 146	-0,708 502 024 3
38	-2,846 153 846	0,153 846 153 8	4,269 230 769	-0,692 307 692 3
39	-2,85	0,15	4,286 585 369	-0,676 829 268 3
40	-2,853 658 537	0,146 341 463 4	4,303 135 888	-0,662 020 905 9
41	-2,857 142 857	0,142 857 142 9	4,318 936 877	-0,647 840 531 5
42	-2,860 465 116	0,139 534 883 7	4,334 038 055	-0,634 249 471 4
43	-2,863 636 364	0,136 363 636 4	4,348 484 848	-0,621 212 121 2
44	-2,866 666 667	0,133 333 333 3	4,362 318 840	-0,608 695 652 1
45	-2,869 565 217	0,130 434 782 6	4,375 578 168	-0,596 669 750 2
46	-2,872 340 425	0,127 659 574 6	4,388 297 872	-0,585 106 382 9
47	-2,875	0,125	4,400 510 204	-0,573 979 591 8
48	-2,877 551 021	0,122 448 979 6	4,412 244 898	-0,563 265 306 1
49	-2,88	0,12	4,423 529 411	-0,552 941 176 4
50	-2,882 352 941	0,117 647 058 8	4,434 389 140	-0,542 986 425 4

N	C_{10}	C_{11}	C_{20}	C_{21}
51	-2,884 615 385	0,115 384 615 4	4,444 847 606	-0,533 381 712 7
52	-2,886 792 453	0,113 207 547 2	4,454 926 625	-0,524 109 014 7
53	-2,888 888 889	0,111 111 111 1	4,464 646 465	-0,515 151 515 2
54	-2,890 909 091	0,109 090 909 1	4,474 025 974	-0,506 493 506 5
55	-2,892 857 143	0,107 142 857 1	4,483 082 706	-0,498 120 300 7
56	-2,894 736 842	0,105 263 157 9	4,491 833 031	-0,490 018 148 9
57	-2,896 551 724	0,103 448 275 9	4,500 292 227	-0,482 174 167 2
58	-2,898 305 085	0,101 694 915 3	4,508 474 576	-0,474 576 271 2
59	-2,9	0,1	4,516 393 442	-0,467 213 114 7
60	-2,901 639 344	0,098 360 655 74	4,524 061 343	-0,460 074 034 9
61	-2,903 225 806	0,096 774 193 54	4,531 490 014	-0,453 149 001 5
62	-2,904 761 905	0,095 238 095 24	4,538 690 476	-0,446 428 571 4
63	-2,906 250	0,093 750	4,545 673 077	-0,439 903 846 2
64	-2,907 692 307	0,092 307 692 30	4,552 447 552	-0,433 566 433 6
65	-2,909 090 909	0,090 909 090 91	4,559 023 067	-0,427 408 412 5
66	-2,910 447 761	0,089 552 238 80	4,565 408 253	-0,421 422 300 3
67	-2,911 764 706	0,088 235 294 12	4,571 611 254	-0,415 601 023 1
68	-2,913 043 478	0,086 956 521 74	4,577 639 752	-0,409 937 888 2
69	-2,914 285 714	0,085 714 285 72	4,583 501 007	-0,404 426 559 4
70	-2,915 492 960	0,084 507 042 26	4,589 201 879	-0,399 061 032 9
71	-2,916 666 667	0,083 333 333 33	4,594 748 858	-0,393 835 616 4
72	-2,917 808 219	0,082 191 780 82	4,600 148 094	-0,388 744 909 4
73	-2,918 918 919	0,081 081 081 08	4,605 405 405	-0,383 783 783 8
74	-2,92	0,08	4,610 526 316	-0,378 947 368 4
75	-2,921 052 632	0,078 947 368 42	4,615 516 062	-0,374 231 032 1
76	-2,922 077 922	0,077 922 077 92	4,620 379 620	-0,369 630 369 6
77	-2,923 076 923	0,076 923 076 92	4,625 121 713	-0,365 141 187 9
78	-2,924 050 633	0,075 949 367 08	4,629 746 835	-0,360 759 493 7
79	-2,925	0,075	4,634 259 260	-0,356 481 481 5
80	-2,925 925 926	0,074 074 074 07	4,638 663 052	-0,352 303 523 0
81	-2,926 829 268	0,073 170 731 70	4,642 962 091	-0,348 222 156 8
82	-2,927 710 843	0,072 289 156 62	4,647 160 068	-0,344 234 079 1
83	-2,928 571 428	0,071 428 571 42	4,651 260 504	-0,340 336 134 4
84	-2,929 411 765	0,070 588 235 30	4,655 266 758	-0,336 525 307 8
85	-2,930 232 558	0,069 767 441 86	4,659 182 037	-0,332 798 716 9
86	-2,931 034 483	0,068 965 517 24	4,663 009 403	-0,329 153 604 9
87	-2,931 818 182	0,068 181 818 18	4,666 751 789	-0,325 587 334 1
88	-2,932 584 270	0,067 415 730 34	4,670 411 986	-0,322 097 378 4
89	-2,933 333 333	0,066 666 666 67	4,673 992 674	-0,318 681 318 7
90	-2,934 065 934	0,065 934 065 93	4,677 496 416	-0,315 336 837 0
91	-2,934 782 608	0,065 217 391 30	4,680 925 664	-0,312 061 711 0
92	-2,935 483 871	0,064 516 129 04	4,684 282 773	-0,308 853 809 2
93	-2,936 170 213	0,063 829 787 24	4,687 569 990	-0,305 711 086 3
94	-2,936 842 105	0,063 157 894 74	4,690 789 473	-0,302 631 578 9
95	-2,937 5	0,062 5	4,693 943 301	-0,299 613 402 2
96	-2,938 144 330	0,061 855 670 10	4,697 033 453	-0,296 654 744 5
97	-2,938 775 510	0,061 224 489 80	4,700 061 841	-0,293 753 865 1
98	-2,939 393 939	0,060 606 060 61	4,703 030 303	-0,290 909 090 9
99	-2,94	0,06	4,705 940 594	-0,288 118 811 9
100	-2,940 594 060	0,059 405 940 59	4,708 794 409	-0,285 381 479 3

N	c_{22}	c_{30}	c_{31}	c_{32}
3	3			
4	2	- 0,2	0,8	- 2
5	1,428 571 429	- 0,5	1,5	- 2,5
6	1,071 428 572	- 0,833 333 333 3	2	- 2,5
7	0,833 333 333 3	- 1,166 666 667	2,333 333 333	-2,333 333 333
8	0,666 666 666 7	- 1,484 848 485	2,545 454 545	-2,121 212 121
9	0,545 454 545 5	-1,781 818 182	2,672 727 273	-1,909 090 909
10	0,454 545 454 5	-2,055 944 056	2,741 258 742	-1,713 286 714
11	0,384 615 384 6	- 2,307 692 308	2,769 230 769	-1,538 461 538
12	0,329 670 329 7	-2,538 461 538	2,769 230 769	-1,384 615 384
13	0,285 714 285 7	- 2,75	2,75	- 1,25
14	0,25	-2,944 117 648	2,717 647 060	-1,132 352 941
15	0,220 588 235 2	-3,122 549 020	2,676 470 588	-1,029 411 765
16	0,196 078 431 4	-3,286 893 705	2,629 514 964	-0,939 112 487 0
17	0,175 438 596 5	-3,438 596 491	2,578 947 368	-0,859 649 122 8
18	0,157 894 736 8	-3,578 947 369	2,526 315 790	-0,789 473 684 2
19	0,142 857 142 9	-3,709 090 909	2,472 727 273	-0,727 272 727 3
20	0,129 870 129 9	-3,830 039 526	2,418 972 332	-0,671 936 758 9
21	0,118 577 075 1	-3,942 687 747	2,365 612 648	-0,622 529 644 3
22	0,108 695 652 2	-4,047 826 087	2,313 043 478	-0,578 260 869 6
23	0,1	-4,146 153 846	2,261 538 461	-0,538 461 538 4
24	0,092 307 692 28	-4,238 290 598	2,211 282 051	-0,502 564 102 5
25	0,085 470 085 50	-4,324 786 325	2,162 393 163	-0,470 085 470 1
26	0,079 365 079 38	-4,406 130 268	2,114 942 528	-0,440 613 026 8
27	0,073 891 625 64	-4,482 758 621	2,068 965 517	-0,413 793 103 4
28	0,068 965 517 22	-4,555 061 179	2,024 471 635	-0,389 321 468 2
29	0,064 516 129 02	-4,623 387 098	1,981 451 613	-0,366 935 484 0
30	0,060 483 870 96	-4,688 049 852	1,939 882 697	-0,346 407 624 5
31	0,056 818 181 82	-4,749 331 550	1,899 732 620	-0,327 540 106 9
32	0,053 475 935 83	-4,807 486 633	1,860 962 568	-0,310 160 427 9
33	0,050 420 168 07	-4,862 745 098	1,823 529 412	-0,294 117 647 1
34	0,047 619 047 62	-4,915 315 315	1,787 387 387	-0,279 279 279 3
35	0,045 045 045 05	-4,965 386 439	1,752 489 332	-0,265 528 686 6
36	0,042 674 253 20	-5,013 130 540	1,718 787 614	-0,252 762 884 4
37	0,040 485 829 96	-5,058 704 454	1,686 234 818	-0,240 890 688 2
38	0,038 461 538 46	-5,102 251 407	1,654 784 240	-0,229 831 144 5
39	0,036 585 365 86	-5,143 902 439	1,624 390 244	-0,219 512 195 1
40	0,034 843 205 57	-5,183 777 652	1,595 008 508	-0,209 869 540 6
41	0,033 222 591 36	-5,221 987 315	1,566 596 194	-0,200 845 666 0
42	0,031 712 473 57	-5,258 632 840	1,539 112 051	-0,192 389 006 4
43	0,030 303 030 30	-5,293 807 641	1,512 516 469	-0,184 453 227 9
44	0,028 985 507 24	-5,327 597 903	1,486 771 508	-0,176 996 608 1
45	0,027 752 081 14	-5,360 083 256	1,461 840 888	-0,169 981 498 6
46	0,026 595 744 68	-5,391 337 386	1,437 689 970	-0,163 373 860 2
47	0,025 510 204 08	-5,421 428 571	1,414 285 714	-0,157 142 857 1
48	0,024 489 795 92	-5,450 420 168	1,391 596 639	-0,151 260 504 2
49	0,023 529 411 76	-5,478 371 040	1,369 592 760	-0,145 701 357 4
50	0,022 624 434 39	-5,505 335 952	1,348 245 539	-0,140 442 243 7

N	C_{22}	C_{30}	C_{31}	C_{32}
51	0,021 770 682 15	- 5,531 365 909	1,327 527 818	-0,135 462 022 3
52	0,020 964 360 59	- 5,556 508 480	1,307 413 760	- 0,130 741 376 0
53	0,020 202 020 20	- 5,580 808 081	1,287 878 788	- 0,126 262 626 3
54	0,019 480 519 48	- 5,604 306 221	1,268 899 522	- 0,122 009 569 4
55	0,018 796 992 48	- 5,627 041 743	1,250 453 721	- 0,117 967 332 1
56	0,018 148 820 33	- 5,649 051 031	1,232 520 225	-0,114 122 243 1
57	0,017 533 606 08	- 5,670 368 206	1,215 078 901	-0,110 461 718 3
58	0,016 949 152 54	- 5,691 025 285	1,198 110 586	-0,106 974 159 5
59	0,016 393 442 62	- 5,711 052 353	1,181 597 038	- 0,103 648 863 0
60	0,015 864 621 89	- 5,730 477 702	1,165 520 888	-0,100 475 938 7
61	0,015 360 983 10	- 5,749 327 957	1,149 865 591	- 0,097 446 236 56
62	0,014 880 952 38	- 5,767 628 207	1,134 615 385	- 0,094 551 282 08
63	0,014 423 076 92	- 5,785 402 099	1,119 755 245	- 0,091 783 216 80
64	0,013 986 013 99	- 5,802 671 956	1,105 270 849	- 0,089 134 745 86
65	0,013 568 521 03	- 5,819 458 856	1,091 148 535	- 0,086 599 090 12
66	0,013 169 446 88	- 5,835 782 722	1,077 375 272	- 0,084 169 943 11
67	0,012 787 723 79	- 5,851 662 406	1,063 938 619	- 0,081 841 432 26
68	0,012 422 360 25	- 5,867 115 739	1,050 826 699	- 0,079 608 083 30
69	0,012 072 434 61	- 5,882 159 623	1,038 028 169	- 0,077 464 788 72
70	0,011 737 089 20	- 5,896 810 084	1,025 532 189	- 0,075 406 778 57
71	0,011 415 525 11	- 5,911 082 314	1,013 328 397	- 0,073 429 593 96
72	0,011 106 997 41	- 5,924 990 742	1,001 406 886	- 0,071 529 063 28
73	0,010 810 810 81	- 5,938 549 075	0,989 758 179 3	- 0,069 701 280 23
74	0,010 526 315 79	- 5,951 770 335	0,978 373 205 7	- 0,067 942 583 73
75	0,010 252 904 99	- 5,964 666 912	0,967 243 283 1	- 0,066 249 539 94
76	0,009 990 009 990	- 5,977 250 597	0,956 360 095 6	- 0,064 618 925 38
77	0,009 737 098 344	- 5,989 532 620	0,945 715 676 8	- 0,063 047 711 78
78	0,009 493 670 886	- 6,001 523 676	0,935 302 391 0	- 0,061 533 052 04
79	0,009 259 259 259	- 6,013 233 966	0,925 112 917 8	- 0,060 072 267 39
80	0,009 033 423 666	- 6,024 673 219	0,915 140 235 7	- 0,058 662 835 62
81	0,008 815 750 806	- 6,035 850 720	0,905 377 608 0	- 0,057 302 380 25
82	0,008 605 851 978	- 6,046 775 336	0,895 818 568 4	- 0,055 988 660 52
83	0,008 403 361 344	- 6,057 455 540	0,886 456 908 3	- 0,054 719 562 24
84	0,008 207 934 343	- 6,067 899 429	0,877 286 664 4	- 0,053 493 089 29
85	0,008 019 246 192	- 6,078 114 748	0,868 302 106 9	- 0,052 307 355 84
86	0,007 836 990 594	- 6,088 108 908	0,859 497 728 2	- 0,051 160 579 06
87	0,007 660 878 450	- 6,097 889 003	0,850 868 232 9	- 0,050 051 072 53
88	0,007 490 636 706	- 6,107 461 827	0,842 408 527 8	- 0,048 977 239 99
89	0,007 326 007 326	- 6,116 833 891	0,834 113 712 4	- 0,047 937 569 68
90	0,007 166 746 296	- 6,126 011 436	0,825 979 070 0	- 0,046 930 628 98
91	0,007 012 622 718	- 6,135 000 448	0,818 000 059 7	- 0,045 955 059 53
92	0,006 863 417 982	- 6,143 806 668	0,810 172 307 9	- 0,045 009 572 66
93	0,006 718 924 974	- 6,152 435 610	0,802 491 601 3	- 0,044 092 945 12
94	0,006 578 947 368	- 6,160 892 566	0,794 953 879 5	- 0,043 204 015 19
95	0,006 443 298 972	- 6,169 182 621	0,787 555 228 2	- 0,042 341 678 94
96	0,006 311 803 074	- 6,177 310 662	0,780 291 873 1	- 0,041 504 886 87
97	0,006 184 291 896	- 6,185 281 385	0,773 160 173 2	- 0,040 692 640 69
98	0,006 060 606 060	- 6,193 099 310	0,766 156 615 7	- 0,039 903 990 40
99	0,005 940 594 059	- 6,200 768 783	0,759 277 810 1	- 0,039 138 031 45
100	0,005 824 111 823	- 6,208 293 988	0,752 520 483 4	- 0,038 393 902 22

N	C_{33}	C_{40}	C_{41}	C_{42}
4	4			
5	2,5	0,071 428 571 43	−0,357 142 857 2	1,071 428 571
6	1,666 666 667	0,214 285 714 3	−0,857 142 857 2	1,928 571 429
7	1,166 666 667	0,409 090 909 1	−1,363 636 364	2,454 545 455
8	0,848 484 848 5	0,636 363 636 4	−1,818 181 818	2,727 272 727
9	0,636 363 636 4	0,881 118 881 1	−2,202 797 203	2,832 167 832
10	0,489 510 489 6	1,132 867 133	−2,517 482 517	2,832 167 832
11	0,384 615 384 6	1,384 615 385	−2,769 230 769	2,769 230 769
12	0,307 692 307 6	1,631 868 132	−2,967 032 967	2,670 329 670
13	0,25	1,871 848 739	−3,119 747 899	2,552 521 008
14	0,205 882 353 0	2,102 941 177	−3,235 294 118	2,426 470 589
15	0,171 568 627 5	2,324 303 405	−3,320 433 436	2,298 761 610
16	0,144 478 844 2	2,535 603 715	−3,380 804 953	2,173 374 613
17	0,122 807 017 5	2,736 842 104	−3,421 052 630	2,052 631 578
18	0,105 263 157 9	2,928 229 666	−3,444 976 077	1,937 799 043
19	0,090 909 090 91	3,110 107 283	−3,455 674 759	1,829 474 872
20	0,079 051 383 40	3,282 891 022	−3,455 674 760	1,727 837 380
21	0,069 169 960 48	3,447 035 573	−3,447 035 573	1,632 806 324
22	0,060 869 565 22	3,603 010 033	−3,431 438 127	1,544 147 157
23	0,053 846 153 84	3,751 282 051	−3,410 256 410	1,461 538 461
24	0,047 863 247 86	3,892 307 693	−3,384 615 385	1,384 615 385
25	0,042 735 042 74	4,026 525 199	−3,355 437 666	1,312 997 347
26	0,038 314 176 24	4,154 351 396	−3,323 481 117	1,246 305 419
27	0,034 482 758 62	4,276 179 883	−3,289 369 141	1,184 172 891
28	0,031 145 717 46	4,392 380 423	−3,253 615 128	1,126 251 391
29	0,028 225 806 46	4,503 299 121	−3,216 642 229	1,072 214 076
30	0,025 659 824 04	4,609 259 101	−3,178 799 380	1,021 756 943
31	0,023 395 721 92	4,710 561 498	−3,140 374 332	0,974 598 930 5
32	0,021 390 374 34	4,807 486 632	−3,101 604 278	0,930 481 283 5
33	0,019 607 843 14	4,900 295 253	−3,062 684 533	0,889 166 477 4
34	0,018 018 018 02	4,989 229 831	−3,023 775 655	0,850 436 903 0
35	0,016 595 542 91	5,074 515 814	−2,985 009 302	0,814 093 446 0
36	0,015 318 962 69	5,156 362 841	−2,946 493 052	0,779 954 043 1
37	0,014 170 040 49	5,234 965 933	−2,908 314 407	0,747 852 276 1
38	0,013 133 208 26	5,310 506 567	−2,870 544 090	0,717 636 022 5
39	0,012 195 121 95	5,383 153 716	−2,833 238 798	0,689 166 194 0
40	0,011 344 299 49	5,453 064 803	−2,796 443 488	0,662 315 563 1
41	0,010 570 824 52	5,520 386 590	−2,760 193 295	0,636 967 683 5
42	0,009 866 102 890	5,585 255 997	−2,724 515 121	0,613 015 902 2
43	0,009 222 661 396	5,647 800 858	−2,689 428 980	0,590 362 459 0
44	0,008 633 980 882	5,708 140 610	−2,654 949 121	0,568 917 668 8
45	0,008 094 357 076	5,766 386 943	−2,621 084 974	0,548 599 180 6
46	0,007 598 784 194	5,822 644 377	−2,587 841 945	0,529 331 307 0
47	0,007 142 857 142	5,877 010 804	−2,555 222 089	0,511 044 417 7
48	0,006 722 689 076	5,929 577 985	−2,523 224 675	0,493 674 392 9
49	0,006 334 841 628	5,980 431 999	−2,491 846 666	0,477 162 127 6
50	0,005 976 265 688	6,029 653 662	−2,461 083 127	0,461 453 086 4

N	C_{33}	C_{40}	C_{41}	C_{42}
51	0,005 644 250 928	6,077 318 907	−2,430 927 563	0,446 496 899 3
52	0,005 336 382 694	6,123 499 143	−2,401 372 213	0,432 246 998 3
53	0,005 050 505 051	6,168 261 563	−2,372 408 293	0,418 660 287 1
54	0,004 784 688 996	6,211 669 457	−2,344 026 210	0,405 696 844 1
55	0,004 537 205 082	6,253 782 470	−2,316 215 730	0,393 319 652 2
56	0,004 306 499 738	6,294 656 864	−2,288 966 132	0,381 494 355 4
57	0,004 091 174 752	6,334 345 746	−2,262 266 338	0,370 189 037 1
58	0,003 889 969 436	6,372 899 282	−2,236 105 011	0,359 374 019 7
59	0,003 701 745 108	6,410 364 887	−2,210 470 651	0,349 021 681 7
60	0,003 525 471 532	6,446 787 416	−2,185 351 666	0,339 106 293 1
61	0,003 360 215 054	6,482 209 321	−2,160 736 440	0,329 603 863 8
62	0,003 205 128 206	6,516 670 830	−2,136 613 387	0,320 492 008 1
63	0,003 059 440 560	6,550 210 049	−2,112 970 984	0,311 749 817 2
64	0,002 922 450 684	6,582 863 143	−2,089 797 823	0,303 357 748 5
65	0,002 793 519 036	6,614 664 415	−2,067 082 630	0,295 297 518 5
66	0,002 672 061 686	6,645 646 448	−2,044 814 292	0,287 552 009 8
67	0,002 557 544 758	6,675 840 208	−2,022 981 881	0,280 105 183 5
68	0,002 449 479 486	6,705 275 129	−2,001 574 665	0,272 941 999 8
69	0,002 347 417 840	6,733 979 218	−1,980 582 123	0,266 048 344 9
70	0,002 250 948 614	6,761 979 132	−1,959 993 951	0,259 410 964 1
71	0,002 159 693 940	6,789 300 259	−1,939 800 074	0,253 017 401 0
72	0,002 073 306 182	6,815 966 796	−1,919 990 647	0,246 855 940 3
73	0,001 991 465 149	6,842 001 810	−1,900 556 058	0,240 915 556 7
74	0,001 913 875 598	6,867 427 310	−1,881 485 934	0,235 185 866 8
75	0,001 840 264 998	6,892 264 298	−1,862 774 135	0,229 657 085 1
76	0,001 770 381 517	6,916 532 835	−1,844 408 756	0,224 319 983 8
77	0,001 703 992 210	6,940 252 082	−1,826 382 127	0,219 165 855 2
78	0,001 640 881 388	6,963 440 362	−1,808 685 808	0,214 186 477 3
79	0,001 580 849 142	6,986 115 193	−1,791 311 588	0,209 374 081 7
80	0,001 523 710 016	7,008 293 336	−1,774 251 477	0,204 721 324 3
81	0,001 469 291 801	7,029 990 839	−1,757 497 710	0,200 221 258 1
82	0,001 417 434 443	7,051 223 066	−1,741 042 732	0,195 867 307 4
83	0,001 367 989 056	7,072 004 744	−1,724 879 206	0,191 653 245 1
84	0,001 320 817 020	7,092 349 982	−1,708 999 996	0,187 573 170 2
85	0,001 275 789 167	7,112 272 313	−1,693 398 170	0,183 621 488 3
86	0,001 232 785 038	7,131 784 720	−1,678 066 993	0,179 792 892 1
87	0,001 191 692 203	7,150 899 663	−1,662 999 922	0,176 082 344 6
88	0,001 152 405 647	7,169 629 101	−1,648 190 598	0,172 485 062 6
89	0,001 114 827 202	7,187 984 527	−1,633 632 847	0,168 996 501 4
90	0,001 078 865 034	7,205 976 979	−1,619 320 669	0,165 612 341 2
91	0,001 044 433 171	7,223 617 068	−1,605 248 237	0,162 328 473 4
92	0,001 011 451 071	7,240 915 001	−1,591 409 890	0,159 140 989 0
93	0,000 979 843 225 0	7,257 880 596	−1,577 800 129	0,156 046 166 7
94	0,000 949 538 795 4	7,274 523 294	−1,564 413 612	0,153 040 462 0
95	0,000 920 471 281 2	7,290 852 189	−1,551 245 147	0,150 120 498 1
96	0,000 892 578 212 2	7,306 876 040	−1,538 289 693	0,147 283 055 7
97	0,000 865 800 865 8	7,322 603 281	−1,525 542 350	0,144 525 064 8
98	0,000 840 084 008 4	7,338 042 040	−1,512 998 359	0,141 843 596 1
99	0,000 815 375 655 2	7,353 200 151	−1,500 653 092	0,139 235 853 9
100	0,000 791 626 849 8	7,368 085 172	−1,488 502 055	0,136 699 168 3

N	C_{43}	C_{44}	C_{50}	C_{51}
5	-2,5	5		
6	-3	3	- 0,023 809 523 81	0,142 857 142 9
7	-2,863 636 364	1,909 090 909	- 0,083 333 333 33	0,416 666 666 7
8	-2,545 454 545	1,272 727 273	- 0,179 487 179 5	0,769 230 769 2
9	- 2,202 797 203	0,881 118 881 1	- 0,307 692 307 7	1,153 846 154
10	-1,888 111 888	0,629 370 629 4	- 0,461 538 461 5	1,538 461 538
11	-1,615 384 615	0,461 538 461 5	- 0,634 615 384 5	1,903 846 154
12	-1,384 615 385	0,346 153 846 2	- 0,821 266 968 5	2,239 819 005
13	-1,191 176 471	0,264 705 882 4	- 1,016 806 723	2,542 016 807
14	-1,029 411 765	0,205 882 353 0	- 1,217 492 260	2,809 597 523
15	-0,893 962 848 2	0,162 538 699 7	- 1,420 407 637	3,043 730 650
16	-0,780 185 758 4	0,130 030 959 7	- 1,623 323 013	3,246 646 027
17	-0,684 210 526 1	0,105 263 157 9	- 1,824 561 403	3,421 052 631
18	-0,602 870 813 5	0,086 124 401 93	- 2,022 883 295	3,569 794 051
19	-0,533 596 837 7	0,071 146 245 03	- 2,217 391 304	3,695 652 173
20	-0,474 308 300 4	0,059 288 537 55	- 2,407 453 416	3,801 242 236
21	-0,423 320 158 1	0,049 802 371 54	- 2,592 642 141	3,888 963 211
22	-0,379 264 214 0	0,042 140 468 22	- 2,772 686 734	3,960 981 048
23	-0,341 025 641 0	0,035 897 435 90	- 2,947 435 897	4,019 230 768
24	-0,307 692 307 7	0,030 769 230 77	- 3,116 828 765	4,065 428 824
25	- 0,278 514 588 8	0,026 525 198 94	- 3,280 872 384	4,101 090 480
26	-0,252 873 563 3	0,022 988 505 75	- 3,439 624 274	4,127 549 129
27	-0,230 255 839 9	0,020 022 246 94	- 3,593 178 929	4,145 975 687
28	-0,210 233 592 9	0,017 519 466 08	- 3,741 657 397	4,157 397 108
29	-0,192 448 680 4	0,015 395 894 43	- 3,885 199 241	4,162 713 473
30	-0,176 599 965 5	0,013 584 612 73	- 4,023 956 357	4,162 713 472
31	-0,162 433 155 1	0,012 032 085 56	- 4,158 088 235	4,158 088 235
32	-0,149 732 620 3	0,010 695 187 17	- 4,287 758 346	4,149 443 561
33	-0,138 314 785 4	0,009 538 950 715	- 4,413 131 398	4,137 310 686
34	-0,128 022 759 6	0,008 534 850 639	- 4,534 371 270	4,122 155 700
35	-0,118 721 960 9	0,007 659 481 347	- 4,651 639 494	4,104 387 789
36	-0,110 296 531 4	0,006 893 533 210	- 4,765 094 116	4,084 366 385
37	-0,102 646 390 8	0,006 220 993 384	- 4,874 888 909	4,062 407 424
38	-0,095 684 803 00	0,005 628 517 824	- 4,981 172 826	4,038 788 778
39	-0,089 336 358 49	0,005 104 934 771	- 5,084 089 620	4,013 754 963
40	-0,083 535 296 24	0,004 640 849 791	- 5,183 777 652	3,987 521 270
41	-0,078 224 101 48	0,004 228 329 810	- 5,280 369 782	3,960 277 336
42	-0,073 352 330 17	0,003 860 648 956	- 5,373 993 360	3,932 190 263
43	-0,068 875 620 22	0,003 532 083 088	- 5,464 770 274	3,903 407 339
44	-0,064 754 856 61	0,003 237 742 831	- 5,552 817 057	3,874 058 412
45	-0,060 955 464 51	0,002 973 437 293	- 5,638 245 011	3,844 257 962
46	-0,057 446 808 51	0,002 735 562 310	- 5,721 160 378	3,814 106 919
47	-0,054 201 680 67	0,002 521 008 403	- 5,801 664 512	3,783 694 247
48	-0,051 195 862 96	0,002 327 084 680	- 5,879 854 060	3,753 098 336
49	-0,048 407 752 07	0,002 151 455 648	- 5,955 821 167	3,722 388 229
50	-0,045 818 036 94	0,001 992 088 563	- 6,029 653 660	3,691 624 690

N	C_{43}	C_{44}	C_{50}	C_{51}
51	-0,043 409 420 76	0,001 847 209 394	-6,101 435 253	3,660 861 152
52	-0,041 166 380 79	0,001 715 265 866	-6,171 245 726	3,630 144 544
53	-0,039 074 960 13	0,001 594 896 332	-6,239 161 120	3,599 516 031
54	-0,037 122 587 04	0,001 484 903 482	-6,305 253 929	3,569 011 658
55	-0,035 297 917 50	0,001 384 232 059	-6,369 593 254	3,538 662 919
56	-0,033 590 697 96	0,001 291 949 922	-6,432 244 994	3,508 497 269
57	-0,031 991 645 18	0,001 207 231 894	-6,493 271 986	3,478 538 564
58	-0,030 492 341 06	0,001 129 345 965	-6,552 734 183	3,448 807 465
59	-0,029 085 140 14	0,001 057 641 460	-6,610 688 791	3,419 321 788
60	-0,027 763 088 32	0,000 991 538 868 6	-6,667 190 404	3,390 096 815
61	-0,026 519 851 11	0,000 930 521 091 5	-6,722 291 149	3,361 145 575
62	-0,025 349 650 35	0,000 874 125 874 3	-6,776 040 811	3,332 479 087
63	-0,024 247 208 01	0,000 821 939 254 5	-6,828 486 947	3,304 106 587
64	-0,023 207 696 61	0,000 773 589 887 0	-6,879 675 007	3,276 035 717
65	-0,022 226 694 94	0,000 728 744 096 5	-6,929 648 433	3,248 272 703
66	-0,021 300 148 87	0,000 687 101 576 5	-6,978 448 774	3,220 822 511
67	-0,020 424 336 30	0,000 648 391 628 6	-7,026 115 774	3,193 688 988
68	-0,019 595 835 88	0,000 612 369 871 4	-7,072 687 465	3,166 874 984
69	-0,018 811 499 13	0,000 578 815 357 9	-7,118 200 254	3,140 382 465
70	-0,018 068 425 36	0,000 547 528 041 3	-7,162 689 006	3,114 212 611
71	-0,017 363 939 28	0,000 518 326 545 7	-7,206 187 117	3,088 365 907
72	-0,016 695 570 84	0,000 491 046 201 2	-7,248 726 592	3,062 842 222
73	-0,016 061 037 11	0,000 465 537 307 6	-7,290 338 111	3,037 640 880
74	-0,015 458 225 99	0,000 441 663 599 6	-7,331 051 094	3,012 760 724
75	-0,014 885 181 44	0,000 419 300 885 6	-7,370 893 763	2,988 200 174
76	-0,014 340 090 29	0,000 398 335 841 8	-7,409 893 200	2,963 957 280
77	-0,013 821 270 15	0,000 378 664 935 6	-7,448 075 405	2,940 029 765
78	-0,013 327 158 59	0,000 360 193 475 3	-7,485 465 343	2,916 415 068
79	-0,012 856 303 26	0,000 342 834 753 7	-7,522 086 993	2,893 110 382
80	-0,012 407 352 99	0,000 326 509 289 2	-7,557 963 401	2,870 112 684
81	-0,011 979 049 63	0,000 311 144 146 2	-7,593 116 719	2,847 418 770
82	-0,011 570 220 69	0,000 296 672 325 4	-7,627 568 248	2,825 025 277
83	-0,011 179 772 63	0,000 283 032 218 5	-7,661 338 474	2,802 928 710
84	-0,010 806 684 71	0,000 270 167 117 6	-7,694 447 108	2,781 125 461
85	-0,010 450 003 40	0,000 258 024 775 3	-7,726 913 130	2,759 611 832
86	-0,010 108 837 31	0,000 246 557 007 5	-7,758 754 806	2,738 384 049
87	-0,009 782 352 480	0,000 235 719 336 9	-7,789 989 730	2,717 438 278
88	-0,009 469 768 142	0,000 225 470 670 0	-7,820 634 849	2,696 770 638
89	-0,009 170 352 790	0,000 215 773 006 8	-7,850 706 503	2,676 377 217
90	-0,008 883 420 600	0,000 206 591 176 8	-7,880 220 438	2,656 254 080
91	-0,008 608 328 137	0,000 197 892 600 8	-7,909 191 835	2,636 397 278
92	-0,008 344 471 335	0,000 189 647 075 8	-7,937 635 345	2,616 802 861
93	-0,008 091 282 715	0,000 181 826 577 9	-7,965 565 098	2,597 466 879
94	-0,007 848 228 821	0,000 174 405 084 9	-7,992 994 727	2,578 385 396
95	-0,007 614 807 873	0,000 167 358 414 8	-8,019 937 409	2,559 554 492
96	-0,007 390 547 597	0,000 160 664 078 2	-8,046 405 846	2,540 970 267
97	-0,007 175 003 215	0,000 154 301 144 4	-8,072 412 329	2,522 628 853
98	-0,006 967 755 600	0,000 148 250 119 2	-8,097 968 722	2,504 526 409
99	-0,006 768 409 564	0,000 142 492 832 9	-8,123 086 494	2,486 659 131
100	-0,006 576 592 290	0,000 137 012 339 4	-8,147 776 722	2,469 023 249

N	C_{52}	C_{53}	C_{54}	C_{55}
6	-0,5	1,333 333 333	-3	6
7	-1,166 666 667	2,333 333 333	-3,5	3,5
8	-1,794 871 795	2,871 794 872	-3,230 769 230	2,153 846 154
9	-2,307 692 308	3,076 923 077	-2,769 230 769	1,384 615 385
10	-2,692 307 692	3,076 923 077	-2,307 692 308	0,923 076 923 1
11	-2,961 538 461	2,961 538 461	-1,903 846 154	0,634 615 384 5
12	-3,135 746 607	2,787 330 317	-1,567 873 304	0,447 963 801 0
13	-3,235 294 117	2,588 235 294	-1,294 117 647	0,323 529 411 7
14	-3,277 863 777	2,383 900 929	-1,072 755 418	0,238 390 092 9
15	-3,277 863 777	2,185 242 518	-0,893 962 848 4	0,178 792 569 7
16	-3,246 646 027	1,997 936 016	-0,749 226 006 1	0,136 222 910 2
17	-3,192 982 456	1,824 561 403	-0,631 578 947 3	0,105 263 157 9
18	-3,123 569 794	1,665 903 890	-0,535 469 107 6	0,082 379 862 71
19	-3,043 478 260	1,521 739 130	-0,456 521 739 1	0,065 217 391 29
20	-2,956 521 740	1,391 304 348	-0,391 304 347 9	0,052 173 913 05
21	-2,865 551 840	1,273 578 596	-0,337 123 745 9	0,042 140 468 24
22	-2,772 686 734	1,167 447 046	-0,291 861 761 4	0,034 336 677 82
23	-2,679 487 179	1,071 794 872	-0,253 846 153 8	0,028 205 128 20
24	-2,587 091 070	0,985 558 502 8	-0,221 750 663 1	0,023 342 175 07
25	-2,496 315 945	0,907 751 252 6	-0,194 518 125 5	0,019 451 812 55
26	-2,407 736 992	0,837 473 736 3	-0,171 301 446 1	0,016 314 423 43
27	-2,321 746 385	0,773 915 461 6	-0,151 418 242 5	0,013 765 294 77
28	-2,238 598 443	0,716 351 501 7	-0,134 315 906 6	0,011 679 644 05
29	-2,158 444 023	0,664 136 622 4	-0,119 544 592 0	0,009 962 049 336
30	-2,081 356 736	0,616 698 292 2	-0,106 736 242 9	0,008 538 899 431
31	-2,007 352 941	0,573 529 411 7	-0,095 588 235 29	0,007 352 941 176
32	-1,936 406 995	0,534 181 240 0	-0,085 850 566 43	0,006 359 300 476
33	-1,868 462 890	0,498 256 770 8	-0,077 315 705 81	0,005 522 550 415
34	-1,803 443 119	0,465 404 675 9	-0,069 810 701 38	0,004 814 531 130
35	-1,741 255 425	0,435 313 856 4	-0,063 190 721 08	0,004 212 714 739
36	-1,681 797 923	0,407 708 587 5	-0,057 334 020 11	0,003 698 969 040
37	-1,624 962 970	0,382 344 228 2	-0,052 137 849 29	0,003 258 615 581
38	-1,570 640 080	0,359 003 446 9	-0,047 515 162 09	0,002 879 706 793
39	-1,518 718 094	0,337 492 909 8	-0,043 391 945 55	0,002 552 467 385
40	-1,469 086 784	0,317 640 385 7	-0,039 705 048 21	0,002 268 859 898
41	-1,421 638 018	0,299 292 214 3	-0,036 400 404 45	0,002 022 244 692
42	-1,376 266 592	0,282 311 095 8	-0,033 431 577 14	0,001 807 112 278
43	-1,332 870 799	0,266 574 159 7	-0,030 758 556 89	0,001 618 871 415
44	-1,291 352 804	0,251 971 278 8	-0,028 346 768 87	0,001 453 680 455
45	-1,251 618 871	0,238 403 594 5	-0,026 166 248 18	0,001 308 312 409
46	-1,213 579 474	0,225 782 227 8	-0,024 190 952 97	0,001 180 046 487
47	-1,177 149 321	0,214 027 149 3	-0,022 398 190 05	0,001 066 580 478
48	-1,142 247 320	0,203 066 190 2	-0,020 768 133 09	0,000 965 959 678 4
49	-1,108 796 494	0,192 834 172 8	-0,019 283 417 28	0,000 876 518 967 5
50	-1,076 723 868	0,183 272 147 7	-0,017 928 797 06	0,000 796 835 424 9

N	C_{52}	C_{53}	C_{54}	C_{55}
51	-1,045 960 329	0,174 326 721 5	- 0,016 690 856 31	0,000 725 689 405 0
52	-1,016 440 472	0,165 949 464 9	- 0,015 557 762 33	0,000 662 032 439 7
53	-0,988 102 439 9	0,158 096 390 4	- 0,014 519 056 26	0,000 604 960 677 5
54	-0,960 887 754 1	0,150 727 490 8	- 0,013 565 474 18	0,000 553 692 823 5
55	-0,934 741 148 4	0,143 806 330 5	- 0,012 688 793 87	0,000 507 551 754 8
56	-0,909 610 403 2	0,137 299 683 5	- 0,011 881 703 38	0,000 465 949 152 1
57	-0,885 446 179 9	0,131 177 211 8	- 0,011 137 687 80	0,000 428 372 607 6
58	-0,862 201 866 2	0,125 411 180 5	- 0,010 450 931 71	0,000 394 374 781 6
59	-0,839 833 421 7	0,119 976 203 1	- 0,009 816 234 799	0,000 363 564 251 8
60	-0,818 299 231 3	0,114 849 014 9	- 0,009 228 938 699	0,000 335 597 770 9
61	-0,797 559 966 9	0,110 008 271 3	- 0,008 684 863 523	0,000 310 173 697 3
62	-0,777 578 453 7	0,105 434 366 6	- 0,008 180 252 581	0,000 287 026 406 4
63	-0,758 319 544 6	0,101 109 272 6	- 0,007 711 724 182	0,000 265 921 523 5
64	-0,739 750 000 7	0,097 016 393 54	- 0,007 276 229 515	0,000 246 651 848 0
65	-0,721 838 378 4	0,093 140 435 93	- 0,006 871 015 765	0,000 229 033 858 8
66	-0,704 554 924 3	0,089 467 291 98	- 0,006 493 593 772	0,000 212 904 713 9
67	-0,687 871 474 4	0,085 983 934 30	- 0,006 141 709 592	0,000 198 119 664 3
68	-0,671 761 360 3	0,082 678 321 27	- 0,005 813 319 465	0,000 184 549 824 3
69	-0,656 199 321 0	0,079 539 311 64	- 0,005 506 567 729	0,000 172 080 241 5
70	-0,641 161 419 9	0,076 556 587 46	- 0,005 219 767 327	0,000 160 608 225 4
71	-0,626 624 966 7	0,073 720 584 32	- 0,004 951 382 529	0,000 150 041 894 8
72	-0,612 568 444 4	0,071 022 428 34	- 0,004 700 013 640	0,000 140 298 914 6
73	-0,598 971 441 0	0,068 453 878 98	- 0,004 464 383 411	0,000 131 305 394 5
74	-0,585 814 585 2	0,066 007 277 20	- 0,004 243 324 963	0,000 122 994 926 5
75	-0,573 079 485 4	0,063 675 498 38	- 0,004 035 771 024	0,000 115 307 743 5
76	-0,560 748 674 6	0,061 451 909 55	- 0,003 840 744 347	0,000 108 189 981 6
77	-0,548 805 556 2	0,059 330 330 40	- 0,003 657 349 134	0,000 101 593 031 5
78	-0,537 234 354 8	0,057 304 997 84	- 0,003 484 763 382	0,000 095 472 969 38
79	-0,526 020 069 5	0,055 370 533 63	- 0,003 322 232 018	0,000 089 790 054 53
80	-0,515 148 430 5	0,053 521 914 85	- 0,003 169 060 748	0,000 084 508 286 61
81	-0,504 605 858 0	0,051 754 446 97	- 0,003 024 610 537	0,000 079 595 014 14
82	-0,494 379 423 5	0,050 063 739 09	- 0,002 888 292 640	0,000 075 020 588 04
83	-0,484 456 814 1	0,048 445 681 41	- 0,002 759 564 131	0,000 070 758 054 64
84	-0,474 826 298 2	0,046 896 424 51	- 0,002 637 923 879	0,000 066 782 883 01
85	-0,465 476 694 6	0,045 412 360 45	- 0,002 522 908 914	0,000 063 072 722 84
86	-0,456 397 341 5	0,043 990 105 21	- 0,002 414 091 139	0,000 059 607 188 63
87	-0,447 578 069 3	0,042 626 482 79	- 0,002 311 074 368	0,000 056 367 667 52
88	-0,439 009 173 6	0,041 318 510 45	- 0,002 213 491 631	0,000 053 337 147 74
89	-0,430 681 391 3	0,040 063 385 23	- 0,002 121 002 748	0,000 050 500 065 42
90	-0,422 585 876 4	0,038 858 471 39	- 0,002 033 292 108	0,000 047 842 167 24
91	-0,414 714 178 6	0,037 701 288 96	- 0,001 950 066 671	0,000 045 350 387 69
92	-0,407 058 222 8	0,036 589 503 18	- 0,001 871 054 140	0,000 043 012 738 84
93	-0,399 610 289 2	0,035 520 914 60	- 0,001 796 001 300	0,000 040 818 211 36
94	-0,392 362 995 0	0,034 493 450 11	- 0,001 724 672 506	0,000 038 756 685 52
95	-0,385 309 278 4	0,033 505 154 64	- 0,001 656 848 306	0,000 036 818 851 26
96	-0,378 442 380 2	0,032 554 183 25	- 0,001 592 324 180	0,000 034 996 135 83
97	-0,371 755 831 0	0,031 638 794 12	- 0,001 530 909 393	0,000 033 280 638 98
98	-0,365 243 434 6	0,030 757 341 87	- 0,001 472 425 940	0,000 031 665 073 99
99	-0,358 899 256 0	0,029 908 271 33	- 0,001 416 707 589	0,000 030 142 714 67
100	-0,352 717 607 0	0,029 090 111 92	- 0,001 363 598 996	0,000 028 707 347 29

N	C_{60}	C_{61}	C_{62}	C_{63}
7	0,007 575 757 576	-0,053 030 303 03	0,212 121 212 1	-0,636 363 636 4
8	0,030 303 030 303	-0,181 818 181 8	0,606 060 606 1	-1,454 545 455
9	0,072 727 272 73	-0,381 818 181 8	1,090 909 091	-2,181 818 182
10	0,136 363 636 4	-0,636 363 636 4	1,590 909 091	-2,727 272 727
11	0,220 588 235 3	-0,926 470 588 2	2,058 823 529	-3,088 235 294
12	0,323 529 411 7	-1,235 294 118	2,470 588 235	-3,294 117 647
13	0,442 724 458 2	-1,549 535 604	2,817 337 461	-3,380 804 953
14	0,575 541 795 6	-1,859 442 724	3,099 071 207	-3,380 804 953
15	0,719 427 244 5	-2,158 281 734	3,320 433 436	-3,320 433 436
16	0,872 033 023 9	-2,441 692 467	3,488 132 095	-3,219 814 242
17	1,031 273 836	-2,707 093 821	3,609 458 427	-3,093 821 509
18	1,195 340 129	-2,953 193 260	3,691 491 575	-2,953 193 260
19	1,362 687 747	-3,179 604 743	3,740 711 462	-2,805 533 597
20	1,532 015 810	-3,386 561 265	3,762 845 850	-2,656 126 482
21	1,702 239 789	-3,574 703 557	3,762 845 850	-2,508 563 900
22	1,872 463 768	-3,744 927 537	3,744 927 537	-2,365 217 392
23	2,041 954 023	-3,898 275 863	3,712 643 679	-2,227 586 207
24	2,210 114 943	-4,035 862 069	3,668 965 518	-2,096 551 724
25	2,376 467 681	-4,158 818 441	3,616 363 862	-1,972 562 106
26	2,540 631 565	-4,268 261 030	3,556 884 191	-1,855 765 665
27	2,702 308 120	-4,365 266 963	3,492 213 570	-1,746 106 785
28	2,861 267 421	-4,450 860 433	3,423 738 795	-1,643 394 621
29	3,017 336 553	-4,526 004 830	3,352 596 170	-1,547 352 079
30	3,170 389 857	-4,591 599 103	3,279 713 645	-1,457 650 509
31	3,320 340 729	-4,648 477 020	3,205 846 220	-1,373 934 095
32	3,467 134 739	-4,697 408 357	3,131 605 571	-1,295 836 788
33	3,610 743 870	-4,739 101 330	3,057 484 729	-1,222 993 892
34	3,751 161 688	-4,774 205 785	2,983 878 615	-1,155 049 787
35	3,888 399 310	-4,803 316 795	2,911 101 088	-1,091 662 908
36	4,022 482 045	-4,826 978 454	2,839 399 091	-1,032 508 760 2
37	4,153 446 577	-4,845 687 673	2,768 964 385	-0,977 281 547 6
38	4,281 338 627	-4,859 897 901	2,699 943 279	-0,925 694 838 4
39	4,406 211 004	-4,870 022 689	2,632 444 697	-0,877 481 565 5
40	4,528 121 979	-4,876 439 055	2,566 546 871	-0,832 393 579 7
41	4,647 133 947	-4,879 490 645	2,502 302 895	-0,790 200 914 1
42	4,763 312 297	-4,879 490 645	2,439 745 323	-0,750 690 868 5
43	4,876 724 494	-4,876 724 494	2,378 889 997	-0,713 666 999 1
44	4,987 439 320	-4,871 452 359	2,319 739 218	-0,678 948 063 9
45	5,095 526 240	-4,863 911 411	2,262 284 377	-0,646 366 964 9
46	5,201 054 890	-4,854 317 898	2,206 508 135	-0,615 769 712 2
47	5,304 094 656	-4,842 869 034	2,152 386 237	-0,587 014 428 4
48	5,404 714 339	-4,829 744 728	2,099 889 012	-0,559 970 403 3
49	5,502 981 872	-4,815 109 138	2,048 982 612	-0,534 517 203 1
50	5,598 964 115	-4,799 112 098	1,999 630 041	-0,510 543 840 2

N	C_{60}	C_{61}	C_{62}	C_{63}
51	5,692 726 671	- 4,781 890 403	1,951 792 001	- 0,487 948 000 3
52	5,784 333 767	- 4,763 568 985	1,905 427 594	- 0,466 635 329 1
53	5,873 848 142	- 4,744 261 961	1,860 494 887	- 0,446 518 772 8
54	5,961 330 988	- 4,724 073 614	1,816 951 390	- 0,427 517 974 1
55	6,046 841 882	- 4,703 099 242	1,774 754 431	- 0,409 558 714 8
56	6,130 438 777	- 4,681 425 975	1,733 861 472	- 0,392 572 408 8
57	6,212 177 960	- 4,659 133 470	1,694 230 353	- 0,376 495 634 0
58	6,292 114 073	- 4,636 294 580	1,655 819 493	- 0,361 269 707 6
59	6,370 300 108	- 4,612 975 940	1,618 588 049	- 0,346 840 296 2
60	6,446 787 415	- 4,589 238 499	1,582 496 034	- 0,333 157 059 8
61	6,521 625 741	- 4,565 138 018	1,547 504 413	- 0,320 173 326 8
62	6,594 863 253	- 4,540 725 519	1,513 575 173	- 0,307 845 797 9
63	6,666 546 550	- 4,516 047 663	1,480 671 365	- 0,296 134 273 0
64	6,736 720 721	- 4,491 147 147	1,448 757 144	- 0,285 001 405 4
65	6,805 429 383	- 4,466 063 032	1,417 797 788	- 0,274 412 475 1
66	6,872 714 701	- 4,440 831 038	1,387 759 699	- 0,264 335 180 8
67	6,938 617 445	- 4,415 483 829	1,358 610 409	- 0,254 739 451 7
68	7,003 177 023	- 4,390 051 268	1,330 318 566	- 0,245 597 273 7
69	7,066 431 525	- 4,364 560 648	1,302 853 925	- 0,236 882 531 8
70	7,128 417 767	- 4,339 036 901	1,276 187 324	- 0,228 570 864 0
71	7,189 171 328	- 4,313 502 797	1,250 290 666	- 0,220 639 529 2
72	7,248 726 593	- 4,287 979 111	1,225 136 889	- 0,213 067 285 0
73	7,307 116 796	- 4,262 484 798	1,200 699 943	- 0,205 834 275 9
74	7,364 374 054	- 4,237 037 127	1,176 954 757	- 0,198 921 930 8
75	7,420 529 411	- 4,211 651 828	1,153 877 213	- 0,192 312 868 9
76	7,475 612 874	- 4,186 343 210	1,131 444 111	- 0,185 990 812 7
77	7,529 653 449	- 4,161 124 275	1,109 633 140	- 0,179 940 509 2
78	7,582 679 179	- 4,136 006 825	1,088 422 849	- 0,174 147 655 8
79	7,634 717 172	- 4,111 001 554	1,067 792 612	- 0,168 598 833 4
80	7,685 793 649	- 4,086 118 142	1,047 722 601	- 0,163 281 444 2
81	7,735 933 963	- 4,061 365 331	1,028 193 755	- 0,158 183 654 6
82	7,785 162 634	- 4,036 750 995	1,009 187 749	- 0,153 294 341 6
83	7,833 503 382	- 4,012 282 220	0,990 686 967 9	- 0,148 603 045 2
84	7,880 979 159	- 3,987 965 357	0,972 674 477 4	- 0,144 099 922 6
85	7,927 612 174	- 3,963 806 087	0,955 133 996 9	- 0,139 775 706 9
86	7,973 423 909	- 3,939 809 461	0,938 049 871 7	- 0,135 621 668 2
87	8,018 435 176	- 3,915 979 969	0,921 407 051 6	- 0,131 629 578 8
88	8,062 666 103	- 3,892 321 567	0,905 191 062 1	- 0,127 791 679 4
89	8,106 136 191	- 3,868 837 728	0,889 387 983 4	- 0,124 100 648 8
90	8,148 864 315	- 3,845 531 475	0,873 984 426 1	- 0,120 549 576 0
91	8,190 868 771	- 3,822 405 427	0,858 967 511 6	- 0,117 131 933 4
92	8,232 167 271	- 3,799 461 817	0,844 324 848 3	- 0,113 841 552 6
93	8,272 776 972	- 3,776 702 531	0,830 044 512 3	- 0,110 672 601 6
94	8,312 714 516	- 3,754 129 136	0,816 115 029 6	- 0,107 619 564 3
95	8,351 996 021	- 3,731 742 903	0,802 525 355 5	- 0,104 677 220 3
96	8,390 637 115	- 3,709 544 830	0,789 264 857 4	- 0,101 840 626 8
97	8,428 652 946	- 3,687 535 664	0,776 323 297 6	- 0,099 105 101 83
98	8,466 058 210	- 3,665 715 926	0,763 690 817 9	- 0,096 466 208 58
99	8,502 867 159	- 3,644 085 925	0,751 357 922 8	- 0,093 919 740 34
100	8,539 093 617	- 3,622 645 777	0,739 315 464 7	- 0,091 461 706 97

N	C_{64}	C_{65}	C_{66}	C_{70}
7	1,590 909 091	- 3,5	7,0	
8	2,727 272 727	- 4,0	4,0	- 0,002 331 002 331
9	3,272 727 273	- 3,6	2,4	- 0,010 489 510 49
10	3,409 090 909	- 3,0	1,5	- 0,027 766 351 30
11	3,308 823 529	- 2,426 470 588	0,970 588 235 2	- 0,056 561 085 97
12	3,088 235 294	- 1,941 176 470	0,647 058 823 5	- 0,098 237 675 65
13	2,817 337 461	- 1,549 535 604	0,442 724 458 2	- 0,153 250 774 0
14	2,535 603 715	- 1,239 628 483	0,309 907 120 7	- 0,221 362 229 1
15	2,263 931 888	- 0,996 130 030 9	0,221 362 229 1	- 0,301 857 585 2
16	2,012 383 901	- 0,804 953 560 5	0,160 990 712 1	- 0,393 727 285 0
17	1,784 897 025	- 0,654 462 242 3	0,118 993 135 0	- 0,495 804 729 2
18	1,582 067 818	- 0,535 469 107 6	0,089 244 851 26	- 0,606 864 988 6
19	1,402 766 798	- 0,440 869 565 2	0,067 826 086 95	- 0,725 691 699 5
20	1,245 059 289	- 0,365 217 391 3	0,052 173 913 05	- 0,851 119 894 6
21	1,106 719 368	- 0,304 347 826 1	0,040 579 710 14	- 0,982 061 416 9
22	0,985 507 246 5	- 0,255 072 463 8	0,031 884 057 98	- 1,117 518 164
23	0,879 310 344 9	- 0,214 942 528 8	0,025 287 356 33	- 1,256 587 091
24	0,786 206 896 7	- 0,182 068 965 5	0,020 229 885 06	- 1,398 459 827
25	0,704 486 466 6	- 0,154 987 022 6	0,016 314 423 44	- 1,542 418 927
26	0,632 647 385 8	- 0,132 554 690 4	0,013 255 469 04	- 1,687 832 159
27	0,569 382 647 3	- 0,113 876 529 5	0,010 845 383 76	- 1,834 145 783
28	0,513 560 819 2	- 0,098 246 417 59	0,008 931 492 508	- 1,980 877 446
29	0,464 205 623 6	- 0,085 104 364 32	0,007 400 379 507	- 2,127 609 108
30	0,420 476 108 3	- 0,074 003 795 07	0,006 166 982 922	- 2,273 980 250
31	0,381 648 359 6	- 0,064 586 645 47	0,005 166 931 638	- 2,419 681 502
32	0,347 099 139 7	- 0,056 564 304 24	0,004 351 100 326	- 2,564 448 772
33	0,316 291 523 7	- 0,049 702 953 72	0,003 681 700 276	- 2,708 057 903
34	0,288 762 446 6	- 0,043 812 233 28	0,003 129 445 235	- 2,850 319 857
35	0,264 111 993 9	- 0,038 736 425 77	0,002 671 477 639	- 2,991 076 393
36	0,241 994 240 7	- 0,034 347 569 64	0,002 289 837 976	- 3,130 196 225
37	0,222 109 442 6	- 0,030 540 048 36	0,001 970 325 701	- 3,267 571 608
38	0,204 197 390 8	- 0,027 226 318 78	0,001 701 644 923	- 3,403 115 319
39	0,188 031 764 0	- 0,024 333 522 41	0,001 474 758 934	- 3,536 757 996
40	0,173 415 329 1	- 0,021 800 784 23	0,001 282 399 072	- 3,668 445 794
41	0,160 175 861 0	- 0,019 577 049 67	0,001 118 688 553	- 3,798 138 322
42	0,148 162 671 4	- 0,017 619 344 71	0,000 978 852 483 8	- 3,925 806 837
43	0,137 243 653 7	- 0,015 891 370 42	0,000 858 992 995 9	- 4,051 432 657
44	0,127 302 762 0	- 0,014 362 362 89	0,000 755 913 836 4	- 4,175 005 764
45	0,118 237 859 4	- 0,013 006 164 54	0,000 666 982 796 8	- 4,296 523 605
46	0,109 958 877 2	- 0,011 800 464 87	0,000 590 023 243 4	- 4,415 990 001
47	0,102 386 237 5	- 0,010 726 177 26	0,000 523 228 159 1	- 4,533 414 237
48	0,095 449 500 56	- 0,009 766 925 638	0,000 465 091 697 1	- 4,648 810 235
49	0,089 086 200 52	- 0,008 908 620 052	0,000 414 354 421 0	- 4,762 195 851
50	0,083 240 843 52	- 0,008 139 104 699	0,000 369 959 304 5	- 4,873 592 245

N	C_{64}	C_{65}	C_{66}	C_{70}
51	0,077 864 042 61	-0,007 447 864 945	0,000 331 016 219 8	-4,983 023 346
52	0,072 911 770 17	-0,006 825 782 740	0,000 296 773 162 6	-5,090 515 375
53	0,068 344 710 13	-0,006 264 931 761	0,000 266 592 840 9	-5,196 096 435
54	0,064 127 696 11	-0,005 758 405 365	0,000 239 933 556 9	-5,299 796 149
55	0,060 229 222 77	-0,005 300 171 603	0,000 216 333 534 8	-5,401 645 355
56	0,056 621 020 50	-0,004 884 950 788	0,000 195 398 031 5	-5,501 675 824
57	0,053 277 684 05	-0,004 508 111 727	0,000 176 788 695 2	-5,599 920 035
58	0,050 176 348 27	-0,004 165 583 630	0,000 160 214 755 0	-5,696 410 966
59	0,047 296 404 03	-0,003 853 781 069	0,000 145 425 700 7	-5,791 181 915
60	0,044 619 249 08	-0,003 569 539 927	0,000 132 205 182 5	-5,884 266 354
61	0,042 128 069 32	-0,003 310 062 589	0,000 120 365 912 3	-5,975 697 797
62	0,039 807 646 28	-0,003 072 870 941	0,000 109 745 390 7	-6,065 509 681
63	0,037 644 187 24	-0,002 855 765 929	0,000 100 202 313 3	-6,153 735 274
64	0,035 625 175 68	-0,002 656 792 763	0,000 091 613 543 54	-6,240 407 602
65	0,033 739 238 74	-0,002 474 210 841	0,000 083 871 553 94	-6,325 559 363
66	0,031 976 029 94	-0,002 306 467 733	0,000 076 882 257 78	-6,409 222 878
67	0,030 326 125 20	-0,002 152 176 627	0,000 070 563 168 10	-6,491 430 043
68	0,028 780 930 52	-0,002 010 096 735	0,000 064 841 830 15	-6,572 212 283
69	0,027 332 599 82	-0,001 879 116 238	0,000 059 654 483 73	-6,651 600 525
70	0,025 973 961 82	-0,001 758 237 415	0,000 054 944 919 23	-6,729 625 164
71	0,024 698 454 77	-0,001 646 563 651	0,000 050 663 496 95	-6,806 316 049
72	0,023 500 068 20	-0,001 543 288 061	0,000 046 766 304 88	-6,881 702 461
73	0,022 373 290 86	-0,001 447 683 526	0,000 043 214 433 63	-6,955 813 102
74	0,021 313 064 02	-0,001 359 093 937	0,000 039 973 351 10	-7,028 676 091
75	0,020 314 739 67	-0,001 276 926 493	0,000 037 012 362 13	-7,100 318 949
76	0,019 374 042 99	-0,001 200 644 918	0,000 034 304 140 51	-7,170 768 605
77	0,018 487 038 61	-0,001 129 763 471	0,000 031 824 323 12	-7,240 051 394
78	0,017 650 100 25	-0,001 063 841 659	0,000 029 551 157 19	-7,308 193 054
79	0,016 859 883 34	-0,001 002 479 550	0,000 027 465 193 15	-7,375 218 737
80	0,016 113 300 42	-0,000 945 313 624 6	0,000 025 549 016 88	-7,441 153 002
81	0,015 407 498 82	-0,000 892 013 089 6	0,000 023 787 015 72	-7,506 019 842
82	0,014 739 840 54	-0,000 842 276 602 2	0,000 022 165 173 74	-7,569 842 663
83	0,014 107 884 04	-0,000 795 829 355 9	0,000 020 670 892 36	-7,632 644 320
84	0,013 509 367 74	-0,000 752 420 481 8	0,000 019 292 832 87	-7,694 447 109
85	0,012 942 195 08	-0,000 711 820 729 4	0,000 018 020 777 96	-7,755 272 780
86	0,012 404 420 87	-0,000 673 820 393 0	0,000 016 845 509 83	-7,815 142 541
87	0,011 894 239 05	-0,000 638 227 461 1	0,000 015 758 702 74	-7,874 077 094
88	0,011 409 971 37	-0,000 604 865 952 2	0,000 014 752 828 10	-7,932 096 611
89	0,010 950 057 25	-0,000 573 574 427 4	0,000 013 821 070 54	-7,989 220 756
90	0,010 513 044 42	-0,000 544 204 652 3	0,000 012 957 253 63	-8,045 468 731
91	0,010 097 580 47	-0,000 516 620 395 9	0,000 012 155 774 02	-8,100 859 228
92	0,009 702 405 050	-0,000 490 696 347 4	0,000 011 411 542 96	-8,155 410 463
93	0,009 326 342 835	-0,000 466 317 141 7	0,000 010 719 934 29	-8,209 140 228
94	0,008 968 297 029	-0,000 443 376 482 3	0,000 010 076 738 23	-8,262 065 838
95	0,008 627 243 429	-0,000 421 776 345 4	0,000 009 478 120 122	-8,314 204 185
96	0,008 302 225 008	-0,000 401 426 264 1	0,000 008 920 583 647	-8,365 571 731
97	0,007 992 346 922	-0,000 382 242 678 9	0,000 008 400 937 997	-8,416 184 525
98	0,007 696 771 961	-0,000 364 148 350 9	0,000 007 916 268 497	-8,466 058 210
99	0,007 414 716 343	-0,000 347 071 828 8	0,000 007 463 910 297	-8,515 208 040
100	0,007 145 445 857	-0,000 330 946 966 0	0,000 007 041 424 809	-8,563 648 883

N	C_{71}	C_{72}	C_{73}	C_{74}
8	0,018 648 018 65	- 0,083 916 083 92	0,279 720 279 7	- 0,769 230 769 2
9	0,073 426 573 42	- 0,283 216 783 2	0,786 713 286 6	- 1,730 769 231
10	0,172 768 408 1	- 0,583 093 377 2	1,388 317 565	- 2,545 248 869
11	0,316 742 081 4	- 0,950 226 244 3	1,979 638 009	- 3,110 859 728
12	0,500 119 076 0	- 1,350 321 505	2,500 595 380	- 3,438 318 648
13	0,715 170 278 6	- 1,755 417 957	2,925 696 594	- 3,575 851 393
14	0,953 560 371 5	- 2,145 510 836	3,250 773 994	- 3,575 851 393
15	1,207 430 341	- 2,507 739 938	3,482 972 136	- 3,482 972 136
16	1,469 915 197	- 2,834 836 452	3,634 405 707	- 3,331 538 565
17	1,735 316 552	- 3,123 569 794	3,718 535 469	- 3,146 453 089
18	1,999 084 668	- 3,373 455 378	3,748 283 753	- 2,945 080 092
19	2,257 707 510	- 3,585 770 751	3,735 177 865	- 2,739 130 434
20	2,508 563 900	- 3,762 845 850	3,689 064 559	- 2,536 231 884
21	2,749 771 967	- 3,907 570 690	3,618 121 010	- 2,341 137 124
22	2,980 048 437	- 4,023 065 390	3,529 004 729	- 2,156 614 001
23	3,198 585 323	- 4,112 466 843	3,427 055 703	- 1,984 084 881
24	3,404 945 666	- 4,178 796 954	3,316 505 519	- 1,824 078 035
25	3,598 977 496	- 4,224 886 626	3,200 671 687	- 1,676 542 312
26	3,780 744 036	- 4,253 337 041	3,082 128 290	- 1,541 064 145
27	3,950 467 841	- 4,266 505 268	2,962 850 880	- 1,417 015 638
28	4,108 486 555	- 4,266 505 268	2,844 336 845	- 1,303 654 388
29	4,255 218 217	- 4,255 218 217	2,727 703 985	- 1,200 189 753
30	4,391 134 276	- 4,234 308 052	2,613 770 403	- 1,105 825 940
31	4,516 738 804	- 4,205 239 577	2,503 118 796	- 1,019 789 139
32	4,632 552 620	- 4,169 297 358	2,396 147 907	- 0,941 343 820 6
33	4,739 101 330	- 4,127 604 384	2,293 113 547	- 0,869 801 690 2
34	4,836 906 423	- 4,081 139 795	2,194 161 180	- 0,804 525 766 0
35	4,926 478 765	- 4,030 755 353	2,099 351 746	- 0,744 931 264 8
36	5,008 313 960	- 3,977 190 498	2,008 682 070	- 0,690 484 461 4
37	5,082 889 168	- 3,921 085 930	1,922 100 946	- 0,640 700 315 3
38	5,150 661 023	- 3,862 995 767	1,839 521 794	- 0,595 139 404 0
39	5,212 064 416	- 3,803 398 357	1,760 832 573	- 0,553 404 522 9
40	5,267 511 909	- 3,742 705 830	1,685 903 527	- 0,515 137 188 9
41	5,317 393 651	- 3,681 272 528	1,614 593 214	- 0,480 014 198 8
42	5,362 077 631	- 3,619 402 401	1,546 753 163	- 0,447 744 336 6
43	5,401 910 209	- 3,557 355 503	1,482 231 460	- 0,418 065 283 5
44	5,437 216 809	- 3,495 353 663	1,420 875 473	- 0,390 740 755 0
45	5,468 302 770	- 3,433 585 460	1,362 533 913	- 0,365 557 879 0
46	5,495 454 224	- 3,372 210 546	1,307 058 351	- 0,342 324 806 3
47	5,518 939 071	- 3,311 363 442	1,254 304 334	- 0,320 868 550 6
48	5,539 007 940	- 3,251 156 834	1,204 132 161	- 0,301 033 040 2
49	5,555 895 159	- 3,191 684 453	1,156 407 411	- 0,282 677 367 0
50	5,569 819 708	- 3,133 023 586	1,111 001 272	- 0,265 674 217 1

N	C_{71}	C_{72}	C_{73}	C_{74}
51	5,580 986 148	- 3,075 237 265	1,067 790 717	- 0,249 908 465 7
52	5,589 585 510	- 3,018 376 176	1,026 658 563	- 0,235 275 920 7
53	5,595 796 161	- 2,962 480 320	0,987 493 440 1	- 0,221 682 200 8
54	5,599 784 610	- 2,907 580 471	0,950 189 696 3	- 0,209 041 733 2
55	5,601 706 294	- 2,853 699 433	0,914 647 254 1	- 0,197 276 858 7
56	5,601 706 294	- 2,800 853 147	0,880 771 429 8	- 0,186 317 033 2
57	5,599 920 035	- 2,749 051 653	0,848 472 732 5	- 0,176 098 114 3
58	5,596 473 931	- 2,698 299 931	0,817 666 645 8	- 0,166 561 724 1
59	5,591 485 987	- 2,648 598 625	0,788 273 400 4	- 0,157 654 680 1
60	5,585 066 369	- 2,599 944 689	0,760 217 745 4	- 0,149 328 485 7
61	5,577 317 944	- 2,552 331 941	0,733 428 718 5	- 0,141 538 875 5
62	5,568 336 756	- 2,505 751 540	0,707 839 418 1	- 0,134 245 406 9
63	5,558 212 506	- 2,460 192 421	0,683 386 783 5	- 0,127 411 095 2
64	5,547 028 980	- 2,415 641 652	0,660 011 380 4	- 0,121 002 086 4
65	5,534 864 443	- 2,372 084 761	0,637 657 193 9	- 0,114 987 362 8
66	5,521 792 018	- 2,329 506 007	0,616 271 430 5	- 0,109 338 479 6
67	5,507 880 036	- 2,287 888 630	0,595 804 330 8	- 0,104 029 327 6
68	5,493 192 356	- 2,247 215 055	0,576 208 988 4	- 0,099 035 919 88
69	5,477 788 667	- 2,207 467 075	0,557 441 180 5	- 0,094 336 199 78
70	5,461 724 771	- 2,168 626 012	0,539 459 207 0	- 0,089 909 867 83
71	5,445 052 839	- 2,130 672 850	0,522 223 737 8	- 0,085 738 225 60
.72	5,427 821 659	- 2,093 588 354	0,505 697 670 1	- 0,081 804 034 87
73	5,410 076 857	- 2,057 353 171	0,489 845 993 1	- 0,078 091 390 31
74	5,391 861 111	- 2,021 947 917	0,474 635 661 2	- 0,074 585 603 90
75	5,373 214 340	- 1,987 353 249	0,460 035 474 3	- 0,071 273 101 65
76	5,354 173 892	- 1,953 549 933	0,446 015 966 5	- 0,068 141 328 22
77	5,334 774 712	- 1,920 518 896	0,432 549 300 9	- 0,065 178 661 79
78	5,315 049 494	- 1,888 241 268	0,419 609 170 6	- 0,062 374 336 17
79	5,295 028 837	- 1,856 698 423	0,407 170 706 9	- 0,059 718 370 34
80	5,274 741 368	- 1,825 872 012	0,395 210 392 2	- 0,057 201 504 14
81	5,254 213 889	- 1,795 743 987	0,383 705 980 2	- 0,054 815 140 03
82	5,233 471 471	- 1,766 296 621	0,372 636 418 0	- 0,052 551 289 72
83	5,212 537 584	- 1,737 512 528	0,361 981 776 7	- 0,050 402 525 87
84	5,191 434 194	- 1,709 374 674	0,351 723 183 9	- 0,048 361 937 79
85	5,170 181 853	- 1,681 866 386	0,341 842 761 4	- 0,046 423 091 05
86	5,148 799 792	- 1,654 971 362	0,332 323 566 6	- 0,044 579 990 64
87	5,127 306 014	- 1,628 673 675	0,323 149 538 7	- 0,042 827 047 30
88	5,105 717 359	- 1,602 957 776	0,314 305 446 2	- 0,041 159 046 52
89	5,084 049 580	- 1,577 808 490	0,305 776 839 2	- 0,039 571 120 37
90	5,062 317 404	- 1,553 211 022	0,297 550 004 2	- 0,038 058 721 46
91	5,040 534 631	- 1,529 150 955	0,289 611 923 4	- 0,036 617 599 51
92	5,018 714 131	- 1,505 614 239	0,281 950 232 1	- 0,035 243 779 01
93	4,996 867 965	- 1,482 587 198	0,274 553 184 9	- 0,033 933 539 70
94	4,975 007 387	- 1,460 056 516	0,267 409 618 2	- 0,032 683 397 79
95	4,953 142 919	- 1,438 009 235	0,260 508 919 3	- 0,031 490 089 15
96	4,931 284 389	- 1,416 432 750	0,253 840 994 6	- 0,030 350 553 71
97	4,909 440 973	- 1,395 314 803	0,247 396 241 6	- 0,029 261 921 05
98	4,887 621 234	- 1,374 643 472	0,241 165 521 5	- 0,028 221 497 19
99	4,865 833 166	- 1,354 407 170	0,235 140 133 7	- 0,027 226 752 32
100	4,844 084 216	- 1,334 594 631	0,229 311 792 3	- 0,026 275 309 53

N	C_{75}	C_{76}	C_{77}
8	1,846 153 846	- 4,0	8,0
9	3,115 384 615	- 4,5	4,5
10	3,665 158 371	- 3,970 588 235	2,647 058 824
11	3,733 031 674	- 3,235 294 118	1,617 647 059
12	3,536 556 323	- 2,554 179 567	1,021 671 827
13	3,218 266 254	- 1,992 260 062	0,664 086 687 3
14	2,860 681 114	- 1,549 535 604	0,442 724 458 2
15	2,507 739 938	- 1,207 430 341	0,301 857 585 2
16	2,180 643 424	- 0,944 945 483 9	0,209 987 885 3
17	1,887 871 853	- 0,743 707 093 8	0,148 741 418 8
18	1,631 121 282	- 0,589 016 018 4	0,107 093 821 5
19	1,408 695 652	- 0,469 565 217 3	0,078 260 869 56
20	1,217 391 304	- 0,376 811 594 2	0,057 971 014 50
21	1,053 511 706	- 0,304 347 826 1	0,043 478 260 87
22	0,913 389 459 1	-0,247 376 311 9	0,032 983 508 25
23	0,793 633 952 2	- 0,202 298 850 6	0,025 287 356 32
24	0,691 229 571 3	- 0,166 407 119 0	0,019 577 308 12
25	0,603 555 232 3	- 0,137 652 947 7	0,015 294 771 97
26	0,528 364 849 8	- 0,114 479 050 8	0,012 050 426 40
27	0,463 750 572 6	- 0,095 694 562 60	0,009 569 456 260
28	0,408 100 503 9	- 0,080 383 432 59	0,007 655 565 009
29	0,360 056 926 0	-0,067 836 812 15	0,006 166 982 923
30	0,318 477 870 6	- 0,057 502 948 86	0,005 000 256 422
31	0,282 403 146 2	-0,048 949 878 67	0,004 079 156 556
32	0,251 025 018 8	- 0,041 837 503 14	0,003 347 000 251
33	0,223 663 291 8	-0,035 896 577 69	0,002 761 275 207
34	0,199 744 328 1	- 0,030 912 812 68	0,002 289 837 976
35	0,178 783 503 6	-0,026 714 776 39	0,001 908 198 314
36	0,160 370 584 6	-0,023 164 640 00	0,001 597 561 379
37	0,144 157 570 9	-0,020 151 058 30	0,001 343 403 887
38	0,129 848 597 2	-0,017 583 664 21	0,001 134 429 949
39	0,117 191 546 0	-0,015 388 788 87	0,000 961 799 304 5
40	0,105 971 078 8	-0,013 506 117 89	0,000 818 552 599 6
41	0,096 002 839 75	-0,011 886 065 87	0,000 699 180 345 5
42	0,087 128 627 67	-0,010 487 705 18	0,000 599 297 439 0
43	0,079 212 369 51	-0,009 277 124 357	0,000 515 395 797 6
44	0,072 136 754 77	-0,008 226 121 158	0,000 444 655 197 7
45	0,065 800 418 23	-0,007 311 157 581	0,000 384 797 767 4
46	0,060 115 575 74	-0,006 512 520 705	0,000 333 975 420 8
47	0,055 006 037 25	-0,005 813 646 213	0,000 290 682 310 7
48	0,050 405 532 31	-0,005 200 570 794	0,000 253 686 380 2
49	0,046 256 296 42	-0,004 661 487 236	0,000 221 975 582 7
50	0,042 507 874 74	-0,004 186 381 603	0,000 194 715 423 4

N	C_{75}	C_{76}	C_{77}
51	0,039 116 107 67	-0,003 766 736 295	0,000 171 215 286 1
52	0,036 042 268 70	-0,003 395 286 182	0,000 150 901 608 1
53	0,033 252 330 13	-0,003 065 817 671	0,000 133 296 420 5
54	0,030 716 336 31	-0,002 773 002 583	0,000 118 000 109 9
55	0,028 407 867 66	-0,002 512 260 405	0,000 104 677 516 9
56	0,026 303 581 16	-0,002 279 643 701	0,000 093 046 681 66
57	0,024 382 815 83	-0,002 071 742 521	0,000 082 869 700 85
58	0,022 627 253 09	-0,001 885 604 424	0,000 073 945 271 54
59	0,021 020 624 01	-0,001 718 667 372	0,000 066 102 591 23
60	0,019 548 456 31	-0,001 568 703 284	0,000 059 196 350 34
61	0,018 197 855 42	-0,001 433 770 427	0,000 053 102 608 42
62	0,016 957 314 55	-0,001 312 173 150	0,000 047 715 387 27
63	0,015 816 549 75	-0,001 202 427 759	0,000 042 943 848 54
64	0,014 766 356 31	-0,001 103 233 517	0,000 038 709 947 97
65	0,013 798 483 54	-0,001 013 447 944	0,000 034 946 480 82
66	0,012 905 525 46	-0,000 932 065 727 8	0,000 031 595 448 40
67	0,012 080 825 14	-0,000 858 200 693 1	0,000 028 606 689 77
68	0,011 318 390 84	-0,000 791 070 327 8	0,000 025 936 732 06
69	0,010 612 822 48	-0,000 729 982 498 3	0,000 023 547 822 53
70	0,009 959 246 898	-0,000 674 324 008 7	0,000 021 407 111 39
71	0,009 353 260 975	-0,000 623 550 731 7	0,000 019 485 960 36
72	0,008 790 881 359	-0,000 577 179 079 1	0,000 017 759 356 28
73	0,008 268 500 140	-0,000 534 778 516 0	0,000 016 205 412 61
74	0,007 782 845 624	-0,000 495 965 652 5	0,000 014 804 944 85
75	0,007 330 947 598	-0,000 460 397 675 3	0,000 013 541 108 10
76	0,006 910 106 524	-0,000 427 768 499 1	0,000 012 399 086 93
77	0,006 517 866 179	-0,000 397 804 039 1	0,000 011 365 829 69
78	0,006 151 989 321	-0,000 370 258 616 5	0,000 010 429 820 18
79	0,005 810 436 033	-0,000 344 911 728 0	0,000 009 580 881 333
80	0,005 491 344 397	-0,000 321 565 212 5	0,000 008 810 005 821
81	0,005 193 013 266	-0,000 300 040 766 5	0,000 008 109 209 905
82	0,004 913 886 831	-0,000 280 177 757 9	0,000 007 471 406 878
83	0,004 652 540 849	-0,000 261 831 303 2	0,000 006 890 297 453
84	0,004 407 670 279	-0,000 244 870 571 1	0,000 006 360 274 573
85	0,004 178 078 195	-0,000 229 177 284 9	0,000 005 876 340 639
86	0,003 962 665 835	-0,000 214 644 399 4	0,000 005 434 035 427
87	0,003 760 423 665	-0,000 201 174 928 6	0,000 005 029 373 215
88	0,003 570 423 313	-0,000 188 680 906 8	0,000 004 658 787 822
89	0,003 391 810 318	-0,000 177 082 466 4	0,000 004 319 084 546
90	0,003 223 797 583	-0,000 166 307 018 2	0,000 004 007 398 028
91	0,003 065 659 494	-0,000 156 288 523 2	0,000 003 721 155 314
92	0,002 916 726 539	-0,000 146 966 841 1	0,000 003 458 043 320
93	0,002 776 380 521	-0,000 138 287 152 4	0,000 003 215 980 288
94	0,002 644 050 158	-0,000 130 199 439 6	0,000 002 993 090 565
95	0,002 519 207 132	-0,000 122 658 025 1	0,000 002 787 682 389
96	0,002 401 362 491	-0,000 115 621 157 0	0,000 002 598 228 247
97	0,002 290 063 387	-0,000 109 050 637 5	0,000 002 423 347 499
98	0,002 184 890 105	-0,000 102 911 490 5	0,000 002 261 790 999
99	0,002 085 453 369	-0,000 097 171 662 36	0,000 002 112 427 443
100	0,001 991 391 880	-0,000 091 801 753 35	0,000 001 974 231 255

TABLE IV

The values of the binomial coefficients.

x	$\binom{x}{2}$	$\binom{x}{3}$	$\binom{x}{4}$	$\binom{x}{5}$	$\binom{x}{6}$	x
4	6	4	1			4
5	10	10	5	1		5
6	15	20	15	6	1	6
7	21	35	35	21	7	7
8	28	56	70	56	28	8
9	36	84	126	126	84	9
10	45	120	210	252	210	10
11	55	165	330	462	462	11
12	66	220	495	792	924	12
13	78	286	715	1 287	1 716	13
14	91	364	1 001	2 002	3 003	14
15	105	455	1 365	3 003	5 005	15
16	120	560	1 820	4 368	8 008	16
17	136	680	2 380	6 188	12 376	17
18	153	816	3 060	8 568	18 564	18
19	171	969	3 876	11 628	27 132	19
20	190	1 140	4 845	15 504	38 760	20
21	210	1 330	5 985	20 349	54 264	21
22	231	1 540	7 315	26 334	74 613	22
23	253	1 771	8 855	33 649	100 947	23
24	276	2 024	10 626	42 504	134 596	24
25	300	2 300	12 650	53 130	177 100	25
26	325	2 600	14 950	65 780	230 230	26
27	351	2 925	17 550	80 730	296 010	27
28	378	3 276	20 475	98 280	376 740	28
29	406	3 654	23 751	118 755	475 020	29
30	435	4 060	27 405	142 506	593 775	30
31	465	4 495	31 465	169 911	736 281	31
32	496	4 960	35 960	201 376	906 192	32
33	528	5 456	40 920	237 336	1 107 568	33
34	561	5 984	46 376	278 256	1 344 904	34
35	595	6 545	52 360	324 632	1 623 160	35
36	630	7 140	58 905	376 992	1 947 792	36
37	666	7 770	66 045	435 897	2 324 784	37
38	703	8 436	73 815	501 942	2 760 681	38
39	741	9 139	82 251	575 757	3 262 623	39
40	780	9 880	91 390	658 008	3 838 380	40
41	820	10 660	101 270	749 398	4 496 388	41
42	861	11 480	111 930	850 668	5 245 786	42
43	903	12 341	123 410	962 598	6 096 454	43
44	946	13 244	135 751	1 086 008	7 059 052	44
45	990	14 190	148 995	1 221 759	8 145 060	45
46	1 035	15 180	163 185	1 370 754	9 366 819	46
47	1 081	16 215	178 365	1 533 939	10 737 573	47
48	1 128	17 296	194 580	1 712 304	12 271 512	48
49	1 176	18 424	211 876	1 906 884	13 983 816	49
50	1 225	19 600	230 300	2 118 760	15 890 700	50
51	1 275	20 825	249 900	2 349 060	18 009 460	51
52	1 326	22 100	270 725	2 598 960	20 358 520	52
53	1 378	23 426	292 825	2 869 685	22 957 480	53
54	1 431	24 804	316 251	3 162 510	25 827 165	54
55	1 485	26 235	341 055	3 478 761	28 989 675	55

x	$\binom{x}{2}$	$\binom{x}{3}$	$\binom{x}{4}$	$\binom{x}{5}$	$\binom{x}{6}$	x
56	1 540	27 720	367 290	3 819 816	32 468 436	56
57	1 596	29 260	395 010	4 187 106	36 288 252	57
58	1 653	30 856	424 270	4 582 116	40 475 358	58
59	1 711	32 509	455 126	5 006 386	45 057 474	59
60	1 770	34 220	487 635	5 461 512	50 063 860	60
61	1 830	35 990	521 855	5 949 147	55 525 372	61
62	1 891	37 820	557 845	6 471 002	61 474 519	62
63	1 953	39 711	595 665	7 028 847	67 945 521	63
64	2 016	41 664	635 376	7 624 512	74 974 368	64
65	2 080	43 680	677 040	8 259 888	82 598 880	65
66	2 145	45 760	720 720	8 936 928	90 858 768	66
67	2 211	47 905	766 480	9 657 648	99 795 696	67
68	2 278	50 116	814 385	10 424 128	109 453 344	68
69	2 346	52 394	864 501	11 238 513	119 877 472	69
70	2 415	54 740	916 895	12 103 014	131 115 985	70
71	2 485	57 155	971 635	13 019 909	143 218 999	71
72	2 556	59 640	1 028 790	13 991 544	156 238 908	72
73	2 628	62 196	1 088 430	15 020 334	170 230 452	73
74	2 701	64 824	1 150 626	16 108 764	185 250 786	74
75	2 775	67 525	1 215 450	17 259 390	201 359 550	75
76	2 850	70 300	1 282 975	18 474 840	218 618 940	76
77	2 926	73 150	1 353 275	19 757 815	237 093 780	77
78	3 003	76 076	1 426 425	21 111 090	256 851 595	78
79	3 081	79 079	1 502 501	22 537 515	277 962 685	79
80	3 160	82 160	1 581 580	24 040 016	300 500 200	80
81	3 240	85 320	1 663 740	25 621 596	324 540 216	81
82	3 321	88 560	1 749 060	27 285 336	350 161 812	82
83	3 403	91 881	1 837 620	29 034 396	377 447 148	83
84	3 486	95 284	1 929 501	30 872 016	406 481 544	84
85	3 570	98 770	2 024 785	32 801 517	437 353 560	85
86	3 655	102 340	2 123 555	34 826 302	470 155 077	86
87	3 741	105 995	2 225 895	36 949 857	504 981 379	87
88	3 828	109 736	2 331 890	39 175 752	541 931 236	88
89	3 916	113 564	2 441 626	41 507 642	581 106 988	89
90	4 005	117 480	2 555 190	43 949 268	622 614 630	90
91	4 095	121 485	2 672 670	46 504 458	666 563 898	91
92	4 186	125 580	2 794 155	49 177 128	713 068 356	92
93	4 278	129 766	2 919 735	51 971 283	762 245 484	93
94	4 371	134 044	3 049 501	54 891 018	814 216 767	94
95	4 465	138 415	3 183 545	57 940 519	869 107 785	95
96	4 560	142 880	3 321 960	61 124 064	927 048 304	96
97	4 656	147 440	3 464 840	64 446 024	988 172 368	97
98	4 753	152 096	3 612 280	67 910 864	1 052 618 392	98
99	4 851	156 849	3 764 376	71 523 144	1 120 529 256	99
100	4 950	161 700	3 921 225	75 287 520	1 192 052 400	100
101	5 050	166 650	4 082 925	79 208 745	1 267 339 920	101
102	5 151	171 700	4 249 575	83 291 670	1 346 548 665	102
103	5 253	176 851	4 421 275	87 541 245	1 429 840 335	103
104	5 356	182 104	4 598 126	91 962 520	1 517 381 580	104
105	5 460	187 460	4 780 230	96 560 646	1 609 344 100	105
106	5 565	192 920	4 967 690	101 340 876	1 705 904 746	106
107	5 671	198 485	5 160 610	106 308 566	1 807 245 622	107
108	5 778	204 156	5 359 095	111 469 176	1 913 554 188	108
109	5 886	209 934	5 563 251	116 828 271	2 025 023 364	109
110	5 995	215 820	5 773 185	122 391 522	2 141 851 636	110

The values of the binomial coefficients.

x	$\binom{x}{7}$	$\binom{x}{8}$	$\binom{x}{9}$	$\binom{x}{10}$	x
4	.	.	.	.	4
5	.	.	.	.	5
6	.	.	.	.	6
7	1	.	.	.	7
8	8	1	.	.	8
9	36	9	1	.	9
10	120	45	10	1	10
11	330	165	55	11	11
12	792	495	220	66	12
13	1 716	1 287	715	286	13
14	3 432	3 003	2 002	1 001	14
15	6 435	6 435	5 005	3 003	15
16	11 440	12 870	11 440	8 008	16
17	19 448	24 310	24 310	19 448	17
18	31 824	43 758	48 620	43 758	18
19	50 388	75 582	92 378	92 378	19
20	77 520	125 970	167 960	184 75[illegible]	20
21	116 280	203 490	293 930	352 716	21
22	170 544	319 770	497 420	646 646	22
23	245 157	490 314	817 190	1 144 066	23
24	346 104	735 471	1 307 504	1 961 256	24
25	480 700	1 081 575	2 042 975	3 268 760	25
26	657 800	1 562 275	3 124 550	5 311 735	26
27	888 030	2 220 075	4 686 825	8 436 285	27
28	1 184 040	3 108 105	6 906 900	13 123 110	28
29	1 560 780	4 292 145	10 015 005	20 030 010	29
30	2 035 800	5 852 925	14 307 150	30 045 015	30
31	2 629 575	7 888 725	20 160 075	44 352 165	31
32	3 365 856	10 518 300	28 048 800	64 512 240	32
33	4 272 048	13 884 156	38 567 100	92 561 040	33
34	5 379 616	18 156 204	52 451 256	131 128 140	34
35	6 724 520	23 535 820	70 607 460	183 579 396	35
36	8 347 680	30 260 340	94 143 280	254 186 856	36
37	10 295 472	38 608 020	124 403 620	348 330 136	37
38	12 620 256	48 903 492	163 011 640	472 733 756	38
39	15 380 937	61 523 748	211 915 132	635 745 396	39
40	18 643 560	76 904 685	273 438 880	847 660 528	40
41	22 481 940	95 548 245	350 343 565	1 121 099 408	41
42	26 978 328	118 030 185	445 891 810	1 471 442 973	42
43	32 224 114	145 008 513	563 921 995	1 917 334 783	43
44	38 320 568	177 232 627	708 930 508	2 481 256 778	44
45	45 379 620	215 553 195	886 163 135	3 190 187 286	45
46	53 524 680	260 932 815	1 101 716 330	4 076 350 421	46
47	62 891 499	314 457 495	1 362 649 145	5 178 066 751	47
48	73 629 072	377 348 994	1 677 106 640	6 540 715 896	48
49	85 900 584	450 978 066	2 054 455 634	8 217 822 536	49
50	99 884 400	536 878 650	2 505 433 700	10 272 278 170	50
51	115 775 100	636 763 050	3 042 312 350	12 777 711 870	51
52	133 784 560	752 538 150	3 679 075 400	15 820 024 220	52
53	154 143 080	886 322 710	4 431 613 550	19 499 099 620	53
54	177 100 560	1 040 465 790	5 317 936 260	23 930 713 170	54
55	202 927 725	1 217 566 350	6 358 402 050	29 248 649 430	55

x	$\binom{x}{7}$	$\binom{x}{8}$	$\binom{x}{9}$	$\binom{x}{10}$	x
56	231 917 400	1 420 494 075	7 575 968 400	35 607 051 480	56
57	264 385 836	1 652 411 475	8 996 462 475	43 183 019 880	57
58	300 674 088	1 916 797 311	10 648 873 950	52 179 482 355	58
59	341 149 446	2 217 471 399	12 565 671 261	62 828 356 305	59
60	386 206 920	2 558 620 845	14 783 142 660	75 394 027 566	60
61	436 270 780	2 944 827 765	17 341 763 505	90 177 170 226	61
62	491 796 152	3 381 098 545	20 286 591 270	107 518 933 731	62
63	553 270 671	3 872 894 697	23 667 689 815	127 805 525 001	63
64	621 216 192	4 426 165 368	27 540 584 512	151 473 214 816	64
65	696 190 560	5 047 381 560	31 966 749 880	179 013 799 328	65
66	778 789 440	5 743 572 120	37 014 131 440	210 980 549 208	66
67	869 648 208	6 522 361 560	42 757 703 560	247 994 680 648	67
68	969 443 904	7 392 009 768	49 280 065 120	290 752 384 208	68
69	1 078 897 248	8 361 453 672	56 672 074 888	340 032 449 328	69
70	1 198 774 720	9 440 350 920	65 033 528 560	396 704 524 216	70
71	1 329 890 705	10 639 125 640	74 473 879 480	461 738 052 776	71
72	1 473 109 704	11 969 016 345	85 113 005 120	536 211 932 256	72
73	1 629 348 612	13 442 126 049	97 082 021 465	621 324 937 376	73
74	1 799 579 064	15 071 474 661	110 524 147 514	718 406 958 841	74
75	1 984 829 850	16 871 053 725	125 595 622 175	828 931 106 355	75
76	2 186 189 400	18 855 883 575	142 466 675 900	954 526 728 530	76
77	2 404 808 340	21 042 072 975	161 322 559 475	1 096 993 404 430	77
78	2 641 902 120	23 446 881 315	182 364 632 450	1 258 315 963 905	78
79	2 898 753 715	26 088 783 435	205 811 513 765	1 440 680 596 355	79
80	3 176 716 400	28 987 537 150	231 900 297 200	1 646 492 110 120	80
81	3 477 216 600	32 164 253 550	260 887 834 350	1 878 392 407 320	81
82	3 801 756 816	35 641 470 150	293 052 087 900	2 139 280 241 670	82
83	4 151 918 628	39 443 226 966	328 693 558 050	2 432 332 329 570	83
84	4 529 365 776	43 595 145 594	368 136 785 016	2 761 025 887 620	84
85	4 935 847 320	48 124 511 370	411 731 930 610	3 129 162 672 636	85
86	5 373 200 880	53 060 358 690	459 856 441 980	3 540 894 603 246	86
87	5 843 355 957	58 433 559 570	512 916 800 670	4 000 751 045 226	87
88	6 348 337 336	64 276 915 527	571 350 360 240	4 513 667 845 896	88
89	6 890 268 572	70 625 252 863	635 627 275 767	5 085 018 206 136	89
90	7 471 375 560	77 515 521 435	706 252 528 630	5 720 645 481 903	90
91	8 093 990 190	84 986 896 995	783 768 050 065	6 426 898 010 533	91
92	8 760 554 088	93 080 887 185	868 754 947 060	7 210 666 060 598	92
93	9 473 622 444	101 841 441 273	961 835 834 245	8 079 421 007 658	93
94	10 235 867 928	111 315 063 717	1 063 677 275 518	9 041 256 841 903	94
95	11 050 084 695	121 550 931 645	1 174 992 339 235	10 104 934 117 421	95
96	11 919 192 480	132 601 016 340	1 296 543 270 880	11 279 926 456 656	96
97	12 846 240 784	144 520 208 820	1 429 144 287 220	12 576 469 727 536	97
98	13 834 413 152	157 366 449 604	1 573 664 496 040	14 005 614 014 756	98
99	14 887 031 544	171 200 862 756	1 731 030 945 644	15 579 278 510 796	99
100	16 007 560 800	186 087 894 300	1 902 231 808 400	17 310 309 456 440	100
101	17 199 613 200	202 095 455 100	2 088 319 702 700	19 212 541 264 840	101
102	18 466 953 120	219 295 068 300	2 290 415 157 800	21 300 860 967 540	102
103	19 813 501 785	237 762 021 420	2 509 710 226 100	23 591 276 125 340	103
104	21 243 342 120	257 575 523 205	2 747 472 247 520	26 100 986 351 440	104
105	22 760 723 700	278 818 865 325	3 005 047 770 725	28 848 458 598 960	105
106	24 370 067 800	301 579 589 025	3 283 866 636 050	31 853 506 369 685	106
107	26 075 972 546	325 949 656 825	3 585 446 225 075	35 137 373 005 735	107
108	27 883 218 168	352 025 629 371	3 911 395 881 900	38 722 819 230 810	108
109	29 796 772 356	379 908 847 539	4 263 421 511 271	42 634 215 112 710	109
110	31 821 795 720	409 705 619 895	4 643 330 358 810	46 897 636 623 981	110