AN APPLICATION OF CHARACTERISTIC FUNCTIONS TO THE DISTRIBUTION PROBLEM OF STATISTICS*

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CONTENTS

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PART 1

The General Theory

*I. Introduction:** By the distribution problem of statistics we mean the problem of determining the distribution law of functions of variables satisfying known distribution laws. Many particular problems of this nature have been solved by various methods. In Part 1 of this paper we develop a general solution for this problem for functions of variables satisfying continuous distribution laws. The general result is then applied in Part 2 to derive the distribution laws of several functions whose distribution laws have been derived by other methods and of some functions whose distribution laws have not been given or given only for special cases; in Part 3 we summarize the results. The method of solution is related to the concept of characteristic function.

The theory of characteristic functions is essentially a development of Laplace's[16] "fonction génératrice." In this paper we shall adopt the term characteristic function, although the same concept has been termed generating function[14] and reciprocal function.[21] Poisson [28, 29] employed the methods of Laplace to discuss, in particular, "Sur la Probabilité des Resultats Moyens des Observations." Cauchy[2] was apparently the next to study and apply this theory; he applied the basic concept of characteristic function in connection with whát he called "coefficient limitateur ou ristricteur" to study the problem of a function of errors. In particular he studied the case of a linear function of the errors. More recently the same concept has been reintroduced under the name of characteristic function by Poincaré[27] and also by P. Lévy[17, 18, 19] who employs it to consider the composition of laws of probability, the notion of the limit of a probability law, the idea of stable and semi-stable laws, etc.

In a series of papers, C. V. L. Charlier[3] further applied and

* The reference numbers correspond with the number of the item in the bibliography.

developed the theory of characteristic functions (though he employed the terminology of reciprocal functions) to develop the Gram-Charlier Type A and Type B series, and to consider the distribution law of functions of variables satisfying general frequency laws. Under the name of "Erzeugenden Funktion," T. Kameda[14] studied the properties of functions which are intimately related to characteristic functions. In particular, he discussed the development of a function as a series of Hermite Polynomials and also considered the problem of finding the distribution law of a function of variables obeying general distribution laws.[15]

II. Characteristic Functions: By the characteristic function of the distribution law of the variable x is meant the mean* value of e^{itx} where $i = \sqrt{-1}$. Thus, for a continuous distribution law, if $f(x)\,dx$ is the probability to within infinitesimals of a higher order that $x - \frac{dx}{2} < x_1 < x + \frac{dx}{2}$ and $\varphi(t)$ is the characteristic function of the d.l.,** of x then

$$(1) \qquad \varphi(t) = \int e^{itx} f(x)\,dx,$$

where the limits of the integral depend upon the range of applicability of $f(x)$. We may also write $\varphi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$ if we agree that $f(x) \equiv 0$ outside the range of applicability. The characteristic function derives its importance from the fact[17] that

$$(2) \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi(t)\,dt.$$

For the case of several variables, we have that the character-

* Also known as probable or expected value.

** We shall designate distribution law hereafter by d.l.

istic function of the d.l. $f(x_1, x_2, \cdots, x_n)$ of $x_1, x_2, \cdots, x_n$ is given by

$$(3) \quad \varphi(t_1, t_2, \cdots, t_n) = \int_R \cdots \int e^{it_1x_1 + it_2x_2 + \cdots + it_nx_n} f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n,$$

where R is the region of applicability of $f(x_1, x_2, \cdots, x_n)$. We may also write $\varphi(t_1, \cdots, t_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{it_1x_1 + \cdots + it_nx_n} \cdot f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n$ provided we agree that $f(x_1, x_2, \cdots, x_n) \equiv 0$ outside the region R. As for the case of a single variable we have here too[38]

$$(4) \quad f(x_1, x_2, \cdots, x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-it_1x_1 - \cdots - it_nx_n} \varphi(t_1, \cdots, t_n)\, dt_1 \cdots dt_n.$$

We shall prove that the following extensions are also possible. Consider the function $u(x_1, x_2, \cdots, x_n)$ * of the variables $x_1, x_2, \cdots, x_n$ whose d.l. is $f(x_1, x_2, \cdots, x_n)$. Then the characteristic function of the d.l. of u is given by

$$(5) \quad \varphi(t) = \int_R \cdots \int e^{it \cdot u(x_1, x_2, \cdots, x_n)} \cdot f(x_1, x_2, \cdots, x_n)\, dx_1 \cdot dx_2 \cdots dx_n,$$

where R is the region of applicability of $f(x_1, x_2, \cdots, x_n)$. The d.l. of u, $P(u)$, is given by

$$(6) \qquad P(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itu} \varphi(t)\, dt, \qquad \text{where}$$

$\varphi(t)$ is defined by (5).

If we consider the several functions $u_1(x_1, x_2, \cdots, x_n)$;

* The conditions which $u(x_1, x_2, \cdots, x_n)$ must satisfy will be developed further in this paper.

$u_2(x_1, x_2, \cdots\cdots x_n)$; . . .; $u_n(x_1, x_2, \cdots\cdots, x_n)$ of the variables $x_1, x_2, \cdots\cdot, x_n$ whose d.l. is $f(x_1, x_2, \cdots\cdots, x_n)$, then the characteristic function of the d.l. of $u_1, u_2, \cdots\cdots, u_n$ is given by

$$(7)\quad \varphi(t_1, t_2, \cdots, t_n) = \int_R \cdots \int e^{it_1 u_1(x_1, x_2, \cdots, x_n) + \cdots + it_n u_n(x_1, x_2, \cdots, x_n)} \cdot f(x_1, x_2, \cdots, x_n)\, dx_1, dx_2, \cdots, dx_n,$$

where R is the region of applicability of $f(x_1, x_2, \cdots, x_n)$. The d.l. of $u_1, u_2, \cdots\cdots, u_n$ is given by

$$(8)\quad P(u_1, u_2, \cdots, u_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-it_1 u_1 - it_2 u_2 - \cdots - it_n u_n} \cdot \varphi(t_1, t_2, \cdots, t_n)\, dt_1 dt_2 \cdots dt_n$$

where $\varphi(t_1, t_2, \cdots\cdot, t_n)$ is defined by (7).

III. Theorems Regarding a Single Function $u(x_1, x_2, \cdots\cdot, x_n)$: We shall now justify our statements and determine the precise conditions the function u must obey.

Consider the function $u(x_1, x_2, \cdots\cdots, x_n)$ of the variables $x_1, x_2, \cdots\cdot, x_n$ satisfying the continuous d.l. $f(x_1, x_2, \cdots\cdot, x_n)$

such that $\int_R \cdots \int f(x_1, x_2, \cdots, x_n)\, dx_1 \cdot dx_2 \cdots dx_n = 1$. The

function u may have at most a denumerable infinity of discontinuities. The probability that $u(x_1, x_2, \cdots\cdot, x_n)$ satisfies the conditions

$$(9)\quad u_1 < u < u_2 \qquad \text{is given by}$$

$$(10)\quad \int_A \cdots \int f(x_1, x_2, \cdots, x_n)\, dx_1 \cdot dx_2 \cdots\cdot, dx_n, \text{ where } A$$

is the region defined by the inequalities $u_1 < u < u_2$. To avoid the difficulty of integrating over the region A we shall avail ourselves of the discontinuity factor (See Whittaker and Watson[40] § 9.7).

$$(11) \qquad F = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{u_1}^{u_2} e^{-it(\theta-u)}\, d\theta\, dt, \quad \text{where } F = 1$$

for $u_1 < u < u_2$; $F = 0$ for $u_1 \geqq u$; $F = 0$ for $u \geqq u_2$.

We are now able to say that the required probability is given* by

$$(12) \qquad \int_R \cdots \int f(x_1, x_2, \cdots, x_n) \cdot F \cdot dx_1\, dx_2 \cdots dx_n .$$

If we set $2\omega = u_1 + u_2$ and $\gamma = u_2 - u_1$, the required probability may also be written as

$$(13) \qquad \frac{1}{2\pi}\int_R \cdots \int f(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n \cdot \int_{-\infty}^{\infty}\int_{\omega-\frac{\gamma}{2}}^{\omega+\frac{\gamma}{2}} e^{-it(\theta-u)}\, d\theta\, dt.$$

Integrating with respect to θ, we obtain

$$(14) \qquad \frac{1}{2\pi}\int_R \cdots \int f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \cdot \int_{-\infty}^{\infty} e^{it(u-\omega)} \cdot \frac{2\sin\frac{\gamma t}{2}}{t}\, dt.$$

We now want to prove that

$$(15) \qquad \int_R z\, dX \int_{-\infty}^{\infty} e^{it(u-\omega)} \cdot \frac{2\sin\frac{\gamma t}{2}}{t}\, dt = \int_{-\infty}^{\infty} \frac{2\sin\frac{\gamma t}{2}}{t}\, dt \int_R e^{it(u-\omega)} \cdot z \cdot dX,$$

where we write $z = f(x_1, x_2, \cdots, x_n)$; $dX = dx_1, dx_2, \cdots dx_n$

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* This method is essentially an application of Cauchy's "Coefficient limitateur ou ristricteur." See C.R. Vol. 37, p. 150 ff, and Whittaker and Robinson, Calculus of Obs., p. 169.

and $\int_R$ as the multiple integral over the region R.

We have that

$$(16)\quad \int_R z\,dX\int_{-\infty}^{\infty}\frac{2\sin\frac{\alpha t}{2}}{t}e^{it(u-\omega)}dt = \int_R z\,dX\int_0^{\infty}\frac{2\sin\frac{\alpha t}{2}}{t}e^{it(u-\omega)}dt + \int_R z\,dX\int_0^{\infty}\frac{2\sin\frac{\alpha t}{2}}{t}e^{-it(u-\omega)}dt.$$

We will now prove that

$$(17)\quad \int_R z\,dX\int_0^{\infty}\frac{2\sin\frac{\alpha t}{2}}{t}e^{it(u-\omega)}dt = \int_0^{\infty}\frac{2\sin\frac{\alpha t}{2}}{t}dt\int_R e^{it(u-\omega)}\cdot z\cdot dX$$

For this, it is sufficient[12] to prove the existence of the $(n+1)$ fold integral* $\int z\,\frac{2\sin\frac{\alpha t}{2}}{t}e^{it(u-\omega)}dX\,dt$,

and the existence of the right-hand member of (17).

Consider** the rectangular region G in $(n+1)$ fold space defined by $0\leqq t\leqq t_1$; $x_j'\leqq x_j\leqq x_j''$, $j=1,2,3,\cdots,n$, where we shall designate the region $x_j'\leqq x_j\leqq x_j''$, $j=1,2,\cdots,n$, by E. Then, over G the multiple integral of

$$z\cdot\frac{2\sin\frac{at}{2}}{t}e^{it(u-\omega)}$$

exists since the integrand is bounded and has at most a denumerable infinity of singularities (those of $u(x_1,x_2,\cdots,x_n)$). Then[10]

$$(18)\quad \int_G z\cdot\frac{2\sin\frac{\alpha t}{2}}{t}\cdot e^{it(u-\omega)}dX\,dt = \int_0^{t_1'}\frac{2\sin\frac{\alpha t}{2}}{t}dt\int_E e^{it(u-\omega)}\cdot z\,dX$$

Now for any positive ϵ there exists a $t_1>0$ such that

$$(19)\quad \left|\int_{t_1}^{t_2}\frac{2\sin\frac{\alpha t}{2}}{t}dt\int_E z\cdot e^{it(u-\omega)}dX\right| < \frac{\epsilon}{2}$$

* For the sake of convenience we shall understand a single integral sign to represent a multiple integral where necessary.

** The proof here given is modeled after a similar one of E. L. Dodd (See Annals of Math. 2nd S. Vol. 27, pp. 12-20).

for every $t_2 > t_1$, since $\left|\int_E z \cdot e^{it(u-\omega)} dX\right| \leqq \left|\int_E z\, dX\right| \leqq 1$,

and $\int_0^\infty \frac{2 \sin \frac{\alpha t}{2}}{t} dt = \pi$ for $\alpha > 0$. Furthermore,

(20) $$\left|\int_0^{t_1} \frac{2 \sin \frac{\alpha t}{2}}{t} e^{it(u-\omega)} dt\right| \leqq \left|\int_0^{t_1} \frac{2 \sin \frac{\alpha t}{2}}{t} dt\right| < 4,$$

and since $\int_R z\, dX = 1$, we can find a rectangular region E_1, such that if E encloses E_1 and E_2 is that portion of E not in E_1,

(21) $$\left|\int_{E_2} z \cdot dX\right| < \frac{\epsilon}{8}$$ Thus

(22) $$\left|\int_0^{t_1} \frac{2 \sin \frac{\alpha t}{2}}{t} dt \int_{E_2} z \cdot e^{it(u-\omega)} dX\right| = \left|\int_{E_2} z\, dX \int_0^{t_1} \frac{2 \sin \frac{\alpha t}{2}}{t} e^{it(u-\omega)} dt\right| < \frac{\epsilon}{2}.$$

Hence, since t_2 and E may now increase without limit (19) and (22) show the convergence of the $(n+1)$ fold integral of $z \cdot \frac{2 \sin \frac{\alpha t}{2}}{t} \cdot e^{it(u-\omega)}$

But since $\left|e^{it(u-\omega)}\right| = 1$, $\int_R e^{it(u-\omega)} z\, dX$ exists for all values of t. Therefore,

$$\int_0^\infty \frac{2 \sin \frac{\alpha t}{2}}{t} dt \int_R z \cdot e^{it(u-\omega)} dX$$

exists being equal to the corresponding multiple integral whose existence has just been proved. We have thus established (17) by using the theorem that if the multiple integral and a corresponding iterated integral both exist they are equal.

We can show in a similar manner that

$$\int_R z\, dX \int_0^\infty \frac{2\sin\frac{\alpha t}{2}}{t} e^{-it(u-\omega)} dt = \int_0^\infty \frac{2\sin\frac{\alpha t}{2}}{t} dt \int_R e^{-it(u-\omega)} \cdot z \cdot dX,$$

so that finally

$$(23)\quad \int_R z\, dX \int_{-\infty}^\infty \frac{2\sin\frac{\alpha t}{2}}{t} e^{it(u-\omega)} dt = \int_{-\infty}^\infty \frac{2\sin\frac{\alpha t}{2}}{t} dt \int_R z\, e^{it(u-\omega)} dX.$$

Let u_1 and u_2 approach an intermediate value v as a limit with $u_2 > u_1$. Then $\alpha \to dv$ and $\omega \to v$ and in the limit

$$(24)\quad P(v)\, dv = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{2\sin\frac{t\,dv}{2}}{t} e^{-itv} dt \int_R e^{it\,u(x_1,x_2,\cdots,x_n)} \cdot z \cdot dX.$$

$P(v)$ exists since $\left|\int_R e^{itu} \cdot z \cdot dX\right| \leqq 1$ and

$$\left|\frac{1}{2\pi}\int_{-\infty}^\infty \frac{2\sin\frac{t\,dv}{2}}{t} e^{-itv} dt\right| \leqq \frac{2}{\pi}\int_0^\infty \frac{\sin\frac{t\,dv}{2}}{t} dt = 1.$$

Therefore, to within infinitesimals of a higher order, the d.l. of $u(x_1,\cdots,x_n)$ is given by $P(v)dv = \frac{dv}{2\pi}\int_{-\infty}^\infty e^{-itv} dt \int_R e^{it\,u(x_1,\cdots,x_n)} \cdot z\, dX$

or $\quad (25)\quad P(v) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-itv} \varphi(t)dt$ where $\varphi(t) = \int_R e^{it\,u(x_1,\cdots,x_n)} z\, dX.$

An application of Fourier's Integral Theorem[88] to (25) yields finally

$$(26)\quad \varphi(t) = \int_{-\infty}^\infty e^{itv} P(v)\, dv = \int_R e^{it\,u(x_1,x_2,\cdots,x_n)} \cdot z\, dX,$$

where $P(v) \equiv 0$ outside the range of applicability.

From (26) we see that $\varphi(t)$ is the characteristic function of the d.l. of $u(x_1, x_2, \cdots\cdots, x_n)$.

We now state

*THEOREM I. *If* $u = u(x_1, x_2, \cdots, x_n)$ *is any function which may have at most a denumerable infinity of discontinuities, of the variables* $x_1, x_2, \cdots, x_n$ *where the distribution law of* $x_1, x_2, \cdots, x_n$ *is given by* $f(x_1, x_2, \cdots, x_n)$ *which is on a certain* n*-dimensional manifold* R *a single valued, non-negative continuous function such that* $\int_R f(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n = 1$ *then the characteristic function of the distribution law of* u *is given by*

$$\varphi(t) = \int_R e^{itu(x_1, \cdots, x_n)} \cdot f(x_1, \cdots, x_n)\, dx_1 \cdots\cdots dx_n .$$

**THEOREM II. *Under the conditions of Theorem I, the distribution law of* u *is given by*

$$P(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itu} \varphi(t)\, dt \quad \text{where}$$

$$\varphi(t) = \int_R e^{itu(x_1, x_2, \cdots, x_n)} f(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n .$$

IV. Theorems Regarding Several Functions $u_j(x_1, x_2, \cdots, x_n)$, $j = 1, 2, \cdots\cdots, k$: The procedure in the case where we consider several functions $u_j(x_1, \cdots, x_n)$, $j = 1, 2, \cdots, k$ of the variables

* Charlier[3] (Arkiv. Vol. 8) considers a function $u(x_1, x_2, \cdots, x_n)$ which may not be infinite for real x_j: nor may the maxima and minima of u be infinitely dense for any values of the variables.

Kameda[15] (Proc. Vol. 9) considers a function $u(x_1, x_2, \cdots\cdot, x_n)$ such that (1) u must be a continuous function of at least one argument, say x_n, (2) the derivative of u with respect to x_n exists, (3) there exists no interval of x_n for which $\frac{\partial u}{\partial x_n}$ is identically zero, (4) the function u and its derivatives have the same sign in the neighborhood of $\pm\infty$.

** Dodd[6] (Annals Vol. 27) considers the distribution of a continuous function $u(x_1, x_2, \cdots\cdots, x_n)$.

$x_1, x_2, \cdots\cdots, x_n$ is similar to that above.

The probability that $u_j(x_1, x_2, \cdots, x_n)$, $j = 1, 2, \cdots, r$, where the u_j, $j = 1, 2, \cdots, r$ and x_k, $k = 1, 2, \cdots, n$ are defined as for the case of a single function u, satisfy the conditions

$$(27) \qquad \begin{cases} u_1' < u_1 < u_1'' \\ u_2' < u_2 < u_2'' \\ \text{-----------} \\ u_r' < u_r < u_r'' \end{cases}$$

is given by

$$(28) \qquad \int_B f(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n$$

where the region B is defined by the set of inequalities (27). We can avoid the difficulty of integrating over the region B by introducing the discontinuity factor[88]

$$(29) \qquad F = \frac{1}{(2\pi)^r} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{u_1'}^{u_1''} \cdots \int_{u_r'}^{u_r''} e^{it_1(\theta_1 - u_1) + it_2(\theta_2 - u_2) + \cdots + it_r(\theta_r - u_r)}\, d\theta_1 \cdots d\theta_r\, dt_1 \cdots dt_r$$

where $F = 1$ for $\begin{cases} u_1' < u_1 < u_1'' \\ \text{---------} \\ u_r' < u_r < u_r'' \end{cases}$

and $F = 0$ for $\begin{cases} u_1' \geqq u_1\ ;\ u_1 \geqq u_1'' \\ \text{-----------} \\ u_r' \geqq u_r\ ;\ u_r \geqq u_r'' \end{cases}$.

We can now say that the probability that $u_1, u_2, \cdots\cdots, u_r$ satisfy the conditions (27) is given by

$$(30) \qquad \frac{1}{(2\pi)^r} \int_R z\, dX \int_{-\infty}^{\infty} \int_{u_j'}^{u_j''} e^{it_1(\theta_1 - u_1) + it_2(\theta_2 - u_2) + \cdots + it_r(\theta_r - u_r)}\, d\theta_1 \cdots d\theta_r\, dt_1 \cdots dt_r .$$

In a manner entirely analagous to the case of a single function

u, we find that

$$(31)\quad P(v_1, v_2, \cdots, v_r) = \frac{1}{(2\pi)^r}\int_{-\infty}^{\infty} e^{-it_1v_1 - it_2v_2 - \cdots - it_rv_r}\,\varphi(t_1, t_2, \cdots, t_r)\,dt_1\cdots dt_r$$

where

$$(32)\quad \varphi(t_1, t_2, \cdots, t_r) = \int_R z \cdot e^{it_1u_1(x_1,\cdots x_n) + \cdots + it_ru_r(x_1,\cdots,x_n)}\,dX\,.$$

An application of Fourier's Integral Theorem[38] to (31) yields

$$\varphi(t_1, t_2, \cdots, t_r) = \int_{-\infty}^{\infty} e^{it_1u_1 + it_2u_2 + \cdots + it_ru_r} \cdot P(u_1, \cdots, u_r)\,du_1\cdots du_r$$

where $P(u_1, \cdots, u_r) \equiv 0$ outside the region of applicability, which shows that $\varphi(t_1, t_2, \cdots, t_r)$ also given as in (32) is the characteristic function of the d.l. of $u_1, u_2, \cdots, u_r$.

We now state

THEOREM III. *If* $u_j = u_j(x_1, x_2, \cdots, x_n)$, $j = 1, 2, \cdots, r$, *which may have a denumerable infinity of discontinuities, are functions of the variables* $x_1, x_2, \cdots, x_n$ *whose distribution law is given by* $f(x_1, x_2, \cdots, x_n)$ *which is on a certain* n*-dimensional manifold* R *a single valued, non-negative continuous function such that* $\int_R f(x_1, x_2, \cdots, x_n)\,dx_1\,dx_2\cdots dx_n = 1$, *then the characteristic function of the distribution law of* $u_1, u_2, \cdots, u_r$ *is given by*

$$\varphi(t_1, t_2, \cdots, t_r) = \int_R e^{it_1u_1(x_1,\cdots, x_n) + \cdots + it_ru_r(x_1\cdots x_n)} \cdot f(x_1, \cdots, x_n)\,dx_1\cdots dx_n\,.$$

THEOREM IV. *Under the conditions of Theorem III, the distribution law of* $u_1, u_2, \cdots, u_r$ *is given by*

$$P(u_1, u_2, \cdots, u_r) = \frac{1}{(2\pi)^r}\int_{-\infty}^{\infty} e^{-it_1u_1 - it_2u_2 - \cdots - it_ru_r} \cdot \varphi(t_1, \cdots, t_r)\,dt_1\cdots dt_r$$

where

$$\varphi(t_1, t_2, \cdots, t_r) = \int_R e^{it_1u_1(x_1,\cdots, x_n) + \cdots + it_ru_r(x_1,\cdots, x_n)} \cdot f(x_1, \cdots, x_n)\,dx_1\cdots dx_n\,.$$

PART 2

Various Special Cases of the Distribution Problem

V. Distribution of the arithmetic mean:[31] If we take $u(x_1,\cdots,x_n) = x_1 + x_2 + \cdots + x_n$ and assume that $x_1, x_2, \cdots\cdots, x_n$ are independently distributed each according to the same distribution law, then we find for the distribution of totals

$$(33)\quad P(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itu}\,dt\left(\int_a^b e^{itx} f(x)\,dx\right)^n, \quad a \leqq x \leqq b.$$

The substitution $u = n\bar{x}$ will then yield the distribution of the arithmetic mean.

This result has been derived previously by Poisson,[28] F. Hausdorff[11] and J. O. Irwin.[18]

Hausdorff applied it in particular to find the distribution of means of samples obeying the law $f(x) = 1/2$ for $-1 \leqq x \leqq 1$ and $f(x) = 0$ elsewhere (a rectangular universe); also to the law $f(x) = \frac{e^{-|x|}}{2}$, $-\infty \leqq x \leqq \infty$. Irwin has applied it to the normal law, Pearson Type III distribution, Pearson Type II distribution and a rectangular universe.

VI. Distribution of the geometric mean:[7]
Let $u = \log x_1 + \log x_2 + \cdots\cdots + \log x_n$ where x_j, $j = 1, 2, \cdots, n$ are distributed independently each according to the same distribution law, then

$$(34)\quad P(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itu}\,dt\left(\int_a^b x^{it} f(x)\,dx\right)^n, \quad 0 \leqq a \leqq x \leqq b.$$

The distribution for the geometric mean g is obtained from that of u by the transformation $u = \log g^n$.

a. Consider, for example, the case for $f(x) = \frac{1}{a}$, $0 \leqq x \leqq a$, Then

$$(35)\qquad P(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{it(n\log a - u)}}{(1+it)^n}\,dt,$$

where $n \log a - u \geqq 0$.

From (35) we have

$$(36)\qquad P(u) = \frac{(n\log a - u)^{n-1} e^{-(n\log a - u)}}{\Gamma n}$$

since* $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ibx}\,dx}{(1+ix)^n} = \frac{2\pi}{\Gamma n}\, b^{n-1} e^{-b}$ for $b > 0$

From (36) we obtain

$$(37)\qquad D(g)\,dg = \frac{n^n g^{n-1}}{a^n \Gamma n}\left(\log\frac{a}{g}\right)^{n-1} dg,\quad 0 \leqq g \leqq a.$$

The result for $n = 2, 3$ has been given by A. T. Craig.[7]

b. Suppose now that $f(x) = \dfrac{x^{p-1} e^{-x}}{\Gamma p}$, $\quad 0 \leqq x \leqq \infty.$

Then

$$(38)\qquad P(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itu}\left(\frac{\Gamma p+it}{\Gamma p}\right)^n dt$$

Let $p + it = -z$, then

$$(39)\qquad P(u) = \frac{e^{pu}}{(\Gamma p)^n\, 2\pi i}\int_{-p-i\infty}^{-p+i\infty} e^{uz}\left(\Gamma{-z}\right)^n dz.$$

By a method similar to that used for the case of the general-

* MacRobert,[20] p. 67.

ized variance (see Section XIII), we may show that

$$\lim_{z\to\infty} \left| z^m e^{uz} (\Gamma_{-z})^n \right| \to 0,$$

so that the integral converges and

$$(40) \qquad P(u) = -\frac{e^{pu}}{(\Gamma_p)^n 2\pi i} \int_C e^{zu} (\Gamma_{-z})^n dz,$$

where C is the contour bounded by the line $x = -p$ and that part of the circle $|z| = m + \frac{1}{2}$, $m \to \infty$ which lies to the right of the straight line. The contour is traversed in a counter-clockwise direction.

Now $(\Gamma_{-z})^n = \dfrac{(-1)^n \pi^n}{\sin^n \pi z (\Gamma_{z+1})^n}$ so that we may also write

$$(41) \qquad P(u) = -\frac{e^{pu}}{(\Gamma_p)^n 2\pi i} \int_C \frac{(-1)^n \pi^n e^{uz}}{\sin^n \pi z (\Gamma_{z+1})^n} dz.$$

The poles of the integrand are of the n^{th} order and are those of $(\Gamma_{-z})^n$ viz., $z = r$, $r = 0, 1, 2, \cdots$. Since the contour is traversed in a counter-clockwise manner, the value of the integral is $2\pi i$ times the sum of the residues at the poles within the contour so that

$$(42) \qquad P(u) = \frac{e^{pu}}{(\Gamma_p)^n} \sum_{r=0}^{\infty} \frac{(-1)^{n+nr+1}}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} \frac{e^{uz}}{(\Gamma_{z+1})^n} \right]_{z=r}.$$

or

$$(43) \qquad D(g) = \frac{n g^{np-1}}{\Gamma_n (\Gamma_p)^n} \sum_{r=0}^{\infty} (-1)^{n+nr+1} \left[\frac{d^{n-1}}{dz^{n-1}} \frac{g^{nz}}{(\Gamma_{z+1})^n} \right]_{z=r}$$

c. If instead of assuming the x_j's each satisfy the same distribution law, we assume x_j to be distributed according to

$$f(x_j) = \frac{x_j^{p_j - 1} e^{-x_j}}{\Gamma p_j}$$ where none of the p_j's are equal

or differ by an integer, then

$$(44) \qquad P(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itu} \frac{\prod_{j=1}^{n} \Gamma p_j + it}{\prod_{j=1}^{n} \Gamma p_j} \, dt,$$

or

$$(45) \qquad P(u) = \frac{1}{\prod_{j=1}^{n} \Gamma p_j \cdot 2\pi i} \int_{-\infty i}^{\infty i} e^{uz} \prod_{j=1}^{n} \Gamma p_j - z \, dz.$$

The same results as to the convergence of the integral and the contour may be shown with respect to this integrand as for Section VI b.

The value of

$$J = \int_C e^{uz} \prod_{j=1}^{n} \Gamma p_j - z \quad dz$$

is $2\pi i$ times the sum of the residues within the contour bounded by the y-axis and that part of the circle $|z| = m + \frac{1}{2}$, $m \to \infty$ which lies to the right of this line.

For the pole $z = p_j + r$, $r = 0, 1, 2, \cdots$ the residue is

$$(-1)^r \frac{e^{u(p_j + r)}}{r!} \Gamma p_1 - p_j - r \; \Gamma p_2 - p_j - r \cdots \Gamma p_{j-1} - p_j - r \; \Gamma p_{j+1} - p_j - r \cdots \Gamma p_n - p_j - r$$

therefore,

$$(46) \quad P(u) = \frac{1}{\prod_{j=1}^{n} \Gamma p_j} \left\{ \sum_{j=1}^{n} \sum_{r=0}^{\infty} \frac{(-1)^{r+1} e^{u(p_j + r)}}{r!} \prod_{K=1}^{n}{}^{j} \Gamma p_K - p_j - r \right\}$$

where $\prod_{K=1}^{n}{}^{j}$ means that in the product K takes all the values $1, 2, \cdots, n$ except j.

Finally,

$$(47) \quad D(g) = \frac{n}{\prod_{j=1}^{n} \Gamma p_j} \left\{ \sum_{j=1}^{n} \sum_{r=0}^{\infty} \frac{(-1)^{r+1} g^{n(p_j + r) - 1}}{r!} \prod_{K=1}^{n}{}^{j} \Gamma p_K - p_j - r \right\}$$

d. Suppose that in the previous case $p_j = p + \frac{j-1}{n}$, $j = 1, 2, \cdots, n.$

Since $\Gamma_p \Gamma_{p+\frac{1}{n}} \cdots \Gamma_{p+\frac{n-1}{n}} = n^{\frac{1}{2}-np} (2\pi)^{\frac{n-1}{2}} \Gamma_{np}$
for this case

$$(48) \quad P(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itu}\, n^{\frac{1}{2}-n(p+it)} (2\pi)^{\frac{n-1}{2}} \Gamma_{np+nit}}{n^{\frac{1}{2}-np} (2\pi)^{\frac{n-1}{2}} \Gamma_{np}} \, dt.$$

Let $np + nit = -z$, then

$$(49) \quad P(u) = \frac{e^{up}\, n^{np}}{n \Gamma_{np} \cdot 2\pi i} \int_{-np-i\infty}^{-np+i\infty} \left(e^{\frac{u}{n}} \cdot n\right)^{z} \Gamma_{-z} \, dz.$$

Now it may be shown that

$$\frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} u^{z} \Gamma_{-z} \, dz = e^{-u}$$

where $a > 0$ and $-\frac{\pi}{2} < \operatorname{amp} u < \frac{\pi}{2}$ (See MacRobert,[20] p. 151.)

Therefore

$$(50) \qquad P(u) = \frac{e^{up}\, n^{np}}{n \Gamma_{np}} \, e^{-ne^{\frac{u}{n}}}$$

Substituting $g^n = e^u$ we obtain for the distribution of g

$$(51) \qquad D(g) = \frac{n^{np}\, g^{np-1}\, e^{-ng}}{\Gamma_{np}}, \quad 0 \leq g \leq \infty.$$

In other words, the distribution of the geometric mean of n independent variables respectively satisfying the distribution law

$$\frac{x^{p-1} e^{-x}}{\Gamma_p}; \quad \frac{x^{p+\frac{1}{n}-1} e^{-x}}{\Gamma_{p+\frac{1}{n}}}; \cdots; \quad \frac{x^{p+\frac{n-1}{n}-1} e^{-x}}{\Gamma_{p+\frac{n-1}{n}}}, \quad 0 \leq x \leq \infty$$

is the same as the distribution of the arithmetic mean of n independent variables each satisfying the Pearson Type III distribution law

$$f(x) = \frac{x^{p-1} e^{-x}}{\Gamma p}, \quad 0 \leq x \leq \infty .$$

e. For the case where $p_j = \frac{N-j}{2}$, $j = 1, 2, \cdots, n$, see the discussion for the generalized variance (see Section XIII).

VII. Lemma: The following geometrical considerations will for certain cases simplify the problem of finding the distribution of statistical parameters calculated about a sample mean.

Consider the sample as a point or points (for multi-variate distributions) in an n-dimensional Euclidean space. (This method has been employed to great advantage by R. A. Fisher[8] and others.) Then, if the probability density at any point (the probability for that particular combination of values to occur) is a function of the distance from the origin, the mean value of a function of the distance from the origin and of other geometric invariants of the system for $x_j, y_j, \cdots$ $j = 1, 2, \cdots, n$ satisfying the conditions $\sum_{j=1}^{n} x_j = 0$, $\sum_{j=1}^{n} y_j = 0$, ... will be the same as for the same function for independent variables in $n-1$ dimensional space. Since the important element is the distance from the origin and the integration is to be carried out over an $n-1$ dimensional space, the final result is independent of the fact that the whole system is immersed in an n-dimensional space.

As an illustration, let us consider the following distributions which have been derived by various methods.

VIII. Distribution of variance of a sample of n from a normal population:[8, 23, 34, 36]

Let $\quad u = x_1^2 + x_2^2 + \cdots\cdots + x_{n-1}^2$ where the

x_j are distributed according to $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$, $-\infty \leq x \leq \infty$.

Then

$$(52) \qquad \varphi(t) = \left[\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2} + itx^2} dx \right]^{n-1} = \frac{1}{(1 - 2\sigma^2 it)^{\frac{n-1}{2}}} .$$

(Compare Rider,[31] Annals p. 600; Romanovsky,[34] Metron p. 6.) Therefore, the distribution of

$$V = x_1^2 + x_2^2 + \cdots + x_n^2 = n s^2, \quad \text{where } \sum_{j=1}^{n} x_j = 0,$$

is given by

$$(53) \qquad P(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itv}\,dt}{(1-2\sigma^2 it)^{\frac{n-1}{2}}} = \frac{v^{\frac{n-3}{2}}\, e^{-\frac{v}{2\sigma^2}}}{(2\sigma^2)^{\frac{n-1}{2}}\,\Gamma\left(\frac{n-1}{2}\right)}$$

(see MacRobert,[20] p. 67.)

We thus have

$$(54) \qquad D(s^2)\,ds^2 = \left(\frac{n}{2\sigma^2}\right)^{\frac{n-1}{2}} \frac{(s^2)^{\frac{n-3}{2}}\, e^{-\frac{ns^2}{2\sigma^2}}\, ds^2}{\Gamma\left(\frac{n-1}{2}\right)} \text{ as is well known.}$$

IX. Distribution of the χ^2 of Goodness of Fit Test:[22, 24] Consider

$$\chi^2 = \sum_{j,k=1}^{n} \frac{R_{jk}}{R}\,\frac{x_j x_k}{\sigma_j \sigma_k},$$

where $R = |\rho_{jk}|$, $\rho_{jj} = 1$ and R_{jk} is the cofactor of ρ_{jk} in R so that[1] $|R_{jk}| = R^{n-1}$ and $x_1, x_2, \cdots, x_n$ are distributed according to

$$\frac{e^{-\frac{\chi^2}{2}}}{(2\pi)^{n/2}\sigma_1\sigma_2\cdots\sigma_n R^{1/2}}.$$

Therefore

$$(55) \qquad P(\chi^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\chi^2} dt \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2R}\sum\limits_{j,k=1}^{n} \frac{R_{jk}(1-2it)\,x_j x_k}{\sigma_j\sigma_k}}\,dx_1\,dx_2\cdots dx_n}{(2\pi)^{n/2}\sigma_1\sigma_2\cdots\sigma_n R^{1/2}}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\chi^2} dt \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\sum\limits_{j,k=1}^{n} R_{jk}(1-2it)\,x_j x_k}\,dx_1\cdot dx_2\cdots dx_n}{(2\pi)^{n/2} R^{-\frac{n-1}{2}}}.$$

$$\therefore \quad P(\chi^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\chi^2} \frac{|R_{jk}|^{1/2}}{|R_{jk}(1-2it)|^{1/2}}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-it\chi^2}\, dt}{(1-2it)^{n/2}}$$

and we have finally,

$$(56) \qquad P(\chi^2)\, d\chi^2 = \frac{1}{\Gamma\frac{n}{2}} \left(\frac{\chi^2}{2}\right)^{\frac{n-2}{2}} e^{-\frac{\chi^2}{2}}\, d\left(\frac{\chi^2}{2}\right).$$

If we restrict the x_j in χ^2 to satisfy $\sum_{j=1}^{n} x_j = 0$, then from the preceding, it is clear that $\varphi(t) = \frac{1}{(1-2it)^{\frac{n-1}{2}}}$ and now

$$(57) \qquad P(\chi^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-it\chi^2}\, dt}{(1-2it)^{\frac{n-1}{2}}}$$

or

$$(58) \qquad P(\chi^2)\, d\chi^2 = \frac{1}{\Gamma\frac{n-1}{2}} \left(\frac{\chi^2}{2}\right)^{\frac{n-3}{2}} e^{-\frac{\chi^2}{2}}\, d\left(\frac{\chi^2}{2}\right).$$

This latter case is the one commonly met with in actual practice and is equivalent to the case wherein the expected values are adjusted according to the total in the sample.

X. Simultaneous distribution of variances and correlation coefficient of a sample of n from a bi-variate normal population:[8] This is a special case of the problem of finding the simultaneous distribution of the variances and covariances from an n-variate normal population which has been solved by J. Wishart.[42] The same method is applicable to the general case, but for its own interest and for the sake of simplicity this special case will be considered.

Let $u_1 = \dfrac{\sum_{j=1}^{n} x_j^2}{2(1-\rho^2)\sigma_x^2}$; $u_2 = \dfrac{\rho \sum_{j=1}^{n} x_j y_j}{(1-\rho^2)\sigma_x \sigma_y}$; $u_3 = \dfrac{\sum_{j=1}^{n} y_j^2}{2(1-\rho^2)\sigma_y^2}$.

where x_j and y_j are distributed according to

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x_j^2}{\sigma_x^2} - 2\rho\frac{x_j y_j}{\sigma_x\sigma_y} + \frac{y_j^2}{\sigma_y^2}\right]}$$

Now consider

$$(59)\quad J = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(1-it_1)x^2}{\sigma_x^2} - \frac{2\rho(1+it_2)xy}{\sigma_x\sigma_y} + \frac{(1-it_3)y^2}{\sigma_y^2}\right]}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, dx\, dy$$

$$= \frac{(1-\rho^2)^{1/2}}{\left[(1-it_1)(1-it_3) - \rho^2(1+it_2)^2\right]^{1/2}}$$

$$= \frac{(1-\rho^2)^{1/2}}{\begin{vmatrix} 1-it_1 & \rho(1+it_2) \\ \rho(1+it_2) & 1-it_3 \end{vmatrix}^{1/2}}$$

Therefore, if we add the conditions $\sum_{j=1}^{n} x_j = 0$; $\sum_{j=1}^{n} y_j = 0$, in which case

$u_1 = \dfrac{n\, s_x^2}{2(1-\rho^2)\sigma_x^2}$; $u_2 = \dfrac{n\rho r\, s_x s_y}{(1-\rho^2)\sigma_x\sigma_y}$; $u_3 = \dfrac{n\, s_y^2}{2(1-\rho^2)\sigma_y^2}$ and

$$(60)\qquad g(t_1 t_2 t_3) = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\left[(1-it_1)(1-it_3) - \rho^2(1+it_2)^2\right]^{\frac{n-1}{2}}}.$$

Therefore

$$(61)\quad P(u_1 u_2 u_3) = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{-it_1u_1 - it_2u_2 - it_3u_3}}{\left[(1-it_1)(1-it_3) - \rho^2(1+it_2)^2\right]^{\frac{n-1}{2}}}\, dt_1\, dt_2\, dt_3 .$$

Integrating with respect to t_1, we find

$$(62)\quad \int_{-\infty}^{\infty} \frac{e^{-it_1u_1}\,dt_1}{\left[(1-it_1)(1-it_3)-\rho^2(1+it_2)^2\right]^{\frac{n-1}{2}}} = \frac{2\pi}{\Gamma\frac{n-1}{2}}\;\frac{u_1^{\frac{n-3}{2}}\,e^{-u_1\left[1-\frac{\rho^2(1+it_2)^2}{1-it_3}\right]}}{(1-it_3)^{\frac{n-1}{2}}}.$$

Integrating with respect to t_2, we find

$$(63)\quad \int_{-\infty}^{\infty} e^{-t_2^2\frac{u_1\rho^2}{1-it_3}-it_2\left(u_2-\frac{2u_1\rho^2}{1-it_3}\right)}\,dt_2 = \sqrt{\frac{\pi(1-it_3)}{u_1\rho^2}}\;e^{-\left(u_2-\frac{2u_1\rho^2}{1-it_3}\right)^2\frac{1-it_3}{4u_1\rho^2}}$$

Integrating with respect to t_3, we find

$$(64)\quad \int_{-\infty}^{\infty} \frac{e^{-it_3\left(u_3-\frac{u_2^2}{4u_1\rho^2}\right)}\,dt_3}{(1-it_3)^{\frac{n-2}{2}}} = \frac{2\pi}{\Gamma\frac{n-2}{2}}\left(u_3-\frac{u_2^2}{4u_1\rho^2}\right)^{\frac{n-4}{2}}\cdot e^{-\left(u_3-\frac{u_2^2}{4u_1\rho^2}\right)}$$

using the facts that $\displaystyle\int_{-\infty}^{\infty} e^{-ax^2\pm 2bx}\,dx = \sqrt{\frac{\pi}{a}}\;e^{b^2/a}$

and $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-ibx}\,dx}{(1-ix)^n} = \frac{2\pi}{\Gamma n}\,b^{n-1}e^{-b}.$

Therefore we finally find that

$$(65)\quad P(u_1,u_2,u_3) = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{2\rho\sqrt{\pi}\,\Gamma\frac{n-1}{2}\,\Gamma\frac{n-2}{2}}\,e^{-(u_1-u_2+u_3)}\left(1-\frac{u_2^2}{4u_1u_3\rho^2}\right)^{\frac{n-4}{2}}u_1^{\frac{n-4}{2}}\,u_3^{\frac{n-4}{2}},$$

or

$$(66)\quad D(S_x,r,S_y)\,dS_x\,dr\,dS_y = \frac{n^{n-1}(1-r^2)^{\frac{n-4}{2}}S_x^{n-2}S_y^{n-2}\,e^{-\frac{n}{2(1-\rho^2)}\left[\frac{S_x^2}{\sigma_x^2}-\frac{2\rho r S_xS_y}{\sigma_x\sigma_y}+\frac{S_y^2}{\sigma_y^2}\right]}}{(1-\rho^2)^{\frac{n-1}{2}}\,\pi\,\Gamma_{n-2}\,\sigma_x^{n-1}\,\sigma_y^{n-1}}$$

since $\displaystyle 2^{n-3}\,\Gamma\frac{n-1}{2}\,\Gamma\frac{n-2}{2} = \pi^{1/2}\,\Gamma_{n-2}$

and $\displaystyle\frac{\partial(u_1,u_2,u_3)}{\partial(S_x,r,S_y)} = \frac{n^3\rho\,S_x^2\,S_y^2}{(1-\rho^2)^3\sigma_x^3\sigma_y^3}.$

X. The distribution of the covariance of a sample of n from a bi-variate normal population:[23]

Let $$u = \frac{\rho}{(1-\rho^2)\sigma_x\sigma_y}\sum_{j=1}^{n} x_j y_j$$

where x_j and y_j are distributed according to

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x_j^2}{\sigma_x^2} - 2\frac{\rho x_j y_j}{\sigma_x\sigma_y} + \frac{y_j^2}{\sigma_y^2}\right]}$$

Consider

(67) $$J = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_x^2} - 2\frac{(1+it)\rho xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right]}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, dx\, dy$$

$$= \frac{(1-\rho^2)^{1/2}}{[1-\rho^2(1+it)^2]^{1/2}}.$$

If we impose the conditions

$$\sum_{j=1}^{n} x_j = 0 ; \quad \sum_{j=1}^{n} y_j = 0 \qquad \text{so that } u = \frac{n\rho S_{xy}}{(1-\rho^2)\sigma_x\sigma_y}$$

then

(68) $$\varphi(t) = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{[1-\rho^2(1+it)^2]^{\frac{n-1}{2}}}$$

and

(69) $$P(u) = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itu}\, dt}{[1-\rho^2(1+it)^2]^{\frac{n-1}{2}}}$$

Consider

(70) $$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itu}\, dt}{\left\{[1-\rho(1+it)][1+\rho(1+it)]\right\}^{\frac{n-1}{2}}}$$

Let $\quad 1-\rho(1+it) = -\frac{\rho z}{u}$ so that

(71) $$I = \frac{u^{\frac{n-3}{2}} e^{\frac{u(\rho-1)}{\rho}}}{(2\rho)^{\frac{n-1}{2}} 2\pi i} \int_{-\frac{1-\rho}{\rho}u-i\infty}^{-\frac{1-\rho}{\rho}u+i\infty} \frac{e^{-z}\,dz}{(-z)^{\frac{n-1}{2}}\left(1+\frac{z\rho}{2u}\right)^{\frac{n-1}{2}}}$$

Since we may show that

$$\lim_{z\to\infty} \left| z^m \frac{e^{-z}}{(-z)^{\frac{n-1}{2}}\left(1+\frac{z\rho}{2u}\right)^{\frac{n-1}{2}}} \right| \to 0$$

the integral is convergent and we may write

(72) $$I = -\frac{u^{\frac{n-3}{2}} e^{u\frac{\rho-1}{\rho}}}{(2\rho)^{\frac{n-1}{2}} 2\pi i} \int_{\infty}^{(0+)} \frac{e^{-z}\,dz}{(-z)^{\frac{n-1}{2}}\left(1+\frac{z\rho}{2u}\right)^{\frac{n-1}{2}}},$$

where $\int_{\infty}^{(0+)}$ means that the path of integration starts at infinity on the real axis, encircles the origin in the positive direction and returns to the starting point. (See Whittaker and Watson,[40] pp. 239, 333.)

Since $\frac{1-\rho}{\rho} < \frac{2}{\rho}$, the point $z = -\frac{2}{u\rho}$ is outside the contour so that

(73) $$-\frac{1}{2\pi i}\int_{\infty}^{(0+)} \frac{e^{-z}\,dz}{(-z)^{\frac{n-1}{2}}\left(1+\frac{z\rho}{2u}\right)^{\frac{n-1}{2}}} = \frac{e^{\frac{u}{\rho}}\, W_{0,-\frac{n-2}{2}}\left(\frac{2u}{\rho}\right)}{\Gamma\frac{n-1}{2}},$$

where $W_{K,m}(z)$ is the confluent hypergeometric function.[40]

Also, since $W_{0,m}(z) = W_{0,-m}(z)$ we have finally

(74) $$P_{(u)} = \frac{(1-\rho^2)^{\frac{n-1}{2}} \cdot u^{\frac{n-3}{2}} \cdot e^{u}\, W_{0,\frac{n-2}{2}}\left(\frac{2u}{\rho}\right)}{\Gamma\frac{n-1}{2}\;(2\rho)^{\frac{n-1}{2}}}$$

If we start with the following definition for the Bessel Function of the second kind and imaginary argument[26, 37]

$$(75) \qquad K_m(x) = \frac{\sqrt{\pi}\, x^m}{2^m \Gamma(m+\frac{1}{2})} \int_1^\infty e^{-xt} (t^2-1)^{m-\frac{1}{2}} dt \cdot$$

then it is possible to show that $K_m(x) = \sqrt{\pi}\, x^{-\frac{1}{2}} 2^{-\frac{1}{2}} W_{0,m}(2x)$, so that

$$(76) \qquad P(u) = \frac{(1-\rho^2)^{\frac{n-1}{2}} u^{\frac{n-2}{2}} e^{u} K_{\frac{n-2}{2}}\left(\frac{u}{\rho}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-2}{2}} \rho^{\frac{n}{2}}} .$$

If we finally set $v = \frac{u}{\rho} = \frac{n \cdot S_{xy}}{(1-\rho^2)\sigma_x \sigma_y}$,

we find for the distribution of v,

$$(77) \qquad D(v)\,dv = \frac{(1-\rho^2)^{\frac{n-1}{2}} e^{\rho v} v^{\frac{n-2}{2}} K_{\frac{n-2}{2}}(v)\,dv}{\sqrt{\pi}\, 2^{\frac{n-2}{2}} \Gamma\left(\frac{n-1}{2}\right)},$$

which is the form found by K. Pearson, G. B. Jeffery, F.R. S. and E. M. Elderton.[25]

XII. Do N samples, each of n-categories, come from the same n-variate normal parent?[30] Consider

$$\chi^2 = \sum_{j=1}^{N} \sum_{j,k=1}^{n} \frac{R_{jk}}{R} \cdot \frac{x_{j\alpha}\, x_{k\alpha}}{\sigma_j\, \sigma_k}$$

where the simultaneous distribution of $x_1, x_2, \cdots, x_n$ is given by

$$(78) \qquad \frac{e^{-\frac{1}{2R}\sum_{j,k=1}^{n} R_{jk} \frac{x_j x_k}{\sigma_j \sigma_k}}}{(2\pi)^{n/2} \sigma_1 \sigma_2 \cdots \sigma_n R^{1/2}},$$

where R_{jk} denotes the cofactor corresponding to ρ_{jk} in the determinant $R = |\rho_{jk}|$ of the population correlations and σ_j is the standard deviation of the j^{th} variate.

Consider

$$J = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2R}\sum_{j,k=1}^{n}(1-2it)\frac{R_{jk}x_j x_k}{\sigma_j \sigma_k}}\, dx_1\, dx_2 \cdots dx_n}{(2\pi)^{n/2}\sigma_1 \sigma_2 \cdots \sigma_n R^{1/2}}$$

$$= \frac{|R_{jk}|^{1/2}}{|R_{jk}(1-2it)|^{1/2}} = \frac{1}{(1-2it)^{n/2}}$$

If we impose the conditions $\sum_{j=1}^{n} x_{j\alpha} = 0, \quad \alpha = 1, 2, \cdots, N$ and $\sum_{\alpha=1}^{N} x_{j\alpha} = 0, \quad j = 1, 2, \cdots, n$ then from the previous results the characteristic function for the distribution of χ^2 becomes

$$\varphi(t) = \frac{1}{(1-2it)^{\frac{(n-1)(N-1)}{2}}}$$

and the distribution for χ^2 is

$$(79) \quad P(\chi^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-it\chi^2}\, dt}{(1-2it)^{\frac{(n-1)(N-1)}{2}}} = \frac{(\chi^2)^{\frac{(n-1)(N-1)-2}{2}} e^{-\frac{\chi^2}{2}}}{2^{\frac{(n-1)(N-1)}{2}} \Gamma\left(\frac{(n-1)(N-1)}{2}\right)}$$

This case is equivalent to applying the χ^2 test to a contingency table. If the table has r rows and c columns then the value of n' to be used in Elderton's tables of "Goodness of Fit" is[9] $n' = (r-1)(c-1)+1$ [as we saw in Section IX, equation 58, the distribution for χ^2 has an exponent $\frac{n-3}{2}$ (our n is equal to the n' of the table) and the exponent in the distribution above is $\frac{(n-1)(N-1)-2}{2}$].

XIII. Distribution of the generalized variance of a sample of N from an n-variate normal population:[41] One of the generalizations considered by Wilks is that of the sample variance. For a sample of N from an n variate normal population the generalized sample variance is defined to be the determinant $|a_{jk}|$ where $a_{jk} = a_{kj} = \frac{1}{N}\sum_{\alpha=1}^{N}(x_{j\alpha}-\bar{x}_j)(x_{k\alpha}-\bar{x}_k)$, $j,k=1,2,\dots,n$ and $\bar{x}_j = \frac{1}{N}\sum_{\alpha=1}^{N} x_{j\alpha}$. Wilks has given the distribution of $u = |a_{jk}|$ as an $(n-1)$-tuple integral and has obtained the explicit form of the distribution for $n = 1, 2$.

By employing the theory of characteristic functions we are enabled to express the distribution of u as a single integral and find the explicit form for any value of n.

The simultaneous distribution of the a_{jk} defined above is given[42] by

$$(80)\qquad \frac{|A_{jk}|^{\frac{N-1}{2}}\, e^{-\sum_{j,k=1}^{n} A_{jk}a_{jk}}\, |a_{jk}|^{\frac{N-n-2}{2}}}{\pi^{\frac{n(n-1)}{4}}\prod_{j=1}^{n}\Gamma\left(\frac{N-j}{2}\right)}.$$

where $|A_{jk}|$ is the n-th order determinant of elements $A_{jk} = \frac{N R_{jk}}{2\sigma_j\sigma_k R}$, where R_{jk} is the cofactor of ρ_{jk} in the determinant of parent correlations $R = |\rho_{jk}|$.

If we write $A_{jk} = N B_{jk}$ and $a_{jk} = \frac{b_{jk}}{N}$, the distribution of the b_{jk}'s is

$$(81)\qquad \frac{|B_{jk}|^{\frac{N-1}{2}}\, e^{-\sum_{j,k=1}^{n} B_{jk}b_{jk}}\, |b_{jk}|^{\frac{N-n-2}{2}}}{\pi^{\frac{n(n-1)}{4}}\prod_{j=1}^{n}\Gamma\left(\frac{N-j}{2}\right)}$$

For the sake of concreteness and the better to follow the discussion for the general case, we shall first consider the cases $n = 3, 4$ in detail.

Case 1, $n = 3$: Let $\xi = \log b$ where we write $b = |b_{jk}|$. The distribution of ξ is then given by

$$(82) \quad P(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\xi}\,dt \int \frac{|B_{jk}|^{\frac{N-1}{2}} e^{-\sum_{j,k=1}^{n} B_{jk} b_{jk}}\, b^{\frac{N-5+2it}{2}}}{\pi^{3/2}\,\Gamma\frac{N-1}{2}\,\Gamma\frac{N-2}{2}\,\Gamma\frac{N-3}{2}}\,db_n$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-it\xi} B^{-it}\,\Gamma\left(\frac{N-1}{2}+it\right)\Gamma\left(\frac{N-2}{2}+it\right)\Gamma\left(\frac{N-3}{2}+it\right)}{\Gamma\frac{N-1}{2}\,\Gamma\frac{N-2}{2}\,\Gamma\frac{N-3}{2}}\,dt,$$

where $B = |B_{jk}|$. (Compare Wilks,[41] Biometrika Vol. 24, p. 477, equation 10.)

Let $\frac{N-3}{2} + it = -z$, then

$$(83) \quad P(\xi) = \frac{(Be^{\xi})^{\frac{N-3}{2}}}{\Gamma\frac{N-1}{2}\,\Gamma\frac{N-2}{2}\,\Gamma\frac{N-3}{2}\;2\pi i}\int_{-\frac{N-3}{2}-i\infty}^{-\frac{N-3}{2}+i\infty} e^{\xi z} B^{z}\,\Gamma(1-z)\,\Gamma\left(\tfrac{1}{2}-z\right)\Gamma(-z)\,dz.$$

The integral is taken along the line $x = -\frac{N-3}{2}$ and since $N > 3$ (since otherwise the distribution of the a_{jk}'s is nugatory) all the poles of the integrand are to the right of the line $x = -\frac{N-3}{2}$.

Now $\Gamma(1-z) = -z\,\Gamma(-z)$ so that $\Gamma(1-z)\,\Gamma(\frac{1}{2}-z)\,\Gamma(-z) = -z(\Gamma(-z))^2\,\Gamma(\frac{1}{2}-z)$

but $\Gamma(-z) = -\dfrac{\pi}{\sin \pi z\,\Gamma(1+z)}$ so that $(\Gamma(-z))^2 = \dfrac{\pi^2}{\sin^2 \pi z \cdot (\Gamma(1+z))^2}$.

Now
$$\lim_{z\to\infty}\left|\frac{1}{(\Gamma_{z+1})^2}\right| = \lim_{z\to\infty}\left|\frac{e^{2z-2z\log z}}{2\pi z^3}\right|$$

If we set $z = re^{i\theta}$

$$\lim_{z\to\infty}\left|\frac{1}{(\Gamma_{z+1})^2}\right| = \lim_{r\to\infty}\frac{e^{2r\cos\theta-2r\cos\theta\log r+2r\theta\sin\theta}}{2\pi r^3}$$

Also
$$\Gamma_{\frac{1}{2}-z} = \frac{\pi}{\cos\pi z\,\Gamma_{z+\frac{1}{2}}}$$

and
$$\lim_{z\to\infty}\left|\frac{1}{\Gamma_{z+\frac{1}{2}}}\right| = \lim_{r\to\infty}\frac{e^{r\cos\theta-r\cos\theta\log r+\theta r\sin\theta}}{\pi^{1/2}\,2^{3/2}}$$

We also have that $\lim_{z\to\infty}|\sin\pi z| \leqq \lim_{z\to\infty} e^{\pm\pi r\sin\theta}$

according as $\sin\theta$ is positive or negative and that

$$\lim_{z\to\infty}|\cos\pi z| \leqq \lim_{r\to\infty} e^{\pm\pi r\sin\theta}$$

according as $\sin\theta$ is positive or negative.

We find therefore that finally,

$$\lim_{z\to\infty}\left|e^{\xi z}B^z\Gamma_{1-z}\Gamma_{\frac{1}{2}-z}\Gamma_{-z}\right| \leqq \sqrt{2\pi}\lim_{r\to\infty}\frac{e^{r\cos\theta[\xi+\log B+3-3\log r]+r\sin\theta[3\theta\mp3\pi]}}{r^2}$$

according as $\sin\theta$ is positive or negative.

Therefore if $\frac{\pi}{2} \geqq \theta \geqq \epsilon$; $-\frac{\pi}{2} \leqq \theta \leqq -\epsilon$; or if

$$-\frac{N-3}{2} \leqq r\cos\theta \leqq 0,$$

$z^n e^{\xi z} B^z \Gamma_{1-z}\Gamma_{\frac{1}{2}-z}\Gamma_{-z}$ tends uniformly to zero as z tends to infinity and the integral is uniformly convergent.*

Next if $-\epsilon \leqq \theta \leqq \epsilon$ let $z = r_m e^{i\theta}$ where $r_m = m + \frac{1}{2}$ and m is an integer. Then,

$$\lim_{z\to\infty} \left| e^{\xi z} B^{z} \Gamma_{1-z} \Gamma_{\frac{1}{2}-z} \Gamma_{-z} \right| \leqq M_1^2 M_2 \lim_{m\to\infty} \frac{e^{r_m \cos\epsilon[\xi + \log B + 3 - 3\log r_m] + r_m \sin\epsilon|\theta|}}{r_m^2}$$

where $2M_1 \geqq |\csc \pi z|$; $2M_2 \geqq |\sec \pi z|$ **

Therefore $z^n e^{\xi z} B^{z} \Gamma_{1-z} \Gamma_{\frac{1}{2}-z} \Gamma_{-z}$ tends to zero uniformly as m tends to infinity.

We can now write

$$(84) \quad P(\xi) = -\frac{(Be^{\xi})^{\frac{N-3}{2}}}{\Gamma_{\frac{N-1}{2}} \Gamma_{\frac{N-2}{2}} \Gamma_{\frac{N-3}{2}} 2\pi i} \int_C e^{\xi z} B^{z} \Gamma_{1-z} \Gamma_{\frac{1}{2}-z} \Gamma_{-z} \, dz,$$

where C is the contour bounded by the line $x = -\frac{N-3}{2}$ and that part of the circle $|z| = m + \frac{1}{2}$, where m may be increased indefinitely, which lies to the right of this line; the contour is traversed in a counter-clockwise direction.

The value of $J = \int_C e^{\xi z} B^{z} \Gamma_{1-z} \Gamma_{\frac{1}{2}-z} \Gamma_{-z} \, dz$ is $2\pi i$ times the sum of the residues at the poles within the contour C.

For $z=0$ there is a simple pole at which the residue is $\Gamma_{\frac{1}{2}} = \pi^{1/2}$

For $z = \frac{1}{2} + r$, $r = 0, 1, 2, \ldots$, there is a simple pole at which the

* MacRobert,[20] p. 139, Rule II.
** MacRobert,[20] p. 114 Lemma.

residue is

$$\frac{(-1)^{n+1}\,\pi^{2}\,e^{\xi(n+\frac{1}{2})}\,B^{n+\frac{1}{2}}}{n!\,\Gamma(n+\frac{1}{2})\,\Gamma(n+\frac{3}{2})}$$

since $\Gamma(-z) = -\dfrac{\pi}{\sin \pi z\,\Gamma(1+z)}$; $\Gamma(1-z) = \dfrac{\pi}{\sin \pi z\,\Gamma(z)}$; $\Gamma(\frac{1}{2}-z) = \dfrac{\pi}{\cos \pi z\,\Gamma(\frac{1}{2}+z)}$;

and the residue of $\dfrac{\pi}{\cos \pi z\,\Gamma(\frac{1}{2}+z)}$ for $z = \frac{1}{2}+n$ is equal to $\dfrac{(-1)^{n}}{\Gamma(n+1)}$.

For $z = n$ where n is an integer other than zero, the integrand has a pole of the second order, viz., that of $\Gamma(1-z)\,\Gamma(-z)$ so that the residue is

$$-\pi\left[\frac{d}{dz}\,\frac{e^{\xi z}\,B^{z}}{\cos \pi z\,\Gamma(z)\,\Gamma(z+\frac{1}{2})\,\Gamma(z+1)}\right]_{z=n}$$

Finally we have

$$(85)\quad P(\xi) = \frac{(Be^{\xi})^{\frac{N-3}{2}}}{\Gamma(\frac{N-1}{2})\,\Gamma(\frac{N-2}{2})\,\Gamma(\frac{N-3}{2})}\left\{-\pi^{1/2}+\pi^{2}(e^{\xi}B)^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^{n}(e^{\xi}B)^{n}}{n!\,\Gamma(n+\frac{1}{2})\,\Gamma(n+\frac{3}{2})}+\pi\sum_{n=1}^{\infty}\left[\frac{d}{dz}\,\frac{(e^{\xi}B)^{z}}{\cos \pi z\,\Gamma(z)\,\Gamma(z+\frac{1}{2})\,\Gamma(z+1)}\right]_{z=n}\right\}.$$

If we make the substitutions $\xi = \log b = \log a N^{3}$ where $a = |a_{ij}|$ and $B = \dfrac{A}{N^{3}}$ where $A = |A_{jk}|$ we have for the distribution of a

$$(86)\quad D(a)\,da = \frac{da}{a}\,\frac{(aA)^{\frac{N-3}{2}}}{\Gamma\!\left(\frac{N-1}{2}\right)\Gamma\!\left(\frac{N-2}{2}\right)\Gamma\!\left(\frac{N-3}{2}\right)}\left\{-\pi^{1/2}+\pi^{2}(aA)^{1/2}\sum_{r=0}^{\infty}\frac{(-1)^{r}(aA)^{r}}{r!\,\Gamma\!\left(r+\frac{1}{2}\right)\Gamma\!\left(r+\frac{3}{2}\right)}+\pi\sum_{r=1}^{\infty}\left[\frac{d}{dz}\,\frac{(aA)^{z}}{\cos\pi z\,\Gamma(z)\,\Gamma\!\left(z+\frac{1}{2}\right)\Gamma(z+1)}\right]_{z=r}\right\}$$

or

$$(87)\quad D(a) = \frac{A^{\frac{N-3}{2}}\,a^{\frac{N-5}{2}}}{\Gamma\!\left(\frac{N-1}{2}\right)\Gamma\!\left(\frac{N-2}{2}\right)\Gamma\!\left(\frac{N-3}{2}\right)}\left\{-\pi^{1/2}+\pi^{2}(aA)^{1/2}\sum_{r=0}^{\infty}\frac{(-1)^{r}(aA)^{r}}{r!\,\Gamma\!\left(r+\frac{1}{2}\right)\Gamma\!\left(r+\frac{3}{2}\right)}+\pi\sum_{r=1}^{\infty}\left[\frac{d}{dz}\,\frac{(aA)^{z}}{\cos\pi z\,\Gamma(z)\,\Gamma\!\left(z+\frac{1}{2}\right)\Gamma(z+1)}\right]_{z=r}\right\}$$

Case 2, $n=4$: With the same notation as before, we find that

$$(88)\quad P(\xi) = \frac{1}{\Gamma\!\left(\frac{N-1}{2}\right)\Gamma\!\left(\frac{N-2}{2}\right)\Gamma\!\left(\frac{N-3}{2}\right)\Gamma\!\left(\frac{N-4}{2}\right)2\pi}\int_{-\infty}^{\infty} e^{-it\xi}\,B^{-it}\,\Gamma\!\left(\tfrac{N-1}{2}+it\right)\Gamma\!\left(\tfrac{N-2}{2}+it\right)\Gamma\!\left(\tfrac{N-3}{2}+it\right)\Gamma\!\left(\tfrac{N-4}{2}+it\right)dt.$$

Let $\frac{N-4}{2}+it=-z$ so that

$$(89)\quad P(\xi) = \frac{(Be^{\xi})^{\frac{N-4}{2}}}{\Gamma\!\left(\frac{N-1}{2}\right)\Gamma\!\left(\frac{N-2}{2}\right)\Gamma\!\left(\frac{N-3}{2}\right)\Gamma\!\left(\frac{N-4}{2}\right)2\pi}\int_{-\frac{N-4}{2}-i\infty}^{-\frac{N-4}{2}+i\infty} e^{\xi z}\,B^{z}\,\Gamma\!\left(\tfrac{3}{2}-z\right)\Gamma(1-z)\,\Gamma\!\left(\tfrac{1}{2}-z\right)\Gamma(-z)\,dz.$$

A similar discussion as for the case $n=3$ applies here with regard to the convergence and we can write here too,

$$(90)\quad P(\xi) = -\frac{(Be^{\xi})^{\frac{N-4}{2}}}{\Gamma\!\left(\frac{N-1}{2}\right)\Gamma\!\left(\frac{N-2}{2}\right)\Gamma\!\left(\frac{N-3}{2}\right)\Gamma\!\left(\frac{N-4}{2}\right)\cdot 2\pi i}\int_{C} e^{\xi z}\,B^{z}\,\Gamma\!\left(\tfrac{3}{2}-z\right)\Gamma(1-z)\,\Gamma\!\left(\tfrac{1}{2}-z\right)\Gamma(-z)\,dz,$$

where the contour C is bounded by the line $x = -\frac{N-4}{2}$, $(N>4)$ and that part of the circle $|z| = m+\frac{1}{2}$ ·, where m may be increased indefinitely, which lies to the right of this line. The contour is traversed in a counter-clockwise direction.

The value of $J = \int_C e^{\xi z} B^z \Gamma\left(\tfrac{3}{2}-z\right) \Gamma(1-z)\, \Gamma\left(\tfrac{1}{2}-z\right) \Gamma(-z) \, dz$

is $2\pi i$ times the sum of the residues at the poles within this contour.

For $z=0$ there is a simple pole at which the residue is $\Gamma\left(\tfrac{3}{2}\right)\Gamma\left(\tfrac{1}{2}\right) = \frac{\pi}{2}$.

For $z=\frac{1}{2}$ there is a simple pole at which the residue is $-2\pi(e^{\xi}B)^{1/2}$.

The integrand may also be written as

$$\frac{\pi^2}{\sin^2 \pi z} \cdot \frac{\pi^{i}}{\cos^2 \pi z} \quad \frac{e^{\xi z} B^z}{\Gamma(z+1)\, \Gamma\left(z+\frac{1}{2}\right) \Gamma(z)\, \Gamma\left(z-\frac{1}{2}\right)}$$

and the poles are those of $\frac{1}{\sin \pi z}$ and $\frac{1}{\cos \pi z}$.

We have already considered the simple poles $z=0, \frac{1}{2}$

For $z = n$, n an integer other than zero, the integrand has a pole of the second order, that of $\frac{1}{\sin^2 \pi z}$ at which the residue is

$$\pi^2 \left[\frac{d}{dz} \frac{(e^{\xi}B)^z}{\cos^2 \pi z \, \Gamma(z+1)\, \Gamma\left(z+\frac{1}{2}\right) \Gamma(z)\, \Gamma\left(z-\frac{1}{2}\right)}\right]_{z=n}$$

For $z = \frac{1}{2}+n$, n an integer other than zero, the integrand has a pole of the second order, that of $\frac{1}{\cos^2 \pi z}$ at which the residue is

$$\pi^2 \left[\frac{d}{dz} \frac{(e^{\xi}B)^z}{\sin^2 \pi z \, \Gamma(z+1)\, \Gamma\left(z+\frac{1}{2}\right) \Gamma(z)\, \Gamma\left(z-\frac{1}{2}\right)}\right]_{z=\frac{1}{2}+n}$$

We thus find that

$$(91)\quad P(\xi) = -\frac{(e^{\xi}B)^{\frac{N-4}{2}}}{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)\Gamma\left(\frac{N-3}{2}\right)\Gamma\left(\frac{N-4}{2}\right)}\Bigg\{\frac{\pi}{2} - 2\pi\left(e^{\xi}B\right)^{1/2}$$

$$+\pi^2\sum_{\lambda=1}^{\infty}\left[\frac{d}{dz}\,\frac{(e^{\xi}B)^{z}}{\cos^2\pi z\cdot\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\,\Gamma(z)\,\Gamma(z-\frac{1}{2})}\right]_{z=\lambda} + \pi^2\sum_{\lambda=1}^{\infty}\left[\frac{d}{dz}\,\frac{(e^{\xi}B)^{z}}{\sin^2\pi z\,\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\,\Gamma(z)\,\Gamma(z-\frac{1}{2})}\right]_{z=\frac{1}{2}+\lambda}\Bigg\}.$$

For the distribution of a, we find

$$(92)\quad D(a) = \frac{A^{\frac{N-4}{2}}\,a^{\frac{N-6}{2}}}{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)\Gamma\left(\frac{N-3}{2}\right)\Gamma\left(\frac{N-4}{2}\right)}\Bigg\{-\frac{\pi}{2} + 2\pi(aA)^{1/2}$$

$$-\pi^2\sum_{\lambda=1}^{\infty}\left[\frac{d}{dz}\,\frac{(aA)^{z}}{\cos^2\pi z\,\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\,\Gamma(z)\,\Gamma(z-\frac{1}{2})}\right]_{z=\lambda} - \pi^2\sum_{\lambda=1}^{\infty}\left[\frac{d}{dz}\,\frac{(aA)^{z}}{\sin^2\pi z\,\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\,\Gamma(z)\,\Gamma(z-\frac{1}{2})}\right]_{z=\frac{1}{2}+\lambda}\Bigg\}$$

Case 3, n even: As is evident from the previous discussion, retaining the same notation,

$$(93)\quad P(\xi) = \frac{1}{\prod_{j=1}^{n}\Gamma\left(\frac{N-j}{2}\right)2\pi}\int_{-\infty}^{\infty} e^{-it\xi}\,B^{-it}\prod_{j=1}^{n}\Gamma\left(\frac{N-j}{2}+it\right)dt.$$

Let $\frac{N-n}{2} + it = -z$ so that

$$(94)\quad P(\xi) = \frac{(e^{\xi}B)^{\frac{N-n}{2}}}{\prod_{j=1}^{n}\Gamma\!\left(\frac{N-j}{2}\right) 2\pi i}\int_{-\frac{N-n}{2}-i\infty}^{-\frac{N-n}{2}+i\infty} e^{\xi z} B^{z}\,\Gamma\!\left(\tfrac{n-1}{2}-z\right)\Gamma\!\left(\tfrac{n-2}{2}-z\right)\cdots\Gamma(-z)\,dz.$$

The same considerations as to the convergence and the contour are applicable here too and we find that

$$(95)\quad P(\xi) = -\frac{(e^{\xi}B)^{\frac{N-n}{2}}}{\prod_{j=1}^{n}\Gamma\!\left(\frac{N-j}{2}\right) 2\pi i}\int_{C} e^{\xi z} B^{z}\,\Gamma\!\left(\tfrac{n-1}{2}-z\right)\Gamma\!\left(\tfrac{n-2}{2}-z\right)\cdots\Gamma(-z)\,dz,$$

where C is the contour bounded by the line $x = -\frac{N-n}{2}$, $(N>n)$ and that part of the circle $|z| = m+\frac{1}{2}$, where m may increase indefinitely, to the right of this line and the contour is traversed in a counter-clockwise direction.

The value of $J = \int_{C} e^{\xi z} B^{z}\,\Gamma\!\left(\frac{n-1}{2}-z\right)\Gamma\!\left(\frac{n-2}{2}-z\right)\cdots\Gamma(-z)\,dz$

is $2\pi i$ times the sum of the residues at the poles within the contour. Let us write $n = 2p$ so that the integrand is

$$e^{\xi z} B^{z}\,\Gamma\!\left(\tfrac{2p-1}{2}-z\right)\Gamma\!\left(\tfrac{2p-2}{2}-z\right)\cdots\Gamma(-z).$$

For $z = r$, $r = 0, 1, 2, \cdots, p-2$ there is a pole of the $(r+1)^{th}$ order, the integrand being representable in the form

$$\frac{(-1)^{1+2+\cdots+\overline{r+1}}\,\pi^{r+1} e^{\xi z} B^{z}\,\Gamma\!\left(\frac{1}{2}-z\right)\Gamma\!\left(\frac{3}{2}-z\right)\cdots\Gamma\!\left(\frac{2p-1}{2}-z\right)\Gamma(r+1-z)\,\Gamma(r+2-z)\cdots\Gamma(p-1-z)}{\sin^{r+1}\pi z\;\Gamma(z+1)\,\Gamma(z)\cdots\Gamma(z-r+1)}.$$

The residue is therefore,

$$\frac{(-1)^{\frac{(n+1)(n+2)}{2}}}{(-1)^{n(n+1)}\, n!}\left[\frac{d^n}{dz^n}\,\frac{e^{\xi z}B^z\,\Gamma(\frac{1}{2}-z)\,\Gamma(\frac{3}{2}-z)\cdots\Gamma(\frac{2p-1}{2}-z)\,\Gamma(n+1-z)\,\Gamma(n+2-z)\cdots\Gamma(p-1-z)}{\Gamma(z+1)\,\Gamma(z)\cdots\Gamma(z-n+1)}\right]_{z=n}$$

For $z=\frac{1}{2}+n$, $n=0,1,2,\cdots,p-2$ there is a pole of the $(n+1)^{th}$ order, the integrand being representable in the form

$$\frac{(-1)^{0+1+2+\cdots+n}\,\pi^{n+1}\,e^{\xi z}B^z\,\Gamma(-z)\,\Gamma(1-z)\cdots\Gamma(p-1-z)\,\Gamma(n+\frac{1}{2}-z)\,\Gamma(n+\frac{3}{2}-z)\cdots\Gamma(\frac{2p-1}{2}-z)}{\cos^{n+1}\pi z\;\Gamma(z+\frac{1}{2})\,\Gamma(z-\frac{1}{2})\cdots\Gamma(z-n+\frac{1}{2})}$$

The residue is therefore of the form

$$\frac{(-1)^{\frac{n(n+1)}{2}}}{(-1)^{(n+1)(n+1)}\, n!}\left[\frac{d^n}{dz^n}\,\frac{e^{\xi z}B^z\,\Gamma(-z)\,\Gamma(1-z)\cdots\Gamma(p-1-z)\,\Gamma(n+\frac{1}{2}-z)\,\Gamma(n+\frac{3}{2}-z)\cdots\Gamma(\frac{2p-1}{2}-z)}{\Gamma(z+\frac{1}{2})\,\Gamma(z-\frac{1}{2})\cdots\Gamma(z-n+\frac{1}{2})}\right]_{z=\frac{1}{2}+n}$$

For $z=p-1+n$, $n=0,1,2,\cdots$ there is a pole of the $p-th$ order, the integrand being representable as

$$\frac{(-1)^p\,\pi^p\,\pi^p\,e^{\xi z}B^z}{\sin^p\pi z\;\cos^p\pi z\;\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+\frac{3}{2})}$$

The residue is therefore of the form

$$\frac{(-1)^p\,\pi^p}{(-1)^{p(p-1+n)}\,(p-1)!}\left[\frac{d^{p-1}}{dz^{p-1}}\,\frac{e^{\xi z}B^z}{\cos^p\pi z\;\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+\frac{3}{2})}\right]_{z=p-1+n}$$

For $z=\frac{2p-1}{2}+n$, $n=0,1,2,\cdots$ there is a pole of the $p-th$ order at which the residue is

$$\frac{(-1)^p\,\pi^p}{(-1)^{p(p+n)}\,(p-1)!}\left[\frac{d^{p-1}}{dz^{p-1}}\,\frac{e^{\xi z}B^z}{\sin^p\pi z\;\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+\frac{3}{2})}\right]_{z=\frac{2p-1}{2}+n}$$

We have therefore that

(96) $$P(\xi) =$$

$$-\frac{(Be^{\xi})^{\frac{N-n}{2}}}{\prod_{j=1}^{n}\Gamma\left(\frac{N-j}{2}\right)}\left\{\sum_{r=0}^{p-2}\frac{(-1)^{\frac{(r+1)(r-2)}{2}}}{r!}\left[\frac{d^r}{dz^r}\,\frac{e^{\xi z}B^z\,\Gamma\left(\frac{1}{2}-z\right)\Gamma\left(\frac{3}{2}-z\right)\cdots\Gamma\left(\frac{2p-1}{2}-z\right)\Gamma(r+1-z)\Gamma(r+2-z)\cdots\Gamma(p-1-z)}{\Gamma(z+1)\,\Gamma(z)\cdots\Gamma(z-r+1)}\right]_{z=r}\right.$$

$$+\sum_{r=0}^{p-2}\frac{(-1)^{\frac{(r+1)(r+2)}{2}}}{r!}\left[\frac{d^r}{dz^r}\,\frac{e^{\xi z}B^z\,\Gamma(-z)\Gamma(1-z)\cdots\Gamma(p-1-z)\Gamma\left(r+\frac{1}{2}-z\right)\Gamma\left(r+\frac{3}{2}-z\right)\cdots\Gamma\left(\frac{2p-1}{2}-z\right)}{\Gamma\left(z+\frac{1}{2}\right)\Gamma\left(z-\frac{1}{2}\right)\cdots\Gamma\left(z-r+\frac{1}{2}\right)}\right]_{z=\frac{1}{2}+r}$$

$$+\sum_{r=0}^{\infty}\frac{(-1)^{p(r+p)}\pi^p}{(p-1)!}\left[\frac{d^{p-1}}{dz^{p-1}}\,\frac{e^{\xi z}B^z}{\cos^p \pi z\,\Gamma(z+1)\,\Gamma\left(z+\frac{1}{2}\right)\cdots\Gamma\left(z-p+\frac{3}{2}\right)}\right]_{z=p-1+r}$$

$$\left.+\sum_{r=0}^{\infty}\frac{(-1)^{p(p+r+1)}\pi^p}{(p-1)!}\left[\frac{d^{p-1}}{dz^{p-1}}\,\frac{e^{\xi z}B^z}{\sin^p \pi z\,\Gamma(z+1)\,\Gamma\left(z+\frac{1}{2}\right)\cdots\Gamma\left(z-p+\frac{3}{2}\right)}\right]_{z=\frac{2p-1}{2}+r}\right\}$$

For the distribution of a, we find

(97) $$D(a) =$$

$$\frac{A^{\frac{N-n}{2}}a^{\frac{N-n-2}{2}}}{\prod_{j=1}^{n}\Gamma\left(\frac{N-j}{2}\right)}\left\{\frac{(-1)^{\frac{r(r-1)}{2}}}{r!}\left[\frac{d^r}{dz^r}\,\frac{(aA)^z\,\Gamma\left(\frac{1}{2}-z\right)\Gamma\left(\frac{3}{2}-z\right)\cdots\Gamma\left(\frac{2p-1}{2}-z\right)\Gamma(r+1-z)\Gamma(r+2-z)\cdots\Gamma(p-1-z)}{\Gamma(z+1)\,\Gamma(z)\cdots\Gamma(z-r+1)}\right]_{z=r}\right.$$

$$+\sum_{r=0}^{p-2}\frac{(-1)^{\frac{r(r+3)}{2}}}{r!}\left[\frac{d^r}{dz^r}\,\frac{(aA)^z\,\Gamma(-z)\,\Gamma(1-z)\cdots\Gamma(p-1-z)\,\Gamma(r+\frac{1}{2}-z)\,\Gamma(r+\frac{3}{2}-z)\cdots\Gamma(\frac{2p-1}{2}-z)}{\Gamma(z+\frac{1}{2})\,\Gamma(z-\frac{1}{2})\cdots\Gamma(z-r+\frac{1}{2})}\right]_{z=\frac{1}{2}+r}$$

$$+\sum_{r=0}^{\infty}\frac{(-1)^{p(r+p)+1}\,\pi^p}{(p-1)!}\left[\frac{d^{p-1}}{dz^{p-1}}\,\frac{(aA)^z}{\cos^p\pi z\,\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+\frac{3}{2})}\right]_{z=p-1+r}$$

$$+\sum_{r=0}^{\infty}\frac{(-1)^{p(p+r+1)+1}\,\pi^p}{(p-1)!}\left[\frac{d^{p-1}}{dz^{p-1}}\,\frac{(aA)^z}{\sin^p\pi z\,\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+\frac{3}{2})}\right]_{z=\frac{2p-1}{2}+r}$$

with $n=2p$.

Case 4, n odd: As before we find that

$$(98)\qquad P(\xi)=-\frac{(e^{\xi}B)^{\frac{N-n}{2}}}{\prod_{j=1}^{n}\Gamma\left(\frac{N-j}{2}\right)2\pi i}\int_C e^{\xi z}B^z\,\Gamma\left(\tfrac{n-1}{2}-z\right)\Gamma\left(\tfrac{n-2}{2}-z\right)\cdots\Gamma(-z)\,dz.$$

Let $n=2p+1$

The integrand is

$$e^{\xi z}B^z\,\Gamma(p-z)\,\Gamma\left(\tfrac{2p-1}{2}-z\right)\cdots\Gamma\left(\tfrac{1}{2}-z\right)\Gamma(-z).$$

The considerations are similar to the case for n even except that the integrand has an additional factor, viz. $\Gamma(p-z)$.

For $z=r$, $r=0,1,2,\cdots,(p-1)$ there is a pole of the $(r+1)$-th order at which the residue is

$$\frac{(-1)^{\frac{(r+1)(r+2)}{2}}}{(-1)^{r(r+1)}\cdot r!}\left[\frac{d^r}{dz^r}\,\frac{e^{\xi z}B^z\,\Gamma(\frac{1}{2}-z)\,\Gamma(\frac{3}{2}-z)\cdots\Gamma(\frac{2p-1}{2}-z)\,\Gamma(r+1-z)\,\Gamma(r+2-z)\cdots\Gamma(p-z)}{\Gamma(z+1)\,\Gamma(z)\cdots\Gamma(z-r+1)}\right]_{z=r}$$

For $z = \frac{1}{2} + r$, $r = 0, 1, \cdots, (p-2)$ there is a pole of the $(r+1)$-th order at which the residue is

$$\frac{(-1)^{\frac{r(r+1)}{2}}}{(-1)^{(r+1)(r+1)}\, r!}\left[\frac{d^r}{dz^r}\,\frac{e^{\xi z}B^z\,\Gamma(-z)\,\Gamma(1-z)\cdots\Gamma(p-z)\,\Gamma(r+\frac{1}{2}-z)\,\Gamma(r+\frac{3}{2}-z)\cdots\Gamma(\frac{2p-1}{2}-z)}{\Gamma(z+\frac{1}{2})\,\Gamma(z-\frac{1}{2})\cdots\Gamma(z-r+\frac{1}{2})}\right]_{z=\frac{1}{2}+r}$$

For $z = p + r$, $r = 0, 1, 2, \cdots$ there is a pole of the $(p+1)$-th order at which the residue is

$$\frac{(-1)^{p+1}\pi^p}{(-1)^{(p+1)(p+r)}\, p!}\left[\frac{d^p}{dz^p}\,\frac{e^{\xi z}B^z}{\cos^p \pi z\;\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+1)}\right]_{z=p+r}$$

For $z = \frac{2p-1}{2} + r$, $r = 0, 1, 2, \cdots$ there is a pole of the p-th order at which the residue is

$$\frac{(-1)^{p+1}\pi^{p+1}}{(-1)^{p(p+r)}\,(p-1)!}\left[\frac{d^{p-1}}{dz^{p-1}}\,\frac{e^{\xi z}B^z}{\sin^{p+1}\pi z\;\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+1)}\right]_{z=\frac{2p-1}{2}+r}$$

since the integrand is representable as

$$\frac{(-1)^{p+1}\pi^{p+1}\pi^p e^{\xi z}B^z}{\sin^{p+1}\pi z\cdot\cos^p\pi z\;\Gamma(z+1)\cdot\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+1)}$$

We have therefore that

(99) $P(\xi) =$

$$-\frac{(e^{\xi}D)^{\frac{N-n}{2}}}{\prod_{j=1}^{n}\Gamma\left(\frac{N-j}{2}\right)}\left\{\sum_{r=0}^{p-1}\frac{(-1)^{\frac{(r+1)(r-2)}{2}}}{r!}\left[\frac{d^r}{dz^r}\,\frac{e^{\xi z}B^z\,\Gamma(\frac{1}{2}-z)\,\Gamma(\frac{3}{2}-z)\cdots\Gamma(\frac{2p-1}{2}-z)\,\Gamma(r+1-z)\,\Gamma(r+2-z)\cdots\Gamma(p-z)}{\Gamma(z+1)\,\Gamma(z)\cdots\Gamma(z-r+1)}\right]_{z=r}\right.$$

$$+\sum_{r=0}^{p-2}\frac{(-1)^{\frac{(r+1)(r+2)}{2}}}{r!}\left[\frac{d^r}{dz^r}\,\frac{e^{\xi z}B^z\,\Gamma(-z)\,\Gamma(1-z)\cdots\Gamma(p-z)\,\Gamma(r+\frac{1}{2}-z)\,\Gamma(r+\frac{3}{2}-z)\cdots\Gamma(\frac{2p-1}{2}-z)}{\Gamma(z+\frac{1}{2})\,\Gamma(z-\frac{1}{2})\cdots\Gamma(z-r+\frac{1}{2})}\right]_{z=r+\frac{1}{2}}$$

$$+\sum_{r=0}^{\infty}\frac{(-1)^{(p+1)(p+1+r)}\,\pi^{p}}{p!}\left[\frac{d^{p}}{dz^{p}}\;\frac{e^{\xi z}B^{z}}{\cos^{p}\pi z\,\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+1)}\right]_{z=p+r}$$

$$\left.+\sum_{r=0}^{\infty}\frac{(-1)^{p(p+r+1)+1}\,\pi^{p+1}}{(p-1)!}\left[\frac{d^{p-1}}{dz^{p-1}}\;\frac{e^{\xi z}B^{z}}{\sin^{p+1}\pi z\,\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+1)}\right]_{z=\frac{2p-1}{2}+r}\right\}$$

The distribution for a is

(100) $D(a) =$

$$\frac{A^{\frac{N-n}{2}}\,a^{\frac{N-n-2}{2}}}{\prod_{j=1}^{n}\Gamma\left(\frac{N-j}{2}\right)}\left\{\sum_{r=0}^{p-1}\frac{(-1)^{\frac{r(r-1)}{2}}}{r!}\left[\frac{d^{r}}{dz^{r}}\;\frac{(aA)^{z}\,\Gamma(\frac{1}{2}-z)\,\Gamma(\frac{3}{2}-z)\cdots\Gamma(\frac{2p-1}{2}-z)\,\Gamma(r+1-z)\,\Gamma(r+2-z)\cdots\Gamma(p-z)}{\Gamma(z+1)\,\Gamma(z)\cdots\Gamma(z-r+1)}\right]_{z=r}\right.$$

$$+\sum_{r=0}^{p-2}\frac{(-1)^{\frac{r(r+3)}{2}}}{r!}\left[\frac{d^{r}}{dz^{r}}\;\frac{(aA)^{z}\,\Gamma(-z)\,\Gamma(1-z)\cdots\Gamma(p-z)\,\Gamma(r+\frac{1}{2}-z)\,\Gamma(r+\frac{3}{2}-z)\cdots\Gamma(\frac{2p-1}{2}-z)}{\Gamma(z+\frac{1}{2})\,\Gamma(z-\frac{1}{2})\cdots\Gamma(z-r+\frac{1}{2})}\right]_{z=r+\frac{1}{2}}$$

$$+\sum_{r=0}^{\infty}\frac{(-1)^{p^{2}+pr+r}\,\pi^{p}}{p!}\left[\frac{d^{p}}{dz^{p}}\;\frac{(aA)^{z}}{\cos^{p}\pi z\,\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+1)}\right]_{z=p+r}$$

$$\left.+\sum_{r=0}^{\infty}\frac{(-1)^{p(p+r+1)}\,\pi^{p+1}}{(p-1)!}\left[\frac{d^{p-1}}{dz^{p-1}}\;\frac{(aA)^{z}}{\sin^{p+1}\pi z\,\Gamma(z+1)\,\Gamma(z+\frac{1}{2})\cdots\Gamma(z-p+1)}\right]_{z=\frac{2p-1}{2}+r}\right\},$$

with $n=2p+1$.

It is of interest to derive from the general formula the distribution when $n = 1, 2$.

For $n = 1$ the value of p in equation (100) is zero. The expression in the brace in equation (100) becomes

$$1 - \frac{aA}{1!} + \frac{(aA)^2}{2!} - \cdots\cdots = e^{-aA},$$

so that

$$D(a) = \frac{A^{\frac{N-1}{2}} a^{\frac{N-3}{2}} e^{-aA}}{\Gamma\frac{N-1}{2}} \tag{101}$$

For $N = 2$ the value of p in equation (97) is 1. The expression in the brace in equation (97) becomes

$$\frac{\pi}{\Gamma\frac{1}{2}} + \frac{\pi a A}{\Gamma 2\,\Gamma 3/2} + \frac{\pi (aA)^2}{\Gamma 3\,\Gamma 5/2} + \cdots \tag{102}$$

$$- \frac{\pi (aA)^{1/2}}{\Gamma\frac{3}{2}\,\Gamma 1} - \frac{\pi (aA)^{3/2}}{\Gamma 5/2\,\Gamma 2} - \frac{\pi (aA)^{5/2}}{\Gamma 7/2\,\Gamma 3} - \cdots$$

$$= \frac{\pi}{\Gamma\frac{1}{2}}\left[1 - \frac{2(aA)^{1/2}}{1!} + \frac{2^2 aA}{2!} - \frac{2^3 (aA)^{3/2}}{3!} + \cdots\right]$$

$$= \pi^{1/2} e^{-2\sqrt{aA}};$$

there is no difficulty about combining the infinite series in equation (102) since each is absolutely convergent for all value of a.

Therefore,

$$(103)\quad D(a) = \frac{\pi^{1/2} A^{\frac{N-2}{2}} a^{\frac{N-4}{2}} e^{-2\sqrt{aA}}}{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)} = \frac{2^{N-3} A^{\frac{N-2}{2}} a^{\frac{N-4}{2}} e^{-2\sqrt{aA}}}{\Gamma(N-2)},$$

The explicit expressions for $N = 1, 2$ have already been obtained otherwise by Wilks.[41]

PART 3

Conclusion

XIV. Summary and Conclusions. By the use of a discontinuity factor derived from Fourier's Integral Theorem we obtain the characteristic function (in the sense of P. Levy) of the distribution law, and the distribution law of very general functions of variables satisfying a continuous distribution law. In the application of the general theory a certain lemma is found to simplify the calculations for a particular class of distribution laws and functions. Several of the distributions derived are presented not because the results are new but as illustrations of a general method of procedure which it is hoped will enable us to find the distribution laws of many functions not yet obtained.

The explicit form of the distribution of the generalized sample variance for an n-variate normal population is derived. The same analysis is applicable to find the explicit form of the other generalizations introduced by Wilks, for general n, since the integrals that must be evaluated are all of the same general nature. The writer hopes to be able to present these further results in the near future.

NOTE

After this paper had been completed, the writer's attention was drawn to the fact that an analysis very similar to that of Sections VIII, X, and XI of this paper had already appeared in two papers by Wishart and Bartlett, viz:

"The distribution of second order moment statistics in a normal system." Proc. Cambridge Phil. Soc. Vol. 28 (1932) p. 455f.

"The generalized product moment distribution in a normal system." Proc. Cambridge Phil. Soc. Vol. 29 (1933) p. 260.

These sections are, however, presented here as illustrations of the Lemma of section VII.

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