CONVEXITY PROPERTIES OF GENERALIZED MEAN VALUE FUNCTIONS[1]

By Nilan Norris

Consider the following generalized mean value functions: (1) the unit weight or simple sample form, $\phi(t) = \left(\frac{x_1^t + x_2^t + \cdots + x_n^t}{n}\right)^{\frac{1}{t}}$; in which the x_i are positive real numbers not all equal each to each, and in which t may take any real value; (2) the weighted sample form, $\omega(t) = \left(\frac{c_1x_1^t + c_2x_2^t + \cdots + c_nx_n^t}{c_1 + c_2 + \cdots + c_n}\right)^{\frac{1}{t}}$, in which the c_i are positive numbers not all equal each to each, and in which the x_i and t are restricted as in $\phi(t)$; (3) the integral form, $\theta(t) = \left[\int_{x=0}^{1} x^t dx\right]^{\frac{1}{t}}$, where $\int_{x=0}^{1} x^t dx$ exists for every real value of t; and (4) the generalized integral form $\Phi(t) = \left[\int_{x=0}^{\infty} x^t d\psi(x)\right]^{\frac{1}{t}}$, where $\psi(x)$ is a non-decreasing function integrable in the Riemann-Stieltjes sense such that $\psi(\infty) - \psi(0) = 1$, and such that $\int_{x=0}^{\infty} x^t d\psi(x)$ exists for every real value of t. The facts that all of these functions are monotonic increasing and that both $\phi(t)$ and $\omega(t)$ have two horizontal asymptotes have been previously demonstrated.[2] Although the existence of $\phi(t)$ and $\omega(t)$ has been known since 1840, there appears to have been no attempt made to investigate the behavior of the second derivatives of them.[3]

When the x_i are price relatives, production relatives, or similar data, $\phi(t)$ and $\omega(t)$ yield common types of index numbers by direct substitution of integral values of t. For any values of t such that $0 < t_1 < t_2 < \infty$, the type bias of $\phi(t_2)$ will be greater than the type bias of $\phi(t_1)$. Similarly, for any values of t such that $-\infty < t_1 < t_2 < 0$, the type bias of $\phi(t_1)$ will be greater than the type bias of $\phi(t_2)$. The second derivatives of $\phi(t)$ and $\omega(t)$ indicate whether

[1] Presented at a joint meeting of the American Mathematical Society, the Econometric Society, and the Institute of Mathematical Statistics at St. Louis on January 2, 1936. The writer is indebted to C. C. Craig, Einar Hille, Dunham Jackson, and J. Shohat for helpful critical reviews of the preliminary draft of this paper.

[2] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge University Press, London, 1934), pp. 12–15; and Nilan Norris, "Inequalities among Averages," *Annals of Mathematical Statistics*, Vol. VI, No. 1, March, 1935, pp. 27–29.

[3] Jules Bienaymé, *Société Philomatique de Paris*, Extraits des procès-verbaux des séances pendant l'année 1840 (Imprimerie D'A. René et Cie., Paris, 1841), Séance du 13 juin 1840 p. 68.

type bias is changing at an increasing or a decreasing rate as between the unlimited number of averages available for use. Considerable interest attaches to $\omega(t)$, the weighted sample form of function.

Let $\omega(t)$ be made arbitrary for the case of $n = 2$, with $x_1 = 1$, and $x_2 = e^{-\lambda}$, where λ is any real number. Also let $c_1 = \alpha$, and $c_2 = \beta$, where $\alpha + \beta = 1$. Then $\omega(t) = [\alpha + \beta e^{-\lambda t}]^{\frac{1}{t}}$. Now for all values of t,

$$\alpha + \beta e^{-\lambda t} = 1 - \frac{\beta\lambda}{1} t + \frac{\beta\lambda^2}{2} t^2 - \frac{\beta\lambda^3}{6} t^3 + \cdots$$

For $| t |$ sufficiently small, it follows that

$$\log (\alpha + \beta e^{-\lambda t}) = - \beta\lambda t + \tfrac{1}{2} \beta\lambda^2 (1 - \beta)t^2 + \beta\lambda^3 \left[-\frac{1}{6} + \frac{\beta}{2} - \frac{\beta^2}{3} \right] t^3 + \cdots,$$

so that for $t \neq 0$

$$\frac{1}{t} \log (\alpha + \beta e^{-\lambda t}) = - \beta\lambda + \tfrac{1}{2} \beta\lambda^2 (1 - \beta)t + \beta\lambda^3 \left[-\frac{1}{6} + \frac{\beta}{2} - \frac{\beta^2}{3} \right] t^2 + \cdots.$$

Therefore $\omega(t) = \exp. \left[\frac{1}{t} \log (\alpha + \beta e^{-\lambda t}) \right]$

$$= e^{-\beta\lambda} \left[1 + \tfrac{1}{2} \beta\lambda^2 (1 - \beta)t + \beta\lambda^3 \left\{ -\frac{1}{6} + \frac{\beta}{2} - \frac{\beta^2}{3} + \frac{1}{8}\beta(1 - \beta)^2\lambda \right\} t^2 + \cdots \right].$$

It follows that $\omega''(0) = 2\beta\lambda^3 e^{-\beta\lambda} \left[-\frac{1}{6} + \frac{\beta}{2} - \frac{\beta^2}{3} + \frac{1}{8} \beta(1 - \beta)^2\lambda \right]$. It is clear that $\omega(0)$ is the weighted geometric mean, and that $\phi(0)$ is the unit weight or simple sample form of geometric mean. As a means of demonstrating the range of values which $\omega''(0)$ may take it is helpful to rewrite the expression for $\omega''(0)$ as follows:

$$\omega''(0) = \tfrac{1}{4} \beta^2 (1 - \beta)^2\lambda^3 \left[\lambda - \frac{4}{3} \frac{1 - 2\beta}{\beta(1 - \beta)} \right] e^{-\beta\lambda} \equiv f(\lambda, \beta).$$

This consideration makes it possible to distinguish three cases of $y = f(\lambda, \beta)$ for fixed β, namely, $0 < \beta < \frac{1}{2}$; $\beta = \frac{1}{2}$; and $\frac{1}{2} < \beta < 1$. In all three cases $f(\lambda, \beta)$ has an absolute minimum $\mu(\beta) \leqq 0$, and $\mu(\frac{1}{2}) = 0$. The corresponding values of λ satisfies the quadratic equation $\lambda^2 - \frac{4}{3} \frac{4 - 5\beta}{\beta(1 - \beta)} \lambda + \frac{4 - 8\beta}{\beta^2(1 - \beta)} = 0.$ It is clear that by taking β near enough to 0, one can make $\mu(\beta)$ as large negative as is desired. Also, by choosing λ properly, one can make $\omega''(0)$ take any value between $\mu(\beta)$ and ∞. For example, when $\alpha = \beta = \frac{1}{2}$, λ may be selected so as to make $\omega''(0)$ any arbitrarily chosen non-negative number. For then $\omega''(0) = \frac{\lambda^4}{64} e^{-\frac{\lambda}{2}}$, and as λ increases from $-\infty$ to 0, $\omega''(0)$ decreases from ∞ to 0. If $\lambda = 0, \omega''(0) = 0$. If $\lambda > 0$, as λ increases from 0 to 8, $\omega''(0)$ increases to

$64e^{-4}$, and as λ increases beyond 8, $\omega''(0)$ decreases, approaching 0 as λ increases indefinitely. It is evident that the case of $\alpha = \beta = \frac{1}{2}$, with $\lambda = -\log 2$, $x_1 = 1$, and $x_2 = e^{-\lambda}$, is one in which $\omega(t)$ becomes the unit weight or simple sample type of generalized mean value function, namely, $\phi(t) = \left(\frac{1^t + 2^t}{2}\right)^{\frac{1}{t}}$. Reference to the first expression above noted for $\omega''(0)$ will make clear that $\phi''(0) = \frac{(\log 2)^4}{64}\sqrt{2}$ in this special case.

Analysis of $\Phi(t)$, the generalized integral form of generalized mean value function, makes it possible to characterize populations of a very general character, as well as samples. But in the case of $\Phi(t)$ it is even more difficult to generalize as to convexity properties. For example, let

$$\Phi(t) = \left[\int_{u=-\infty}^{\infty} e^{-ut}\, dE(u)\right]^{\frac{1}{t}},$$

where

$$E(u) = \frac{1}{\sqrt{\pi}} \int_{v=-\infty}^{u} e^{-v^2}\, dv.$$

This expression is obviously of the required generalized integral type. Now

$$[\Phi(t)]^t = \frac{1}{\sqrt{\pi}} \int_{u=-\infty}^{\infty} e^{-ut-u^2}\, du = \frac{1}{\sqrt{\pi}}\, e^{\frac{t^2}{4}} \int_{u=-\infty}^{\infty} e^{-\left(u+\frac{t}{2}\right)^2} du = e^{\frac{t^2}{4}}.$$

Therefore $\Phi(t) = e^{\frac{t}{4}}$, and $\Phi''(t) = \frac{e^{\frac{t}{4}}}{16} > 0$ for all t. That is, in this particular case, $\Phi(t)$ has only one horizontal asymptote.

The foregoing examples indicate that the following conclusions may be drawn as to the diverse convexity attributes of the various means as functions of t: (1) The unit weight form, $\phi(t)$, and the weighted sample form, $\omega(t)$, must always have a point of inflection, since both of them not only increase with t, but are doubly asymptotic (have two horizontal asymptotes). (2) Points of inflection for $\phi(t)$ and $\omega(t)$ do not necessarily occur at $t = 0$. (3) The generalized integral form, $\Phi(t)$, need not always have a point of inflection. That is, the second derivatives of certain forms of $\Phi(t)$ do not change their sign, since such forms are concave upward.

UNIVERSITY OF MICHIGAN.