# INEQUALITIES AMONG AVERAGES

By Nilan Norris

Numerous inequalities among averages of various types are condensed in the monotonic character of the function

$$\phi(t) = \left(\frac{x_1^t + x_2^t + \cdots + x_n^t}{n}\right)^{\frac{1}{t}}$$

of the positive numbers $x_1, x_2, \cdots, x_n$, not all equal each to each. For $t = -1$ this function is the harmonic mean; for $t = 0$ it is the geometric mean; for $t = 1$ the arithmetic mean; and for $t = 2$ the root mean square. The relations among these four means which customarily are proved by special and disconnected methods appear easily as applications of the theorem that $\phi(t)$ is an increasing function of $t$. That is, for any values of $t_1$ and $t_2$ such that $-\infty < t_1 < t_2 < +\infty$, it will be true that $\phi(t_1) < \phi(t_2)$. Several proofs of this theorem have been published, many of them very complex. An extremely simple proof is herewith presented.[1]

That $\phi(t)$, $\phi'(t)$ and $\phi''(t)$ all exist and are continuous for all real values of $t$ may be shown by expanding each of the quantities $x_i^t$ in a series of powers of $t$ and considering the remainders after each of the first three terms. The ordinary rule for evaluating forms reducing to $0/0$, which requires the function under consideration to be continuous and to have at least a continuous first derivative for $t = 0$, may then be applied to $[\log \phi(t)]/t$ to show that $\phi(0)$ is the geometric mean. It is clear that $\phi(-\infty)$ and $\phi(+\infty)$ are respectively the least and the greatest of the $x_i$. This fact and the monotonic property of $\phi(t)$ make it evident that for each real value of $t$, the function may be regarded as an average in the usual sense that it lies within the range of the observations.

For a simple demonstration of the increasing character of $\phi(t)$, consider the auxiliary function

$$F(t) = t^2 \frac{\phi'(t)}{\phi(t)} = t^2 \frac{d}{dt}\left\{\frac{1}{t} \log \frac{\Sigma x^t}{n}\right\} = t\, \frac{\Sigma x^t \log x}{\Sigma x^t} - \log \frac{\Sigma x^t}{n}\,.$$

It is clear that $\phi'(t)$ has the same sign as $F(t)$. The theorem will be proved by showing that the sign of $F(t)$ is positive for all values of $t$ except zero, when $\phi'(t)$ vanishes.

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[1] Professor Harold Hotelling rendered invaluable assistance in condensing for publication the material herein presented from a more extended study of generalized mean value functions.

Differentiating the last expression with respect to $t$, one obtains upon simplification

$$F'(t) = \frac{t}{(\Sigma x^t)^2} [(\Sigma x^t) (\Sigma x^t \log^2 x) - (\Sigma x^t \log x)^2] .$$

By Cauchy's inequality (known as Schwarz' inequality when applied to integrals instead of sums), the expression in square brackets is positive. Hence $F'(t)$ has the same sign as $t$. Consequently $F(t)$, since it diminishes for negative values of $t$ and increases for positive values, has a minimum for $t = 0$. But by direct substitution, $F(0) = 0$. It follows that $F(t)$ and $\phi'(t)$ are positive for all values of $t$ other than zero. Therefore $\phi(t)$ is an increasing function.

By direct general methods it is possible to show that

$$\phi'(0) = (\Pi x)^{\frac{1}{n}} \frac{1}{2n^2} [n\Sigma(\log x)^2 - (\Sigma \log x)^2].$$

This expression obviously vanishes only when $n\Sigma(\log x)^2 = (\Sigma \log x)^2$, a condition which is satisfied only in the trivial case when $x_1 = x_2 = \cdots = x_n$.

A proof exactly parallel to that given above may be applied to integrals or, more generally, to Stieltjes integrals. The monotonic increasing character of $\left[\int_{x=0}^{\infty} x^t d\psi(x)\right]^{\frac{1}{t}}$ appears in this way if one assumes that $\psi(x)$ is a non-decreasing function integrable in the Riemann-Stieltjes sense, such that $\psi(\infty) - \psi(0) = 1$, and such that $\int_{x=0}^{\infty} x^t d\psi(x)$ exists for every real value of $t$. In terms of statistical theory, this consideration extends the theorem from samples to populations of a very general character.

Proof of the increasing character of $\phi(t)$ has also been derived from Hölder's inequality, the demonstration being expressed in terms of Stieltjes integrals.[2] The simplest general proof of the monotonic attribute of $\phi(t)$ heretofore published appears to be that of Paul Lévy.[3] As early as 1840 Bienaymé[4] presented a generalized form of $\phi(t)$, namely,

$$\left(\frac{c_1 a_1^m + c_2 a_2^m + \cdots + c_n a_n^m}{c_1 + c_2 + \cdots + c_n}\right)^{\frac{1}{m}},$$

and announced, without proof, its increasing character. In 1858 a proof of the monotonic quality of $\phi(t)$ for special cases was published by Schlömilch.[5] Of

[2] J. Shohat, "Stieltjes Integrals in Mathematical Statistics," *Annals of Mathematical Statistics* (American Statistical Association, Ann Arbor, 1930), Vol. 1, No. 1, p. 84.

[3] *Calcul des Probabilités* (Gauthier-Villars et Cie., Paris, 1925), pp. 157 *f.*

[4] Jules Bienaymé, *Société Philomatique de Paris*, Extraits des Procès-Verbaux des Seances Pedant L'Anée 1840 (Imprimerie D'A. René et Cie., Paris, 1841), Seance du 13 juin 1840, p. 68.

[5] O. Schlömilch, "Ueber Mittelgrössen verschiedener Ordnungen," *Zeitschrift für Mathematik und Physik* (B. G. Teubner, Leipzig, 1858), Vol. 3, pp. 303 *f.*

the more recent general proofs of the increasing character of $\phi(t)$ which have appeared, those of Jensen,[6] Pólya,[7] Jessen,[8] and Carathéodory[9] may be mentioned. A recent application of $\phi(t)$ to index number theory is that of Professor John B. Canning.[10]

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[6] J. L. W. V. Jensen, "Sur Les Fonctions Convexes Et Les Inegalités Entre Les Valeurs Moyennes," *Acta Mathematica* (Beijers Bokförlagsaktielbolag, Stockholm, 1905), Vol. 30, pp. 183–185.

[7] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze Aus Der Analysis* (Julius Springer, Berlin, 1925), Vol. I, pp. 54 *f.* and 210.

[8] Børge Jessen, "Bemaerkninger om koveskse Funktioner og Uligheder imellem Middelvaerdier," *Matematisk Tidsskrift* (Charles Johansens Bogtrykkeri, Copenhagen, 1931), No. 2, 1931, pp. 26–28.

[9] Attributed to Professor Constantin Carathéodory in an unpublished manuscript of Professor Harold Hotelling.

[10] "A Theorem Concerning a Certain Family of Averages of a Certain Type of Frequency Distribution," a paper presented before a joint meeting of the American Statistical Association and the Econometric Society at Berkeley, California, June 22, 1934.