APPLICATIONS OF TWO OSCULATORY FORMULAS

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INTRODUCTION

The main purpose of this paper is to illustrate how Mr. Jenkins' osculatory formulas[1] (A) and (B) can be applied in a convenient manner. The first section of this paper will be little more than a summary of some of the formulas contained in the other three articles. The second section will contain the applications.

I. SOME MATHEMATICS OF THE FORMULAS

The Woolhouse notation will in this paper be used to stand for the differences of u_{x+n} which represents the given values of a function. The general formulas are

$$y_x = y_0 + x\Delta y_0 + \tfrac{1}{2}x(x-1)B + \tfrac{1}{6}x(x-1)(x-\tfrac{1}{2})C; \tag{1}$$

and

$$y_x = u_0 + xa_1 + \tfrac{1}{2}x(x-1)B + \tfrac{1}{6}x(x-1)(x-\tfrac{1}{2})C. \tag{2}$$

The special formulas belonging to (2) are

$$B = b - \tfrac{1}{6}d \text{ and } C = c_1 - \tfrac{1}{6}e_1\,, \tag{A}$$

where b and d are defined by $b = \frac{1}{2}(b_0 + b_1)$ and by $d = \frac{1}{2}(d_0 + d_1)$; and

$$B = b \text{ and } C = 0. \tag{B}$$

The special formulas belonging to (1) are

$$y_0 = u_0 + \tfrac{1}{4}b_0\,, \qquad B = b, \qquad \text{and } C = 0; \tag{C}$$

and

$$y_0 = u_0 - \tfrac{1}{36}d_0\,, \qquad B = b - \tfrac{1}{6}d, \qquad \text{and } C = c_1 - \tfrac{1}{6}e_1\,. \tag{D}$$

Formula (C) is equivalent to Mr. Jenkins' formula (A). Also (D) is equivalent to his formula (B).

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[1] This paper presupposes a knowledge of three other articles. The first one by Mr. Wilmer A. Jenkins is entitled "Graduation Based on a Modification of Osculatory Interpolation," and is printed in the October 1927 issue of the Transactions of the Actuarial Society of America. The other two papers are mine. One of them is entitled "Some Practical Interpolation Formulas," and is printed in the September 1935 issue of these Annals. The other one entitled "A Family of Osculatory Formulas" is printed in the October 1935 issue of the Transactions.

II. APPLICATIONS OF (C) AND (D)

First, there is the problem of selecting suitable examples to which (C) and (D) can be applied. Secondly, we will then apply in a convenient manner the formulas to these examples.

The problem of selecting suitable examples will now be considered. "The non-reproducing characteristic of" formula (D) "raises the question of what will happen in the graduation of a series whose fourth differences are all positive, say. The answer is that the graduated series will lie everywhere below the observed points and that the observations will not be correctly represented by the interpolated series." On the other hand, if we select a series whose fourth differences change frequently in sign, (D) because of its non-reproducing characteristic has valuable smoothing possibilities. In like manner, (C) may be valuable when the second differences change frequently in sign. Mr. Jenkins gives at quinquennial ages rates of mortality which were graphically determined from the published American Men Ultimate Experience. Since the fourth differences of these rates change frequently in sign, we will apply (D) to a few of these rates. So far as I know no suitable actuarial examples have been found to which (C) can be applied. However, there is the possibility that (C) might be valuable in some sciences. Since I do not know of any suitable real example to which (C) can be applied, we will apply it to a trivial series whose second differences change frequently in sign.

We are now ready to apply in a convenient manner (C) and (D) to the examples selected in the preceding paragraph.

First, we will apply (C). I have in my other article applied (B) in a convenient manner. This method with little change can be applied to (C). If it is desired to apply (C) at either end of the table where values of u_x are not available for the calculation of the second differences, it can be assumed they vanish. It is convenient if S and S^2 represent respectively the major differences Δu_x and $\Delta^2 u_x$ in such a manner that they are arranged centrally in the working illustration. It is convenient if s and s^2 represent respectively the minor differences δy_x and $\delta^2 y_x$. The quantity y_0 can be computed by $y_0 = u_0 + \frac{1}{4}b_0$, and y_1 can be computed in like manner. Since we wish in the working illustration of (C) to interpolate four values between y_0 and y_1, the middle $s = \delta y_{.4} = .2\Delta y_0$, and $s^2 = .04B = .02(b_0 + b_1)$. We can by the use of the foregoing method apply (C) to suitable functions, whose given values can be represented by $f(r)$. Then, it follows from the definition of u_x that $f(r) = u_x$. It might prevent confusion if it is stated that x and r are related to each other in such a way that we always interpolate between y_0 and y_1. We shall now apply (C) to the case when $f(r)$ represents the trivial series shown at top of page 3.

Finally, we will apply (D). Mr. Henderson has applied (A) in a very convenient manner. His method with little change can be applied to (D). If it is desired to apply (D) at either end of the table where values of u_x are not available for the calculation of the differences required, it can be assumed that the fourth differences that can not be computed vanish, and the required

r	$f(r)$	S	S^2	y_x	s	s^2
0	1		0	1.0	.18	
1				1.18	.14	
2		1		1.32	.10	−.04
3				1.42	.06	
4				1.48	.02	
5	2		−2	1.5		
6				1.5		
7		−1		1.5	00	.00
8				1.5		
9				1.5		
10	1		2	1.5	.02	
11				1.52	.06	
12		1		1.58	.10	.04
13				1.68	.14	
14				1.82	.18	
15	2		0	2.0		

differences can be filled in consistently with that assumption. It is convenient if S, S^2, and S^3 represent respectively the major differences Δu_x, $\Delta^2 u_x$, and $\Delta^3 u_x$ in such a manner that they are arranged centrally in the working illustration. It is convenient if s, s^2, and s^3 represent the minor differences so that by definition $s = s_x = \delta y_x$, $s^2 = s_x^2 = \delta^2 y_{x-.2}$, and $s^3 = \delta^3 y_x$. The first $s^2 = \delta^2 y_{-.2} = .04(b_0 - \frac{1}{6}d_0)$. The last $s^2 = \delta^2 y_{.8} = .04(b_1 - \frac{1}{6}d_1)$. The quantity y_0 can be computed by $y_0 = u_0 - \frac{1}{36}d_0$, and y_1 can be computed in like manner. The middle $s = \delta y_{.4} = .2\Delta y_0 - s^3$. We are now in position to apply (D) to the quinquennial rates of mortality.

Age	Rate	S	S^2	S^3	s^4
72	.07010				
		.03808			
77	.10818		.00861		
		.04669		.01799	
82	.15487		.02660		−.03745
		.07329		−.01946	
87	.22816		.00714		.14518
		.08043		.12572	
92	.30859		.13286		.00000
		.21329		.12572	
97	.52188		.25858		.00000

Age	y_x	s	s^2	s^3
82	.15591	12612	.001314	
83	.168522	13527	915	
84	.182049	.014043	516	−.000399
85	.196092	14160	117	
86	.210252	13878	−.000282	
87	.22413	13460	−.000682	
88	.237590	13977	.000517	
89	.251567	.015693	1716	.001199
90	.267260	18608	2915	
91	.285868	22722	4114	
92	.30859	28006	.005314	
93	.336596	34326	6320	
94	.370922	.041652	7326	.001006
95	.412574	49984	8332	
96	.462558	59322	9338	
97	.52188		.010343	