EDITORIAL

NOTE ON THE COMPUTATION AND MODIFICATION OF MOMENTS.

For the purpose of this note we shall deviate from the usual practice in the calculus of finite differences and define

$$\Delta u_x = u_{x+1} - M \cdot u_x ,$$

where M is a constant. It follows that this generalized Δ and the symbol E are connected by the operator relation

$$\Delta = (E - M), \text{ so that}$$

$$\Delta^n = (E - M)^n, \text{ and therefore}$$

$$(1) \quad \Delta^n u_x = u_{x+n} - \binom{n}{1} M \cdot u_{x+n-1} + \binom{n}{2} M^2 \cdot u_{x+n-2} - \binom{n}{3} M^3 \cdot u_{x+n-3} + \cdots .$$

If the n-th unmodified moments about an arbitrary origin, and about the arithmetic mean, be designated by ν_n and $\bar{\nu}_n$, respectively, the usual relation may be written

$$(2) \quad \bar{\nu}_n = \nu_n - \binom{n}{1} M \cdot \nu_{n-1} + \binom{n}{2} M^2 \cdot \nu_{n-2} - \binom{n}{3} M^3 \cdot \nu_{n-3} + \cdots ,$$

where $M = \nu_1$ equals the distance of the mean from the provisional mean. From (1) and (2) it follows that

$$(3) \qquad \bar{\nu}_n = \Delta^n \nu_0 ,$$

that is, the n-th moment about the mean is simply equal to the n-th leading difference of ν_0. Since computing machines are ideally adapted to computations of the type $(A-B\cdot C)$, formula (3) is very effective.

As an illustration, let us compute the first seven moments about the mean for the distribution of weights of 7749 adult males born in the British Isles. (See Yule's "Theory of Statistics", p. 95.) The provisional mean is taken in the 150-lb. class, and the class interval of ten pounds is taken as the unit of x.

TABLE 1

n	$\Sigma x^n f$	ν_n	$\Delta\nu_n$	$\Delta^2\nu_n$	$\Delta^3\nu_n$	$\Delta^4\nu_n$
0	7749	1.000000	.000000	4.55356	6.92736	91.3249
1	1726	.222738	4.55356	7.94161	92.8679	456.762
2	35670	4.60317	8.95586	94.6368	477.447	4471.10
3	77344	9.98116	96.6316	498.526	4577.45	
4	766026	98.8548	520.050	4688.49		
5	4200496	542.069	4804.32			
6	38164290	4925.06				

n	$\Delta^5\nu_n$	$\Delta^6\nu_n$
1	436.420	4272.15
2	4369.36	

It is very important that the provisional mean be so chosen that $M=\nu_1$ is less than unity—otherwise particular attention must be paid to the number of digits which are significant in the values of the various differences.

Let us now discuss the modification of moments.

Designating the general modified or corrected moment by μ_n, Sheppard's formula for continuous variates may be written

$$(4) \qquad \mu_n = \nu_n - \binom{n}{2}\frac{1}{12}\nu_{n-2} + \binom{n}{4}\frac{7}{240}\nu_{n-4} - \binom{n}{6}\frac{31}{1344}\nu_{n-6} + \cdots;$$

so that for moments about the mean,

$$
(5)\qquad
\begin{cases}
\bar{\mu}_2 = \bar{\nu}_2 - \frac{1}{12} \\
\bar{\mu}_3 = \bar{\nu}_3 \\
\bar{\mu}_4 = \bar{\nu}_4 - \frac{1}{2}\bar{\nu}_2 + \frac{7}{240} \\
\bar{\mu}_5 = \bar{\nu}_5 - \frac{5}{6}\bar{\nu}_3 \\
\bar{\mu}_6 = \bar{\nu}_6 - \frac{5}{4}\bar{\nu}_4 + \frac{7}{16}\bar{\nu}_2 - \frac{31}{1344}
\end{cases}
$$

In many distributions discrete variates are grouped into classes, each class containing k different values of the variable. The formula, corresponding to Sheppard's, for grouped-discrete distributions may be obtained by employing the calculus of finite differences, and was given without proof in an Editorial on page 111, Vol. 1, No. 1 of the Annals as follows,

$$
(6)\qquad
\begin{aligned}
\mu_n = \nu_n - \binom{n}{2}\frac{1-\frac{1}{k^2}}{12}\nu_{n-2} + \binom{n}{4}\frac{\left(1-\frac{1}{k^2}\right)\left(7-\frac{3}{k^2}\right)}{240}\nu_{n-4} \\
-\binom{n}{6}\frac{\left(1-\frac{1}{k^2}\right)\left(31-\frac{16}{k^2}+\frac{3}{k^4}\right)}{1344}\nu_{n-6} + \cdots\cdots.
\end{aligned}
$$

Obviously the limit of (6) as k approaches infinity is (4). So

$$
(7)\qquad
\begin{cases}
\bar{\mu}_2 = \bar{\nu}_2 - \frac{1-\frac{1}{k^2}}{12} \\
\bar{\mu}_3 = \bar{\nu}_3 \\
\bar{\mu}_4 = \bar{\nu}_4 - \frac{1-\frac{1}{k^2}}{2}\bar{\nu}_2 + \frac{\left(1-\frac{1}{k^2}\right)\left(7-\frac{3}{k^2}\right)}{240} \\
\bar{\mu}_5 = \bar{\nu}_5 - \frac{5\left(1-\frac{1}{k^2}\right)}{6}\bar{\nu}_3 \\
\bar{\mu}_6 = \bar{\nu}_6 - \frac{5\left(1-\frac{1}{k^2}\right)}{4}\bar{\nu}_4 + \frac{\left(1-\frac{1}{k^2}\right)\left(7-\frac{3}{k^2}\right)}{16}\bar{\nu}_2 - \frac{\left(1-\frac{1}{k^2}\right)\left(31-\frac{16}{k^2}+\frac{3}{k^4}\right)}{1344}.
\end{cases}
$$

In both (5) and (7) above it is to be understood that the class interval, λ , is chosen as the unit of x .

The modified moments, $\bar{\mu}_n$, of the following table are obtained by applying formulae (5) to the values of $\bar{\nu}_n$, which were obtained as the leading differences of table 1.

TABLE 2

n	$\bar{\nu}_n$	$\bar{\mu}_n$	σ^n	$\alpha_n = \frac{\bar{\mu}_n}{\sigma^n}$
0	1.00000	1.00000	1.00000	1.000000
1	.00000	.00000	2.114292	.000000
2	4.55356	4.47023	4.47023	1.000000
3	6.92736	6.92736	9.45137	.732948
4	91.3249	89.0773	19.9830	4.45765
5	436.420	430.647	42.2499	10.1929
6	4272.15	4159.96	89.3286	46.5692

Tables 3 and 4 shed an interesting light on the subject of modification, and are obtained from the results of formulae (5) and (7). If the class interval be denoted by λ , and the unmodified values of the standard deviation and skewness by σ' and α_3' , respectively, that is

$$(8) \qquad \begin{cases} \sigma_v' = \lambda \sqrt{\bar{\nu}_{2:x}} \\ \alpha_3' = \dfrac{\bar{\nu}_{3:x}}{(\sigma_x')^3} \end{cases}$$

it follows that

$$(9) \qquad \begin{cases} \sigma_v = \sigma_v' \, \omega & \text{and} \\ \alpha_3 = \dfrac{\alpha_3'}{\omega^3} , & \text{where} \\ \omega = \sqrt{1 - \dfrac{1 - \frac{1}{k^2}}{12} \cdot \left(\dfrac{\lambda}{\sigma_v'} \right)^2} \end{cases}$$

As mentioned before, the case of continuous variates is the special case for which $k = \infty$.

To illustrate: for our distribution taken from Yule, $\lambda = 10\ lbs.$, and by (8)

$$\sigma_v' = 10\sqrt{4.55356\ lbs^2} = 21.3391\ lbs$$

$$\alpha_3' = \frac{6.92736}{(2.13391)^3} = .712919$$

$$\frac{\lambda}{\sigma_v'} = .468623$$

From table 3, $\frac{\lambda}{\sigma_v'} = .47,\ k = \infty$ we have $\omega = .9907$ and consequently $\sigma_v = .9907\ (21.3391\ lbs.) = 21.14\ lbs.$, agreeing with the more exact value deduced from table 2, i.e. $21.14292\ lbs.$

From table 4, $\frac{\lambda}{\sigma'} = .47,\ k = \infty$ we have $\omega^{-3} = 1.028$ and therefore $\alpha_3 = 1.028(.712919) = .7329$, again agreeing with the more accurate value of table 2.

By either interpolation of tables 3 and 4, or by direct computation of ω and ω^{-3}, greater accuracy may be obtained.

For an illustration of grouped-discrete variates we may refer to pages 32 and 37 of Vol. 1, No. 1 of the Annals. For the so-called D(4.1) of Table IX, $\lambda = 4,\ \sigma_v' = 5.089,\ \alpha_3' = .096$. Hence $\frac{\lambda}{\sigma_v'} = .79$, and since $k = 4$ the modified or adjusted values are

$$\sigma_v = 5.089(.9753) = 4.963$$

$$\alpha_3 = .096(1.078) = .103.$$

It should be observed that the factors ω and ω^{-3} are independent of the number of variates, N, and are just as properly applied even if the frequency distribution method for computing the standard deviation, skewness, etc. is not employed. Thus, if I compute the standard deviation for the weights of ten individuals and those variates are recorded to the nearest pound, then the resulting

$$\sigma_v' = \sqrt{\frac{\Sigma(v-M_v)^2}{N}}$$

theoretically, if not practically, should be modified. If it developed that $\sigma_v' = 20\,lbs.$, then for weights to the nearest pound $\lambda = 1$, and

$$\omega = \sqrt{1-\frac{1}{12}\cdot\frac{1}{20^2}} = .999896$$

If, on the other hand, the weights had been taken to the nearest half pound, or to the nearest tenth pound, the corresponding values for ω would be .999974 and .999999 respectively. Only if the variates be discrete and $k=1$, or if the variates be continuous and be measured with absolute accuracy (which is impossible from a practical point of view) so that $\lambda = 0$, can $\omega = 1$, and modification be ignored.

It should be clearly understood, however, that Sheppard's corrections are merely *expectations*. We have no assurance that the use of one of these corrections in any single instance will increase the accuracy of that determination—it is quite likely that in any isolated case the modification will introduce a still greater error into the calculation. As pointed out in pages 36-38 of Vol. 1 of the Annals, and clearly revealed in the included table IX, modifying eliminates only the *systematic* errors, and ignores the *accidental* errors which may be numerically greater and of opposite algebraic sign than the correction itself.

Lastly, one must remember that the mathematical theory underlying Sheppard's Corrections assumes that both the frequency function and a sufficient number of its derivatives vanish at the limits of the distribution. Consequently the case of J-shaped or U-shaped distributions, or of data not actually classified into such distributions, is not covered by formulae (5) or (7). In any event, it is evident that the practical necessity for modification in all cases depends upon the ratio of the limits of accuracy of the measurements *employed* in the computations to the unmodified standard deviation. If we throw accurately determined variates into frequency distributions with large class intervals, λ,—and thus simplify certain computations, then we must realize that the introduction of a systematic error is the penalty paid for such procedure, that the greater the value $\frac{\lambda}{\sigma'_v}$., the greater the penalty, and that the practical necessity for modification rests entirely upon the accuracy demanded of the final results.

The chief value of tables 3 and 4 is that an inspection of these tables gives a rough idea of the value of modification so far as the standard deviation and the skewness are concerned.

H. C. Carver.

Table 3

$$\omega = \sqrt{1 - \frac{1-\frac{1}{K^2}}{12}\cdot\left(\frac{\lambda}{\sigma_v'}\right)^2}$$

$\frac{\lambda}{\sigma_v'}$	2	3	4	5	6	7	8	9	10	∞
01	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
02	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
03	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
04	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
05	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
06	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9998
07	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
08	.9998	.9998	.9998	.9997	.9997	.9997	.9997	.9997	.9997	.9997
09	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
10	.9997	.9996	.9996	.9996	.9996	.9996	.9996	.9996	.9996	.9996
11	.9996	.9996	.9995	.9995	.9995	.9995	.9995	.9995	.9995	.9995
12	.9995	.9995	.9994	.9994	.9994	.9994	.9994	.9994	.9994	.9994
13	.9995	.9994	.9993	.9993	.9993	.9993	.9993	.9993	.9993	.9993
14	.9994	.9993	.9992	.9992	.9992	.9992	.9992	.9992	.9992	.9992
15	.9993	.9992	.9991	.9991	.9991	.9991	.9991	.9991	.9991	.9991
16	.9992	.9991	.9990	.9990	.9990	.9990	.9989	.9989	.9989	.9989
17	.9991	.9989	.9989	.9988	.9988	.9988	.9988	.9988	.9988	.9988
18	.9990	.9988	.9987	.9987	.9987	.9987	.9987	.9987	.9987	.9987
19	.9989	.9987	.9986	.9986	.9985	.9985	.9985	.9985	.9985	.9985
20	.9987	.9985	.9984	.9984	.9984	.9984	.9984	.9984	.9983	.9983
21	.9986	.9984	.9983	.9982	.9982	.9982	.9982	.9982	.9982	.9982
22	.9985	.9982	.9981	.9981	.9980	.9980	.9980	.9980	.9980	.9980
23	.9983	.9980	.9979	.9979	.9979	.9978	.9978	.9978	.9978	.9978
24	.9982	.9979	.9977	.9977	.9977	.9976	.9976	.9976	.9976	.9976
25	.9980	.9977	.9976	.9975	.9975	.9974	.9974	.9974	.9974	.9974
26	.9979	.9975	.9974	.9973	.9973	.9972	.9972	.9972	.9972	.9972
27	.9977	.9973	.9971	.9971	.9970	.9970	.9970	.9970	.9970	.9970
28	.9975	.9971	.9969	.9969	.9968	.9968	.9968	.9968	.9968	.9967
29	.9974	.9969	.9967	.9966	.9966	.9966	.9965	.9965	.9965	.9965
30	.9972	.9967	.9965	.9964	.9963	.9963	.9963	.9963	.9963	.9962
31	.9970	.9964	.9962	.9961	.9961	.9961	.9961	.9960	.9960	.9960
32	.9968	.9962	.9960	.9959	.9958	.9958	.9958	.9958	.9958	.9957
33	.9966	.9960	.9957	.9956	.9956	.9955	.9955	.9955	.9955	.9954
34	.9964	.9957	.9955	.9954	.9953	.9953	.9952	.9952	.9952	.9952
35	.9962	.9955	.9952	.9951	.9950	.9950	.9950	.9949	.9949	.9949
36	.9959	.9952	.9949	.9948	9947	.9947	.9947	.9947	.9946	.9946
37	.9957	.9949	.9946	.9945	.9944	.9944	.9944	.9944	.9943	.9943
38	.9955	.9946	.9943	.9942	.9941	.9941	.9941	.9940	.9940	.9940
39	.9952	.9944	.9940	.9939	.9938	.9938	.9937	.9937	.9937	.9936
40	.9950	.9941	.9937	.9936	.9935	.9934	.9934	.9934	.9934	.9933
41	.9947	.9938	.9934	.9933	.9932	.9931	.9931	.9931	.9930	.9930
42	.9945	.9935	.9931	.9929	.9928	.9928	.9927	.9927	.9927	.9926
43	.9942	.9931	.9928	.9926	.9925	.9924	.9924	.9924	.9923	.9923
44	.9939	.9928	.9924	.9922	.9921	.9921	.9920	.9920	.9920	.9919
45	.9937	.9925	.9921	.9919	.9918	.9917	.9917	.9916	.9916	.9915
46	.9934	.9921	.9917	.9915	.9914	.9913	.9913	.9913	.9912	.9911
47	.9931	.9918	.9913	.9911	.9910	.9909	.9909	.9909	.9908	.9907
48	.9928	.9914	.9910	.9907	.9906	.9906	.9905	.9905	.9905	.9904
49	.9925	.9911	.9906	.9903	.9902	.9902	.9901	.9901	.9901	.9900
50	.9922	.9907	.9902	.9899	.9898	9897	.9897	.9897	9896	.9895

Table 3

$$\omega = \sqrt{1 \quad \frac{1-\frac{1}{k^2}}{12}\cdot\left(\frac{\lambda}{\sigma_v'}\right)^2}$$

$\frac{\lambda}{\sigma_v'}$	2	3	4	5	6	7	8	9	10	∞
51	.9918	.9903	.9898	.9895	.9894	.9893	.9893	.9892	.9892	.9891
52	.9915	.9899	.9894	.9891	.9890	.9889	.9888	.9888	.9888	.9887
53	.9912	.9895	.9890	.9887	.9886	.9885	.9884	.9884	.9883	.9882
54	.9908	.9891	.9885	.9883	.9881	.9880	.9880	.9879	.9879	.9878
55	.9905	.9887	.9881	.9878	.9877	.9876	.9875	.9875	.9874	.9873
56	.9902	.9883	.9877	.9874	.9872	.9871	.9871	.9870	.9870	.9868
57	.9898	.9879	.9872	.9869	.9868	.9867	.9866	.9865	.9865	.9864
58	.9894	.9875	.9868	.9865	.9863	.9862	.9861	.9861	.9860	.9859
59	.9891	.9870	.9863	.9860	.9858	.9857	.9856	.9856	.9855	.9854
60	.9887	.9866	.9858	.9855	.9853	.9852	.9851	.9851	.9850	.9849
61	.9883	.9861	.9854	.9850	.9848	.9847	.9846	.9846	.9845	.9844
62	.9879	.9857	.9849	.9845	.9843	.9842	.9841	.9841	.9840	.9839
63	.9875	.9852	.9844	.9840	.9838	.9837	.9836	.9835	9835	.9833
64	.9871	.9847	.9839	.9835	.9833	.9832	.9831	.9830	.9830	.9828
65	.9867	.9842	.9834	.9830	.9827	.9826	.9825	.9825	.9824	.9822
66	.9863	.9837	.9829	.9824	.9822	.9821	.9820	.9819	.9819	.9817
67	.9859	.9832	.9823	.9819	.9816	.9815	.9814	.9814	.9813	.9811
68	.9854	.9827	.9818	.9813	.9811	.9808	.9808	.9808	.9807	.9805
69	.9850	.9822	.9812	.9808	.9805	.9804	.9803	.9802	.9802	.9800
70	.9846	.9817	.9807	.9802	.9799	.9798	9797	.9796	.9796	.9794
71	.9841	.9811	.9801	.9796	.9794	.9792	.9791	.9790	.9790	.9788
72	.9837	.9806	.9795	.9790	.9788	.9786	.9785	.9784	.9784	.9782
73	.9832	.9801	.9790	.9785	.9782	.9780	.9779	.9778	.9778	.9775
74	.9827	.9795	.9784	.9779	.9776	.9774	.9773	.9772	.9772	.9769
75	.9823	.9789	.9778	.9772	.9769	.9768	.9767	.9766	.9765	.9763
76	.9818	.9784	.9772	.9766	.9763	.9761	.9760	.9759	.9759	.9756
77	.9813	.9778	.9766	.9760	.9757	.9755	.9754	.9753	.9752	.9750
78	.9808	.9772	.9759	.9754	.9750	.9749	.9747	.9746	.9746	.9743
79	.9803	.9766	.9753	.9747	.9744	.9742	.9741	.9740	.9739	.9736
80	.9798	.9760	.9747	.9741	.9737	.9735	.9734	.9733	.9732	.9730
81	.9793	.9754	.9740	.9734	.9731	.9729	.9727	.9726	.9726	.9723
82	.9788	.9748	.9734	.9727	.9724	.9722	.9720	.9719	.9719	.9716
83	.9782	.9742	.9727	.9721	.9717	.9715	.9713	.9712	.9712	.9709
84	.9777	.9735	.9720	.9714	.9710	.9708	.9707	.9706	.9705	.9702
85	.9772	.9729	.9714	.9707	.9703	.9701	.9699	.9698	.9697	.9694
86	.9766	.9722	.9707	.9700	.9696	.9694	.9692	.9691	.9690	.9687
87	.9761	.9716	.9700	.9693	.9689	.9687	.9685	.9684	.9683	.9680
88	.9755	.9709	.9693	.9685	.9681	.9679	.9677	.9676	.9675	.9672
89	.9749	.9702	.9686	.9678	.9674	.9672	.9670	.9669	.9668	.9664
90	.9744	.9695	.9678	.9671	.9666	.9664	.9662	.9661	.9660	.9657
91	.9738	.9688	.9671	.9663	.9659	.9656	.9654	.9653	.9652	.9649
92	.9732	.9681	.9664	.9656	.9651	.9649	.9647	.9646	.9645	.9641
93	.9726	.9674	.9656	.9648	.9643	.9641	.9639	.9638	.9637	.9633
94	.9720	.9667	.9649	.9640	.9635	.9633	.9631	.9630	.9629	.9625
95	.9714	.9660	.9641	.9632	.9627	.9625	.9623	.9622	.9621	.9617
96	.9708	.9653	.9633	.9624	.9619	.9617	.9615	.9613	.9612	.9608
97	.9702	.9645	.9625	.9616	.9611	.9608	.9606	.9605	.9604	.9600
98	.9695	.9638	.9618	.9608	.9603	.9600	.9598	.9597	.9596	.9591
99	.9689	.9630	.9610	.9600	.9595	.9592	.9590	.9588	.9587	.9583
100	.9682	.9623	.9601	.9592	.9586	.9583	.9581	.9580	.9579	.9574

Table 4

$$\omega^{-3} = \left[1 - \frac{1-\frac{1}{k^2}}{12}\left(\frac{\lambda}{\sigma'_v}\right)\right]^{-3}$$

$\frac{\lambda}{\sigma'_v}$	2	3	4	5	6	7	8	9	10	∞
01	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
02	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
03	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
04	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
05	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
06	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.001
07	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001
08	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001
09	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001
10	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001
11	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.002
12	1.001	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002
13	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002
14	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002
15	1.002	1.002	1.003	1.003	1.003	1.003	1.003	1.003	1.003	1.003
16	1.002	1.003	1.003	1.003	1.003	1.003	1.003	1.003	1.003	1.003
17	1.003	1.003	1.003	1.004	1.004	1.004	1.004	1.004	1.004	1.004
18	1.003	1.004	1.004	1.004	1.004	1.004	1.004	1.004	1.004	1.004
19	1.003	1.004	1.004	1.004	1.004	1.004	1.004	1.004	1.004	1.005
20	1.004	1.004	1.005	1.005	1.005	1.005	1.005	1.005	1.005	1.005
21	1.004	1.005	1.005	1.005	1.005	1.005	1.005	1.005	1.005	1.005
22	1.005	1.005	1.006	1.006	1.006	1.006	1.006	1.006	1.006	1.006
23	1.005	1.006	1.006	1.006	1.006	1.007	1.007	1.007	1.007	1.007
24	1.005	1.006	1.007	1.007	1.007	1.007	1.007	1.007	1.007	1.007
25	1.006	1.007	1.007	1.008	1.008	1.008	1.008	1.008	1.008	1.008
26	1.006	1.008	1.008	1.008	1.008	1.008	1.008	1.008	1.008	1.008
27	1.007	1.008	1.009	1.009	1.009	1.009	1.009	1.009	1.009	1.009
28	1.007	1.009	1.009	1.009	1.010	1.010	1.010	1.010	1.010	1.010
29	1.008	1.009	1.010	1.010	1.010	1.010	1.010	1.011	1.011	1.011
30	1.008	1.010	1.011	1.011	1.011	1.011	1.011	1.011	1.011	1.011
31	1.009	1.011	1.011	1.012	1.012	1.012	1.012	1.012	1.012	1.012
32	1.010	1.011	1.012	1.012	1.013	1.013	1.013	1.013	1.013	1.013
33	1.010	1.012	1.013	1.013	1.013	1.014	1.014	1.014	1.014	1.014
34	1.011	1.013	1.014	1.014	1.014	1.014	1.014	1.014	1.014	1.015
35	1.012	1.014	1.015	1.015	1.015	1.015	1.015	1.015	1.015	1.015
36	1.012	1.015	1.015	1.016	1.016	1.016	1.016	1.016	1.016	1.016
37	1.013	1.015	1.016	1.017	1.017	1.017	1.017	1.017	1.017	1.017
38	1.014	1.016	1.017	1.018	1.018	1.018	1.018	1.018	1.018	1.018
39	1.014	1.017	1.018	1.019	1.019	1.019	1.019	1.019	1.019	1.019
40	1.015	1.018	1.019	1.020	1.020	1.020	1.020	1.020	1.020	1.020
41	1.016	1.019	1.020	1.021	1.021	1.021	1.021	1.021	1.021	1.021
42	1.017	1.020	1.021	1.022	1.022	1.022	1.022	1.022	1.022	1.022
43	1.018	1.021	1.022	1.023	1.023	1.023	1.023	1.023	1.023	1.024
44	1.018	1.022	1.023	1.024	1.024	1.024	1.024	1.024	1.024	1.025
45	1.019	1.023	1.024	1.025	1.025	1.025	1.025	1.026	1.026	1.026
46	1.020	1.024	1.025	1.026	1.026	1.027	1.027	1.027	1.027	1.027
47	1.021	1.025	1.026	1.027	1.027	1.028	1.028	1.028	1.028	1.028
48	1.022	1.026	1.028	1.028	1.029	1.029	1.029	1.029	1.029	1.029
49	1.023	1.027	1.029	1.030	1.030	1.030	1.030	1.030	1.030	1.031
50	1.024	1.028	1.030	1.031	1.031	1.031	1.032	1.032	1.032	1.032

Table 4

$$\omega^{1.3} = \left[1 - \frac{1-\frac{1}{k^2}}{12}\cdot\left(\frac{\lambda}{\sigma_V'}\right)^2\right]^{-3}$$

$\frac{\lambda}{\sigma_V'}$	2	3	4	5	6	7	8	9	10	∞
51	1.025	1.030	1.031	1.032	1.032	1.033	1.033	1.033	1.033	1.033
52	1.026	1.031	1.033	1.033	1.034	1.034	1.034	1.034	1.034	1.035
53	1.027	1.032	1.034	1.035	1.035	1.035	1.036	1.036	1.036	1.036
54	1.028	1.033	1.035	1.036	1.037	1.037	1.037	1.037	1.037	1.038
55	1.029	1.035	1.037	1.037	1.038	1.038	1.038	1.039	1.039	1.039
56	1.030	1.036	1.038	1.039	1.039	1.040	1.040	1.040	1.040	1.041
57	1.031	1.037	1.039	1.040	1.041	1.041	1.041	1.042	1.042	1.042
58	1.032	1.039	1.041	1.042	1.042	1.043	1.043	1.043	1.043	1.044
59	1.034	1.040	1.042	1.043	1.044	1.044	1.044	1.045	1.045	1.045
60	1.035	1.041	1.044	1.045	1.045	1.046	1.046	1.046	1.046	1.047
61	1.036	1.043	1.045	1.046	1.047	1.047	1.048	1.048	1.048	1.048
62	1.037	1.044	1.047	1.048	1.048	1.049	1.049	1.049	1.050	1.050
63	1.038	1.046	1.048	1.050	1.050	1.050	1.051	1.051	1.051	1.052
64	1.040	1.047	1.050	1.051	1.052	1.052	1.053	1.053	1.053	1.053
65	1.041	1.049	1.052	1.053	1.054	1.054	1.054	1.054	1.055	1.055
66	1.042	1.050	1.053	1.055	1.055	1.056	1.056	1.056	1.056	1.057
67	1.044	1.052	1.055	1.056	1.057	1.058	1.058	1.058	1.058	1.059
68	1.045	1.054	1.057	1.058	1.059	1.060	1.060	1.060	1.060	1.061
69	1.046	1.055	1.059	1.060	1.061	1.061	1.062	1.062	1.062	1.063
70	1.048	1.057	1.060	1.062	1.063	1.063	1.064	1.064	1.064	1.065
71	1.049	1.059	1.062	1.064	1.065	1.065	1.065	1.066	1.066	1.066
72	1.051	1.061	1.064	1.066	1.066	1.067	1.067	1.068	1.068	1.068
73	1.052	1.062	1.066	1.068	1.068	1.069	1.069	1.070	1.070	1.070
74	1.054	1.064	1.068	1.070	1.070	1.071	1.071	1.072	1.072	1.072
75	1.055	1.066	1.070	1.072	1.073	1.073	1.073	1.074	1.074	1.074
76	1.057	1.068	1.072	1.074	1.075	1.075	1.076	1.076	1.076	1.077
77	1.058	1.070	1.074	1.076	1.077	1.077	1.078	1.078	1.078	1.079
78	1.060	1.072	1.076	1.078	1.079	1.079	1.080	1.080	1.080	1.081
79	1.062	1.074	1.078	1.080	1.081	1.082	1.082	1.082	1.083	1.083
80	1.063	1.076	1.080	1.082	1.083	1.084	1.084	1.085	1.085	1.086
81	1.065	1.078	1.082	1.084	1.085	1.086	1.087	1.087	1.087	1.088
82	1.066	1.080	1.084	1.087	1.088	1.088	1.089	1.089	1.089	1.090
83	1.068	1.082	1.087	1.089	1.090	1.091	1.091	1.092	1.092	1.093
84	1.070	1.084	1.089	1.091	1.092	1.093	1.093	1.094	1.094	1.095
85	1.072	1.086	1.091	1.093	1.095	1.095	1.096	1.096	1.097	1.098
86	1.074	1.088	1.093	1.096	1.097	1.098	1.098	1.099	1.099	1.100
87	1.075	1.090	1.096	1.098	1.100	1.100	1.101	1.101	1.102	1.103
88	1.077	1.093	1.098	1.101	1.102	1.103	1.103	1.104	1.104	1.105
89	1.079	1.095	1.101	1.103	1.105	1.105	1.106	1.106	1.107	1.108
90	1.081	1.097	1.103	1.106	1.107	1.108	1.109	1.109	1.109	1.110
91	1.083	1.100	1.106	1.108	1.110	1.111	1.111	1.112	1.112	1.113
92	1.085	1.102	1.108	1.111	1.112	1.113	1.114	1.114	1.115	1.116
93	1.087	1.104	1.111	1.114	1.115	1.116	1.117	1.117	1.117	1.119
94	1.089	1.107	1.113	1.116	1.118	1.119	1.119	1.120	1.120	1.122
95	1.091	1.109	1.116	1.119	1.121	1.122	1.122	1.123	1.123	1 124
96	1.093	1.112	1.118	1.122	1.123	1.124	1.125	1.126	1.126	1.127
97	1.095	1.114	1.121	1.125	1.126	1.127	1.128	1.129	1.129	1.130
98	1.097	1.117	1.124	1.127	1.129	1.130	1.131	1.131	1.132	1.133
99	1.099	1.120	1.127	1.130	1.132	1.133	1.134	1.134	1.135	1.136
100	1.102	1.122	1.130	1.133	1.135	1.136	1.137	1.137	1.138	1.139