NOTE ON ZOCH'S PAPER ON THE POSTULATE OF THE ARITHMETIC MEAN

By Albert Wertheimer

1. **Introduction.** There appeared recently a paper by Richmond T. Zoch[1] entitled "On The Postulate of the Arithmetic Mean." The stated purpose of his paper, was to show that the derivation of the Postulate as given by Whittaker & Robinson, is not correct. It is the purpose of this paper to show, that Zoch has not proven any error to exist in the Whittaker & Robinson derivation, but that there are a few errors in his paper. As this paper is intended to be read with Zoch's paper as a reference, the terms used there will not be redefined here, and except where otherwise stated, the symbols used will have the same meaning.

2. Zoch introduces the function

$$f \equiv \bar{x} + a\mu_3/\mu_2$$

and claims that it satisfies all the four axioms of Whittaker & Robinson, and obviously it is not the arithmetic mean. He therefore concludes that their derivation must have errors somewhere, and proceeds to find them. Let us first examine the f function. Considering only the part μ_3/μ_2, the partial derivatives with respect to x_i are given by

$$\frac{3\mu_2\{(x_i - \bar{x})^2 - \mu_2\} - 2\mu_3(x_i - \bar{x})}{n\mu_2^2}$$

It is then stated (p. 172) "... clearly these partial derivatives are single valued and continuous. Therefore the function μ_3/μ_2 satisfies axiom IV." Now, the condition that a function be continuous and single valued means of course that this be true throughout the region of definition of the function. It is not shown how these derivatives are clearly continuous and single valued for the very important case where all the x's are equal and the derivatives become indeterminate. As a matter of fact they are not continuous in this case, and therefore the f function does not satisfy axiom IV. To prove this, we only have to consider the very simple case where we let

$$x_i = k + c_i z$$

[1] This Journal Vol. VI no. 4, Dec. 1935, pp. 171–182.

where k is a fixed constant, c_i is a set of arbitrary constants not all equal, and z is a parameter. We then have

$$\bar{x} = k + \bar{c}z$$
$$\mu_2 = \mu_2' z^2$$
$$\mu_3 = \mu_3' z^3$$

where

$$\bar{c} = 1/n \sum c_i$$
$$\mu_2' = 1/n \sum (c_i - \bar{c})^2$$
$$\mu_3' = 1/n \sum (c_i - \bar{c})^3$$

Substituting these values in f and the derivatives, we get taking $a = 1$,

$$f = k + z\bar{c} + z^3\mu_3'/z^2\mu_2'$$
$$\partial f/\partial x_i = 1/n + \frac{3z^2\mu_2'\{z^2(c_i - \bar{c})^2 - z^2\mu_2'\} - 2z^4\mu_3'(c_i - \bar{c})}{nz^4\mu_2'^2}$$

Now going to the limit when z approaches zero, and all the x's approach k, we get

$$\lim_{z\to 0} f \doteq k,$$
$$\lim_{z\to 0} \partial f/\partial x_i = 1/n\{-2 + 3(c_i - \bar{c})^2/\mu_2' - 2\,\mu_3'(c_i - \bar{c})/\mu_2'^2\}$$

Thus, when all the x's approach the same value, the function f also approaches the same value independent of the c's, that is regardless of the mode of approach, while the derivatives can take on any value depending on the c's that is on how the limiting value of f is approached. The f function then does not have continuous single valued partial derivatives, and therefore does not satisfy axiom IV.

In part 2 of the paper it is stated "Now when the x_i all approach a then both f and $\partial f/\partial x_i$ become indeterminate forms. However, in this case f takes an indeterminate form which can be evaluated and it can be shown that μ_3/μ_2 will always have the value zero, i.e., f will have the value a when all the $x_i \to a$; while the $\partial f/\partial x_i$ can take any value whatever and in general the $\partial f/\partial x_i$ will not be equal when the $x_i \to a$." This statement really amounts to saying that the f function does not satisfy axiom IV, but it is there used to demonstrate that one of Schiaparelli's propositions is false.

3. Having exhibited a function different from the arithmetic mean, and supposedly satisfying all the four axioms, the question is asked "Where is the proof given by Whittaker & Robinson lacking in rigor?" After numbering the various steps in the derivation "... for the sake of rigor and careful reasoning

. . ." it is stated (p. 174), "The sixth step involves the tacit assumption that the partial derivatives are functions of k. These partial derivatives are not necessarily functions of k . . ." and it is therefore concluded that the sixth step is not valid. Now, how can any function that by definition is to be evaluated at $\theta k x_i$ not be a function of k? What is shown (pp. 174–5) is that these derivatives do not necessarily involve k explicitly, but this is neither implied nor necessary for the sixth step, and there is no ground for doubting its validity.

4. In order to overcome the supposed defect in the sixth step, it is proposed to change axiom IV so as to require the partial derivatives to be constants. But even then (p. 175) ". . . there remains an objection in the seventh step." Now, the seventh step consists of the statement that if

$$\phi(x_i) = \sum c_i x_i$$

where the c's are independent of the x's then due to the condition that ϕ be a symmetric function, all the c's must be equal. To show the defect in this step it is stated, that under certain conditions ". . . the function $f \equiv \bar{x} + \mu_3/\mu_2$ will have partial derivatives with respect to x_i which are unequal and constant; yet at the same time the function f is a symmetrical expression of the n variables." Granting that all that is correct, what has this got to do with the seventh step? The f function certainly is not of the type $\sum c_i x_i$ to which the seventh step is applied.

5. One more point should be mentioned. On p. 181 it is supposedly proven that any function satisfying the first three axioms must have continuous first partial derivatives. The proof is essentially as follows: Assuming all the x's are given the same increment Δx, the increment of the function then is $\Delta\phi$. It is then stated ". . . but by axiom I, $\Delta\phi = \Delta x$. Therefore $\Delta\phi/\Delta x = 1 = d\phi/dx$. In other words, the total derivative of ϕ exists and is constant. Therefore the total derivative of ϕ is continuous." From this, the continuity of the first partial derivatives is proven by means of Euler's Theorem for homogeneous functions. Now, just what does the symbol $d\phi/dx$ (which is called the total derivative) mean for a function of many independent variables? Besides, (whatever this symbol means) is it considered rigorous to deduce a general Theorem from the very special case where all the differentials are made equal? This is one place where the f function could be used effectively as an exhibit of a function satisfying the first three axioms, and not having continuous partial derivatives.

It is also stated (p. 181) that ". . . it would seem more satisfactory to postulate that the function ϕ is single valued, for the single-valuedness of a derivative does not insure the single-valuedness of the integral while the single-valuedness of a function does insure the single-valuedness of the derivative where the derivative exists." This statement is certainly not self evident and requires proof. For a single variable at least, it is easy to imagine a function represented by a curve with corners defined in a certain interval. The function then could be single valued everywhere in the interval, while the derivatives at the corners may exist and have two distinct values, depending on whether the corner is approached from the right or the left. On the other hand it is hard to imagine a curve representing a single valued function such that the integral i.e. the function represented by the area under the curve should not be single valued.

6. **In Conclusion:** It is stated in the Introduction that "Since this book has had wide circulation, it is believed that the errors in this proof should be called to the attention of the users of the book. The present paper has been prepared for this purpose." It is for the same reason, that this paper was prepared to show that no error has been proven to exist.

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