A NOTE ON SHEPPARD'S CORRECTIONS

By Solomon Kullback

In this note we shall derive a simple relation between the characteristic function of the grouped distribution and the characteristic function of the original continuous distribution, assuming that the frequency curve has high contact with the x-axis at both ends.

If we set $p_s = \int_{x_s - \frac{w}{2}}^{x_s + \frac{w}{2}} f(x)\, dx$, then the characteristic function of the grouped distribution is given by

$$\psi(t) = \sum e^{itx_s} p_s \tag{1}$$

where $i = \sqrt{-1}$. Replacing p_s by its value as given above, we have

$$\begin{aligned} \psi(t) &= \sum e^{itx_s} \int_{x_s - \frac{w}{2}}^{x_s + \frac{w}{2}} f(x)\, dx \\ &= \sum e^{itx_s} \int_{-\frac{w}{2}}^{\frac{w}{2}} f(x + x_s)\, dx \\ &= \int_{-\frac{w}{2}}^{\frac{w}{2}} dx \sum e^{itx_s} f(x + x_s) \\ &= \sum e^{itx_s} f(x_s) \int_{-\frac{w}{2}}^{\frac{w}{2}} e^{-itx}\, dx\,. \end{aligned} \tag{2}$$

There is no difficulty about justifying the inversion of the order of integration and summation.

Because of the assumption of high-contact with the axis of x at both ends of the frequency curve, we have

$$\varphi(t) = \int e^{itx} f(x)\, dx = w \sum e^{itx_s} f(x_s) \tag{3}$$

so that

$$\psi(t) = \frac{2}{wt} \sin \frac{tw}{2}\, \varphi(t)\,. \tag{4}$$

This is the desired result, from which there follows the desired moment relations by equating coefficients of $(it)^r$ on both sides of the equation. For example:

$$1 + M_1 it + \frac{M_2}{2!}(it)^2 + \frac{M_3}{3!}(it)^3 + \cdots = \left(1 + \frac{(it)^2 w^2}{4}\frac{1}{3!} + \frac{(it)^4 w^4}{16}\frac{1}{5!} + \cdots\right)$$

$$\left(1 + m_1 it + \frac{m_2}{2!}(it)^2 + \cdots\right)$$

$$= 1 + m_1 it + \frac{(it)^2}{2!}\left(m_2 + \frac{w^2}{12}\right) + \frac{(it)^3}{3!}\left(m_3 + \frac{m_1 w^2}{4}\right) + \cdots$$

or

$$M_1 = m_1\,; \qquad M_2 = m_2 + \frac{w^2}{12}\,; \qquad M_3 = m_3 + \frac{m_1 w^2}{4}\,; \cdots$$

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