A METHOD FOR DETERMINING THE COEFFICIENTS OF A CHARACTERISTIC EQUATION

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For the characteristic equation

$$\begin{vmatrix} a_{11} - x & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} - x \end{vmatrix} \equiv (-1)^n(x^n - c_1x^{n-1} + c_2x^{n-2} - \cdots + c_n) \qquad (1)$$

$$\equiv (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$

it is well known that

$$c_i = A_i$$

where A_i is the sum of all i^{th} order co-axial minors of the determinant

$$A = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}. \qquad (2)$$

If n exceeds 3 or 4, the process of calculating all possible principal minors is very cumbersome.

But another more systematic method of calculating the c's may be adopted. Suppose we define

$$A^p = \begin{vmatrix} a_{11}^{(p)} & \cdots & a_{1n}^{(p)} \\ \cdots & \cdots & \cdots \\ a_{n1}^{(p)} & \cdots & a_{nn}^{(p)} \end{vmatrix} \qquad (3)$$

and

$$\sum_1^n \alpha_i^{(p)} = S_p. \qquad (4)$$

It may be proved[1] that

$$S_p = \sum_1^n a_{ii}^{(p)}. \qquad (5)$$

But from Newton's identities[2] we have

$$S_p + c_1S_{p-1} + c_2S_{p-2} + \cdots + c_{p-1}S_1 + pc_p = 0. \qquad (6)$$

[1] Muir, L. & Metzler, W. H., "A Treatise on the Theory of Determinants," p. 606, ¶ 650 and 651.

[2] Dickson, L. E., "First Course in the Theory of Equations," p. 134, ¶ 106.

Newton's identities are ordinarily employed for calculating the sums of the powers of the roots of a polynomial when the coefficients are known. They may be employed equally well, however, for calculating the coefficients when the sums of the powers are given. Thus by means of equations (5) and (6) the coefficients of (1) may be readily calculated.

If in (2) $a_{ij} = a_{ji}$, the calculation of the successive A^p values is straightforward. The determinant A is used as a constant multiplier so that

$$A \cdot A = A^2, \qquad A \cdot A^2 = A^3, \cdots A \cdot A^{n-1} = A^n$$

and the multiplication is column by column. That is,

$$a_{ij}^{(1+p)} = \sum_{k=1}^{n} a_{ki} a_{kj}^{(p)} .$$