CONTRIBUTIONS TO THE THEORY OF COMPARATIVE STATISTICAL ANALYSIS. I. FUNDAMENTAL THEOREMS OF COMPARATIVE ANALYSIS[1]

By William G. Madow

This is the first of several papers in which there will be presented a general approach to the statistical examination of hypotheses which are false if any of several things are true. Phenomena requiring such a statistical theory are investigated quite frequently. As examples may be cited the studies of lag correlation in time series, periodogram analysis in geophysics, factor analysis in psychology, and analysis into components in agriculture.[2]

The theorems of this paper have one purpose: to permit the reduction of the distributions by which the hypotheses are to be tested to essentially the joint distribution of the statistics which contain the information offered by the data concerning the truth or falsity of the things which will negate the hypotheses. In order to do this it has been necessary to generalize the theorem of Poincare on the probability that at least one of several events occur.[3] As illustrations there are stated, after Theorems III, VI, and IX, generalizations of a distribution derived by Jordan, (5) page 109.[4]

In a second paper, we shall give a complete derivation of the joint distributions necessary for the applications of the analysis of variance. A reconsideration of the Schuster periodogram will be included. In other papers these results will be extended to problems arising in the theory of regression, and to problems of the distributions of medians, etc.

The fundamental theorems of comparative analysis are now obtained in such a form that they are applicable to problems in the theory of probability no matter what the distributions may be. Some special cases of these theorems[5]

---

[1] Presented to the American Mathematical Society, March 27, 1937. Research under a grant-in-aid from the Carnegie Corporation of New York.

[2] Naturally these techniques are also useful in other branches of science then those in which they were first applied. It should be noted that by analysis into components we here refer to the work of Fisher, (2), chapter 6.

[3] See, Poincaré, (7), page 60. This theorem is attributed to Poincaré by Jordan, (5), and Fréchet, (3).

[4] This distribution states the probability that in r trials of an experiment which has exactly n possible results, these results being mutually exclusive, each of the possible results occurs at least once. Jordan's derivation has been simplified by Fréchet, (3), page 12.

[5] The theorems are, of course, part of the theory of measure and integration.

have been used in connection with the derivation of distributions of positional statistics such as the k^{th} in order of N elements,[6] and others.

Let Ω be a collection of elements x, and let Δ be a set of subsets of Ω. Then, the axioms which the elements of Δ are to satisfy are[7]

I. Δ is a field;[8]
II. $\Omega \epsilon \Delta$;
III. To every $A \epsilon \Delta$ there is ordered a non-negative real number $P(A)$;
IV. $P(\Omega) = 1$;
V. If $A \epsilon \Delta$ and $B \epsilon \Delta$, and $AB = 0$, then $P(A + B) = P(A) + P(B)$.

We shall regard Ω as the set of possible results of an experiment ϵ. By events we shall mean elements of Δ. The complement $\bar{A}$ of A with respect to Ω will be an element of Δ if A is an element of Δ. $\bar{A}$ consists of all elements of Ω which are not elements of A and hence is the event which occurs if and only if A does not occur.[9]

Let the subsets of Ω

$$E_1, E_2, \cdots, E_k \tag{1}$$

be elements of Δ. Then, if $\alpha_1, \alpha_2, \cdots, \alpha_k$ is a permutation of 1, 2, $\cdots$, k, the set

$$E_{\alpha_1} E_{\alpha_2} \cdots E_{\alpha_j} \bar{E}_{\alpha_{j+1}} \cdots \bar{E}_{\alpha_k} \tag{2}$$

is an element of Δ and is the event which occurs whenever all the events $E_{\alpha_1}, E_{\alpha_2}, \cdots, E_{\alpha_j}$ occur, while none of the events $E_{\alpha_{j+1}}, E_{\alpha_{j+2}}, \cdots, E_{\alpha_k}$ occur.

The events (1) are said to be independent if and only if

$$P(E_{\alpha_1} \cdots E_{\alpha_j} \bar{E}_{\alpha_{j+1}} \cdots \bar{E}_{\alpha_k}) = \prod_{\nu=1}^{j} P(E_{\alpha_\nu}) \cdot \prod_{\nu=j+1}^{k} P(\bar{E}_{\alpha_\nu}). \tag{3}$$

for all selections of the sets (1) and their complements.[10]

Theorem I. The probability that the first j of the k events (1) *occur, while the remaining k − j events do not occur, is*

---

[6] See, for example, Gumbel, (4). It is noted that Theorems I, II, and III are stated by Arne Fisher, (1), page 42, who assumes, however, that the events are independent.

[7] These axioms are stated by Kolmogoroff, (6), page 2.

[8] A set of sets is a field if the fact that A and B are elements of the set implies that $A + B$, AB, and $A - AB$ are also elements of the set.

[9] The event A will be said to have occurred if the result of the performance of the experiment E is an element of A.

[10] See Kolmogoroff, (6), page 9 for a discussion of various equivalent definitions of independence.

$$(4)\quad P(E_1 \cdots E_j\bar{E}_{j+1} \cdots \bar{E}_k) = \sum_{\nu=0}^{k-j} (-1)^\nu \sum_{\substack{\alpha_1,\cdots,\alpha_\nu=j+1 \\ \alpha_1<\alpha_2<\cdots<\alpha_\nu}}^{k} P(E_1 \cdots E_jE_{\alpha_1} \cdots E_{\alpha_\nu}).$$ [11]

Proof. Let $k = j + 1$. Then it follows from Axiom V that

$$(5)\quad P(E_1E_2 \cdots E_j) = P(E_1E_2 \cdots E_jE_{j+1}) + P(E_1E_2 \cdots E_j\bar{E}_{j+1}).$$

Hence the theorem is true for $k = j + 1$ and any $j > 0$. Let the theorem be true for $k = j, j + 1, \cdots, k - 1$. From Axiom V it follows that

$$(6)\quad P(E_1 \cdots E_j\bar{E}_{j+1} \cdots \bar{E}_k) = P(E_1 \cdots E_j\bar{E}_{j+1} \cdots \bar{E}_{k-1}) - P(E_1 \cdots E_j\bar{E}_{j+1} \cdots \bar{E}_{k-1}E_k).$$

Substituting from (4) the theorem is proved.

Let $n \geq n_1 + \cdots + n_t$, $n_i \geq 0$ $(i = 1, \cdots, t)$; and let

$$\frac{n!}{n_1!\, n_2! \cdots n_t!\, (n - n_1 - \cdots - n_t)!} \equiv (n; n_1, n_2, \cdots, n_t).$$

COROLLARY. If, for each value of ν, $(\nu = 1, 2, \cdots, k - j)$, the $(k - j; \nu)$ terms

$$P(E_1 \cdots E_jE_{\alpha_1} \cdots E_{\alpha_\nu})$$

which can be obtained by selecting $\alpha_1, \alpha_2, \cdots, a_\nu$ without repetition from $j + 1, j + 2, \cdots, k$, are all equal, then

$$(7)\quad P(E_1 \cdots E_j\bar{E}_{j+1} \cdots \bar{E}_k) = \sum_{\nu=0}^{k-j} (-1)^\nu(k - j; \nu)P(E_1 \cdots E_{j+\nu}).$$

Let

$$(8)\quad S(\nu) = \sum_{\substack{\alpha_1,\cdots,\alpha_\nu=1 \\ \alpha_1<\cdots<\alpha_\nu}}^{k} P(E_{\alpha_1}E_{\alpha_2} \cdots E_{\alpha_\nu})$$

where the summation extends over the $(k; \nu)$ terms

$$(9)\quad P(E_{\alpha_1}E_{\alpha_2} \cdots E_{\alpha_\nu})$$

which can be obtained by selecting ν of the k events (1) without repetition. If all the terms (9) which can be obtained by selecting ν of the k events (1) without repetition are equal, then

$$(10)\quad S(\nu) = (k; \nu)P(E_1 \cdots E_\nu).$$

---

[11] By definition

$$\sum_{\nu=0}^{k-j} (-1)^\nu \sum_{\substack{\alpha_1,\cdots,\alpha_\nu=j+1 \\ \alpha_1<\cdots<\alpha_\nu}}^{k} P(E_1 \cdots E_j\bar{E}_{j+1} \cdots \bar{E}_{\alpha_\nu}) = P(E_1 \cdots E_j) + \sum_{\nu=1}^{k-j} (-1)^\nu \sum_{\substack{\alpha_1,\cdots,\alpha_\nu=j+1 \\ \alpha_1<\cdots<\alpha_\nu}}^{k} P(E_1 \cdots E_jE_{\alpha_1} \cdots E_{\alpha_\nu}).$$

Theorem II. The probability that exactly j of the k events (1) *occur is*

$$P_{(j)} = \sum_{\nu=0}^{k-j} (-1)^{\nu}(j + \nu; \nu)S(j + \nu). \tag{11}$$

Proof. If $A_{(j)}$ is the subset of Ω defined by the requirement that exactly j of the events (1) occur, then $A_{(j)}$ is the sum of $(k; j)$ disjunct sets:

$$A_{(j)} = \sum_{\alpha_1, \cdots, \alpha_j = 1}^{k} E_{\alpha_1} \cdots E_{\alpha_j} \bar{E}_{\alpha_{j+1}} \cdots \bar{E}_{\alpha_k}, \tag{12}$$

where $\alpha_{j+1}, \cdots, \alpha_k$ have those of the values $1, \cdots, k$ which remain after the selection of $\alpha_1, \cdots, \alpha_j$. By Axiom V we may replace A by P in (12). Upon substituting from (4) we note that the resulting terms of (12) which depend on the same number ν, $\nu = j, \cdots, k$, of events have the same sign, that all $S(\nu)$, $\nu = j, \cdots, k$, occur, that no term depending on fewer than j events occurs, and that any particular $P(E_{\alpha_1}E_{\alpha_2} \cdots E_{\alpha_{j+t}})$ will occur in those of the terms of (12) the j occurring events of which are a subset of $E_{\alpha_1}, E_{\alpha_2}, \cdots, E_{\alpha_{j+t}}$ and will occur in no other term of (12). Hence the coefficient of $S(j + t)$ in (11) is $(-1)^t (j + t; t)$. This completes the proof of the theorem.

COROLLARY. If (10) is true for $\nu = j, \cdots, k$, then

$$P_{(j)} = \sum_{\nu=0}^{k-j} (-1)^{\nu} (k; j, \nu)\, P(E_1E_2 \cdots E_{j+\nu}). \tag{13}$$

Theorem III. The probability that at least j of the k events (1) *occur is*

$$P^{(j)} = \sum_{\nu=0}^{k-j} (-1)^{\nu} (j + \nu - 1; \nu)\, S(j + \nu). \tag{14}$$

Proof. If $A^{(j)}$ is the subset of Ω defined by the requirement that at least j of the events (1) occur, then $A^{(j)}$ is the sum of $k - j + 1$ disjunct sets:

$$A^{(j)} = A_{(j)} + A_{(j+1)} + \cdots + A_{(k)}. \tag{15}$$

By Axiom V we may replace A by P in (15). Substituting from (11)

$$P^{(j)} = \sum_{\nu=0}^{k-j} c_{\nu}\, S(j + \nu), \tag{16}$$

where

$$c_{\nu} = (j + \nu; j + \nu) - (j + \nu; 1) + \cdots + (-1)^{\nu}(j + \nu; \nu), \quad (\nu = 0, \cdots, k - j).$$

It is easy to prove that

$$(-1)^{\nu}(j + \nu - 1; \nu) = \sum_{\mu=0}^{\nu} (-1)^{\nu-\mu}(j + \nu; j + \mu). \tag{17}$$

COROLLARY. If (10) is true for $\nu = j, \cdots, k$, then

$$P^{(j)} = \sum_{\nu=0}^{k-j} (-1)^{\nu}(j + \nu - 1; \nu)(k; j + \nu)P(E_1E_2 \cdots E_{j+\nu}). \tag{18}$$

To provide examples illustrating these theorems let us consider r experiments

$$E^{(1)}, E^{(2)}, \cdots, E^{(r)} \tag{19}$$

Let $E^{(i)}$ have k mutually exclusive outcomes

$$O_1^{(i)}, O_2^{(i)}, \cdots, O_k^{(i)}. \tag{20}$$

Then, it is easy to define the spaces $\Omega^{(i)}$, $\Delta^{(i)}$ the probability function $P_i(E^{(i)})$, the combinatory product

$$\Omega = \Omega^{(1)} \times \Omega^{(2)} \times \cdots \times \Omega^{(r)},$$

the set Δ and the probability function $P(E)$ so that Axioms I, $\cdots$, V are satisfied and hence Theorems I, II, and III are valid.

We shall assume that the experiments (19) are independent.

Let

$$\bar{O}_j \qquad (j = 1, \cdots, k)$$

be the event which occurs when neither $O_j^{(1)}$ nor $O_j^{(2)}$ nor $\cdots$ nor $O_j^{(r)}$ occur. Then O_j occurs if upon performance of the experiments (19) at least one of $O_j^{(1)}, O_j^{(2)}, \cdots, O_j^{(r)}$ occur.

It is an immediate result of the definition of independence that

$$P(\bar{O}_{\alpha_1}\bar{O}_{\alpha_2} \cdots \bar{O}_{\alpha_j}) = \prod_{i=1}^{k} \{1 - P(O_{\alpha_1}^{(i)}) - \cdots - P(O_{\alpha_j}^{(i)})\}. \tag{21}$$

From Theorem I, the probability that $O_1, O_2, \cdots, O_j$ each occur while not one of $O_{j+1}, O_{j+2}, \cdots, O_k$ occurs is

$$\begin{aligned} P(O_1 \cdots O_j\bar{O}_{j+1} \cdots \bar{O}_k) &= \sum_{\nu=0}^{j} (-1)^\nu \sum_{\substack{\alpha_1,\cdots,\alpha_\nu=1 \\ \alpha_1<\cdots<\alpha_\nu}}^{j} \\ &\prod_{i=1}^{r} \{1 - P(O_{j+1}^{(i)}) - \cdots - P(O_k^{(i)}) - P(O_{\alpha_1}^{(i)}) - \cdots - P(O_{\alpha_\nu}^{(i)})\}. \end{aligned} \tag{22}$$

From Theorem II, the probability that exactly j of $O_1, O_2, \cdots, O_k$ occur is

$$P_{(j)} = \sum_{\nu=0}^{j} (-1)^\nu (k - j + \nu; \nu) S(k - j + \nu), \tag{23}$$

where

$$S(k - j + \nu) = \sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_{k-j+\nu}=1 \\ \alpha_1<\alpha_2<\cdots<\alpha_{k-j+\nu}}}^{k} \prod_{i=1}^{r} \{1 - P(O_{\alpha_1}^{(i)}) - \cdots - P(O_{\alpha_{k-j+\nu}}^{(i)})\}.$$

Since the probability that at least j of $O_1, O_2, \cdots, O_k$ occur is equal to 1 minus the probability that at least $k - j + 1$ of $\bar{O}_1, \bar{O}_2, \cdots, \bar{O}_k$ occur,[12] it follows at once from Theorem III that

$$\begin{aligned} P\{\text{at least } j \text{ of } O_1, \cdots, O_k \text{ occur}\} &= \\ 1 - \sum_{\nu=0}^{j-1} (-1)^\nu (k - j + \nu; \nu) S(k - j + \nu + 1). \end{aligned} \tag{24}$$

[12] There are, of course, other ways of computing these probabilities.

The case treated by Fréchet and Jordan is that which occurs when we assume $P(O_t^{(h)}) = P(O_t^{(i)})$, $(t = 1, \cdots, k)$, $(i, h = 1, \cdots, r)$ and in (24) let $j = 1$.

It is not difficult to obtain further generalizations of Jordan's distribution by defining events which occur if and only if fewer than j' of r events occur and then proceeding as above.

Certain useful generalizations of Theorems I, II, and III will now be derived. Let the subsets of Ω

$$E_1^{(s)}, E_2^{(s)}, \cdots, E_{k^{(s)}}^{(s)} \qquad (s = 1, \cdots, p) \tag{25}$$

be elements of Δ, and let $N = k^{(1)} + k^{(2)} + \cdots + k^{(p)}$.

Let $j^{(s)} \leq k^{(s)}$, $(s = 1, \cdots, p)$; and let

$$Q^{(t)} = \prod_{s=1}^{t} \prod_{i=1}^{j^{(s)}} E_i^{(s)} \qquad (t = 1, \cdots, p), \tag{26}$$

Let

$$Q^{(t)'} = \prod_{s=1}^{t} \prod_{i=j^{(s)}+1}^{k^{(s)}} \bar{E}_i^{(s)} \qquad (t = 1, \cdots, p). \tag{27}$$

Furthermore, let for each value of s, $(s = h, \cdots, p)$, the $(k^{(s)} - j^{(s)}; \nu^{(s)})$ possible distinct selections of $\nu^{(s)}$ of the $k^{(s)} - j^{(s)}$ sets

$$E_{j^{(s)}+1}^{(s)}, E_{j^{(s)}+2}^{(s)}, \cdots, E_{k^{(s)}}^{(s)} \tag{28}$$

be arranged in some order, and, if the intersection of the $\nu^{(s)}$ sets of the i_sth selection be denoted by

$$q^{i_s}(\nu^{(s)}) \qquad (s = h, \cdots, p), \quad (i_s = 1, 2, \cdots, (k^{(s)} - j^{(s)}; \nu^{(s)})), \tag{29}$$

let

$$q^{i_h \cdots i_p}(\nu^{(h)}, \cdots, \nu^{(p)}) = \prod_{s=h}^{p} q^{i_s}(\nu^{(s)}). \tag{30}$$

There are $\prod_{s=h}^{p} (k^{(s)} - j^{(s)}; \nu^{(s)})$ sets (30), for each value of h, $(h = 1, \cdots, p)$, and any set of fixed values of $\nu^{(h)}, \cdots, \nu^{(p)}$.

Let for each value of s, $(s = h, \cdots, p)$ the $(k^{(s)}; \nu^{(s)})$ possible distinct selections of $\nu^{(s)}$ of the $k^{(s)}$ sets

$$E_i^{(s)}, \qquad (i = 1, \cdots, k^{(s)}), \tag{31}$$

be arranged in some order, and if the intersection of the sets of the i_sth selection be denoted by

$$\dot{q}^{i_s}(\nu^{(s)}) \tag{32}$$

let

$$\dot{q}^{i_h \cdots i_p}(\nu^{(h)}, \cdots, \nu^{(p)}) = \prod_{s=h}^{p} \dot{q}^{i_s}(\nu^{(s)}). \tag{33}$$

There are $\prod_{s=h}^{p} (k^{(s)}; \nu^{(s)})$ sets (33), for each value of h, $(h = 1, \cdots, p)$, and any set of fixed values of $\nu^{(h)}, \cdots, \nu^{(p)}$.

It is clear that the various sets that have been defined are elements of Δ. The fact that the sets are the events which occur if and only if certain sets of events occur is also too obvious to require further comment.

Theorem IV. The probability that of the N events (25) the first $j^{(s)}$ of superscript s occur and the remaining $k^{(s)}$ of superscript s do not occur, $s = 1, \cdots, p$, is

$$(34)\quad P(Q^{(p)} Q^{(p)\prime}) = \sum_{\nu^{(1)}=0}^{k^{(1)}-j^{(1)}} \sum_{\nu^{(2)}=0}^{k^{(2)}-j^{(2)}} \cdots \sum_{\nu^{(p)}=0}^{k^{(p)}-j^{(p)}} (-1)^{\nu^{(1)}+\nu^{(2)}+\cdots+\nu^{(p)}} \sum_{i_1=1}^{(k^{(1)}-j^{(1)};\nu^{(1)})} \cdots \sum_{i_p=1}^{(k^{(p)}-j^{(p)};\nu^{(p)})} P[q^{i_1\cdots i_p}(\nu^{(1)} \cdots \nu^{(p)})].$$

Proof. Theorem I is a proof of Theorem IV for $p = 1$. The theorem may then be proved either by regarding it as a special case of Theorem I and collecting terms, or by induction.

COROLLARY. If, for each possible set of values of $\nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(p)}$ the

$$\prod_{s=1}^{p} (k^{(s)} - j^{(s)}; \nu^{(s)})$$

terms

$$(35)\quad P[q^{i_1\cdots i_1}(\nu^{(1)}, \cdots, \nu^{(p)})]$$

are all equal, then

$$(36)\quad P(Q^{(p)} Q^{(p)\prime}) = \sum_{\nu^{(1)}=0}^{k^{(1)}-j^{(1)}} \cdots \sum_{\nu^{(p)}=0}^{k^{(p)}-j^{(p)}} (-1)^{\nu^{(1)}+\cdots+\nu^{(p)}} \prod_{s=1}^{p} (k^{(s)} - j^{(s)}; \nu^{(s)}) P[q^{1\cdots 1}(\nu^{(1)}, \cdots, \nu^{(p)})].$$

Let, for each value of h, $(h = 1, \cdots, p)$,

$$(37)\quad S(\nu^{(h)}, \nu^{(h+1)}, \cdots, \nu^{(p)}) = \sum_{i_h=1}^{(k^{(h)};\nu^{(h)})} \cdots \sum_{i_p=1}^{(k^{(p)};\nu^{(p)})} P[Q^{(h-1)} Q^{(h-1)\prime} q^{i_h\cdots i_p}(\nu^{(h)} \cdots \nu^{(p)})].$$

It is apparent that by using (34) it is possible to obtain an expression for (37) which does not depend explicitly on $Q^{(h-1)\prime}$. In fact

$$(38)\quad S(\nu^{(h)}, \cdots, \nu^{(p)}) = \sum_{\nu^{(1)}=0}^{k^{(1)}-j^{(1)}} \cdots \sum_{\nu^{(h-1)}=0}^{k^{(h-1)}-j^{(h-1)}} (-1)^{\nu^{(1)}+\cdots+\nu^{(h-1)}} \sum_{i_1=1}^{(k^{(1)}-j^{(1)};\nu^{(1)})} \cdots \sum_{i_{h-1}=1}^{(k^{(h-1)}-j^{(h-1)};\nu^{(h-1)})} \sum_{i_h=1}^{(k^{(h)};\nu^{(h)})} \cdots \sum_{i_p=1}^{(k^{(p)};\nu^{(p)})} P[q^{i_1\cdots i_{h-1}}(\nu^{(1)}, \cdots, \nu^{(h-1)}) q^{i_h\cdots i_p}(\nu^{(h)}, \cdots, \nu^{(p)})].$$

If the different terms of (37) are all equal, then

$$S(\nu^{(h)}, \cdots, \nu^{(p)}) = \prod_{s=h}^{p} (k^{(s)}; \nu^{(s)}) P[Q^{(h-1)} Q^{(h-1)\prime} \dot{q}^{1,\cdots,1}(\nu^{(h)}, \cdots, \nu^{(p)})]. \tag{39}$$

If the different terms of (38) are all equal, then

$$\begin{aligned} S(\nu^{(h)}, \cdots, \nu^{(p)}) = & \sum_{\nu^{(1)}=0}^{k^{(1)}-j^{(1)}} \cdots \sum_{\nu^{(h-1)}=0}^{k^{(h-1)}-j^{(h-1)}} (-1)^{\nu^{(1)}+\cdots+\nu^{(h-1)}} \\ & \prod_{s=1}^{h-1} (k^{(s)} - j^{(s)}; \nu^{(s)}) \prod_{s=h}^{p} (k^{(s)}; \nu^{(s)}) \\ & P[q^{1\cdots1}(\nu^{(1)}, \cdots, \nu^{(h-1)}) \dot{q}^{1\cdots1}(\nu^{(h)}, \cdots, \nu^{(p)})]. \end{aligned} \tag{40}$$

Theorem V. The probability that of the N events (25) *the first* $j^{(s)}$ *of superscript s occur and the remaining* $k^{(s)}$ *do not occur,* ($s = 1, \cdots, h - 1$), *and exactly* $j^{(s)}$ *events of superscript s occur* ($s = h, \cdots, p$), *is*

$$\begin{aligned} P_{(j^{(h)}\cdots j^{(p)})}(Q^{(h-1)} Q^{(h-1)\prime}) = & \sum_{\nu^{(h)}=0}^{k^{(h)}-j^{(h)}} \cdots \sum_{\nu^{(p)}=0}^{k^{(p)}-j^{(p)}} (-1)^{\nu^{(h)}+\cdots+\nu^{(p)}} \\ & \prod_{s=h}^{p} (j^{(s)} + \nu^{(s)}; \nu^{(s)}) S(j^{(h)} + \nu^{(h)}, \cdots, j^{(p)} + \nu^{(p)}). \end{aligned} \tag{41}$$

Proof. The theorem may be proved, either by induction using Theorem II, or by obtaining disjunct sets as in Theorem II and using Theorem IV.

COROLLARY I. If (39) is true for all sets of possible values of $\nu^{(h)}, \cdots, \nu^{(p)}$ then

$$\begin{aligned} P_{(j^{(h)}\cdots j^{(p)})}(Q^{(h-1)} Q^{(h-1)\prime}) = & \sum_{\nu^{(h)}=0}^{k^{(h)}-j^{(h)}} \cdots \sum_{\nu^{(p)}=0}^{k^{(p)}-j^{(p)}} (-1)^{\nu^{(h)}+\cdots+\nu^{(p)}} \\ & \prod_{s=h}^{p} (k^{(s)}; j^{(s)}, \nu^{(s)}) P[Q^{(h-1)} Q^{(h-1)\prime} \dot{q}^{1\cdots1}(\nu^{(h)}, \cdots, \nu^{(p)}). \end{aligned} \tag{42}$$

COROLLARY II. If (40) is true for all sets of possible values of $\nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(p)}$ then

$$\begin{aligned} P_{(j^{(h)}\cdots j^{(p)})}(Q^{(h-1)} Q^{(h-1)\prime}) = & \sum_{\nu^{(1)}=0}^{k^{(1)}-j^{(1)}} \cdots \sum_{\nu^{(p)}=0}^{k^{(p)}-j^{(p)}} (-1)^{\nu^{(1)}+\cdots+\nu^{(p)}} \\ & \prod_{s=1}^{h-1} (k^{(s)} - j^{(s)}; \nu^{(s)}) \prod_{s=h}^{p} (k^{(s)}; j^{(s)}, \nu^{(s)}) \\ & P[q^{1\cdots1}(\nu^{(1)}, \cdots, \nu^{(h-1)}) \dot{q}^{1\cdots1}(\nu^{(h)} \cdots \nu^{(p)})]. \end{aligned} \tag{43}$$

Theorem VI. The probability that of the N events (25) *the first* $j^{(s)}$ *events of superscript s occur and the remaining* $k^{(s)}$ *do not occur,* $s = 1, \cdots, g - 1$, *exactly* $j^{(s)}$ *events of superscript s occur* ($s = g, \cdots, h - 1$), *and at least* $j^{(s)}$ *events of superscript s occur* ($s = h, \cdots, p$) *is*

$$P_{(j^{(g)}\cdots j^{(h-1)})}^{(j^{(h)}\cdots j^{(p)})}(Q^{(g-1)}Q^{(g-1)\prime}) = \sum_{\nu^{(g)}=0}^{k^{(g)}-j^{(g)}} \cdots \sum_{\nu^{(p)}=0}^{k^{(p)}-j^{(p)}} (-1)^{\nu^{(g)}+\cdots+\nu^{(p)}}$$

$$\prod_{s=g}^{h-1}(j^{(s)}+\nu^{(s)};\nu^{(s)})\prod_{s=h}^{p}(j^{(s)}+\nu^{(s)}-1;\nu^{(s)}) \tag{44}$$

$$S(j^{(g)}+\nu^{(g)},\cdots,j^{(p)}+\nu^{(p)}).$$

Proof. The theorem may be proved either by induction using Theorem III or by obtaining disjunct sets as in Theorem III and using Theorem V.

COROLLARY I. If (39) is true for all sets of possible values of

$$\nu^{(g)}, \nu^{(g+1)}, \cdots, \nu^{(p)}$$

then

$$P_{(j^{(g)}\cdots j^{(h-1)})}^{(j^{(h)}\cdots j^{(p)})}(Q^{(g-1)}Q^{(g-1)\prime}) = \sum_{\nu^{(g)}=0}^{k^{(g)}-j^{(g)}} \cdots \sum_{\nu^{(p)}=0}^{k^{(p)}-j^{(p)}} (-1)^{\nu^{(g)}+\cdots+\nu^{(p)}}$$

$$\prod_{s=g}^{h-1}(k^{(s)};j^{(s)},\nu^{(s)})\prod_{s=h}^{p}[(j^{(s)}+\nu^{(s)}-1;\nu^{(s)})(k^{(s)};j^{(s)}+\nu^{(s)})] \tag{45}$$

$$P[Q^{(h-1)}Q^{(h-1)\prime}\dot{q}^{1\cdots1}(\nu^{(g)},\cdots,\nu^{(p)})].$$

COROLLARY II. If (40) is true for all sets of possible values of $\nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(p)}$ then

$$P_{(j^{(g)}\cdots j^{(h-1)})}^{(j^{(h)}\cdots j^{(p)})}(Q^{(g-1)}Q^{(g-1)\prime}) = \sum_{\nu^{(1)}=0}^{k^{(1)}-j^{(1)}} \cdots \sum_{\nu^{(p)}=0}^{k^{(p)}-j^{(p)}} (-1)^{\nu^{(1)}+\cdots+\nu^{(p)}}$$

$$\prod_{s=1}^{g-1}(k^{(s)}-j^{(s)};\nu^{(s)})\prod_{s=g}^{h-1}(k^{(s)};j^{(s)},\nu^{(s)})\prod_{s=h}^{p}[(j^{(s)}+\nu^{(s)}-1;\nu^{(s)})(k^{(s)};j^{(s)}+\nu^{(s)})] \tag{46}$$

$$P[q^{1\cdots1}(\nu^{(1)},\cdots,\nu^{(g-1)})\dot{q}^{1\cdots1}(\nu^{(g)},\cdots,\nu^{(p)})]\cdot$$

Let us again consider the experiments (19), and let us assume that $E^{(i)}$, $(i = 1, \cdots, r)$ has as its mutually exclusive results

$$O_{ts}^{(i)} \qquad (t = 1, \cdots, k^{(s)});\ (s = 1, 2). \tag{47}$$

Let O_{ts} be the event which occurs if, upon performance of the experiments (19) at least one of the events $O_{ts}^{(1)}, O_{ts}^{(2)}, \cdots, O_{ts}^{(r)}$ occur, and let $\bar{O}_{ts}$ be the event which occurs if and only if O_{ts} does not occur.

We may state the probability that the event E_1, which occurs if and only if at least $j^{(1)}$ of the events O_{t1}, $(t = 1, \cdots, k^{(1)})$ occur, and the event E_2, which occurs if and only if at least $j^{(2)}$ of the events O_{t2}, $(t = 1, \cdots, k^{(2)})$ occur, both occur.

It is apparent that

$$P(E_1E_2) = 1 - P(\bar{E}_1) - P(\bar{E}_2) + P(\bar{E}_1\bar{E}_2), \tag{48}$$

where $\bar{E}_3$ is the event which occurs if and only if E_s does not occur, $(s = 1, 2)$.

From Theorem III

$$P(\bar{E}_s) = \sum_{\nu^{(s)}=0}^{j^{(s)}-1} (-1)^{\nu^{(s)}} (k^{(s)} - j^{(s)} + \nu^{(s)}; \nu^{(s)}) S^{(s)}(k^{(s)} - j^{(s)} + \nu^{(s)} + 1) \tag{49}$$

$(s = 1, 2)$,

where

$$S^{(s)}(k^{(s)} - j^{(s)} + \nu^{(s)} + 1) = \sum_{\substack{\alpha_1, \cdots, \alpha_{k^{(s)} - j^{(s)} + \nu^{(s)} + 1} = 1 \\ \alpha_1 < \cdots < \alpha_{k^{(s)} - j^{(s)} + \nu^{(s)} + 1}}}^{k^{(s)}} \prod_{i=1}^{r} \{1 - P(O_{\alpha_1 s}^{(i)}) - \cdots - P(O_{\alpha_{k^{(s)} - j^{(s)} + \nu^{(s)} + 1} s}^{(i)})\}, \quad (s = 1, 2). \tag{50}$$

From Theorem VI

$$P(\bar{E}_1 \bar{E}_2) = \sum_{\nu^{(1)}=0}^{j^{(1)}-1} \sum_{\nu^{(2)}=0}^{j^{(2)}-1} (-1)^{\nu^{(1)}+\nu^{(2)}} \prod_{s=1}^{2} (k^{(s)} - j^{(s)} + \nu^{(s)}; \nu^{(s)}) S(k^{(1)} - j^{(1)} + \nu^{(1)} + 1, k^{(2)} - j^{(2)} + \nu^{(2)} + 1), \tag{50}$$

where

$$S(k^{(1)} - j^{(1)} + \nu^{(1)} + 1, k^{(2)} - j^{(2)} + \nu^{(2)} + 1) = \sum_{i_1=1}^{(k^{(1)}; j^{(1)} - \nu^{(1)} - 1)} \sum_{i_2=1}^{(k^{(2)}; j^{(2)} - \nu^{(2)} - 1)} P[q^{i_1 i_2}(k^{(1)} - j^{(1)} + \nu^{(1)} + 1, k^{(2)} - j^{(2)} + \nu^{(2)} + 1)],$$

and

$$P[q^{i_1 i_2}(k^{(1)} - j^{(1)} + \nu^{(1)} + 1, k^{(2)} - j^{(2)} + \nu^{(2)} + 1)] = \prod_{i=1}^{r} \left\{1 - \sum_{\nu=1}^{k^{(1)} - j^{(1)} + \nu^{(1)} + 1} P(O_{\alpha_\nu 1}^{(i)}) - \sum_{\mu=1}^{k^{(2)} - j^{(2)} + \nu^{(2)} + 1} P(O_{\beta_\mu 2}^{(i)})\right\},$$

the subscripts α_ν, $(\nu = 1, \cdots, k^{(1)} - j^{(1)} + \nu^{(1)} + 1)$, being those of the i_2^{th} selection of $k^{(1)} - j^{(1)} + \nu^{(1)} + 1$ events from $k^{(1)}$ events, and the subscripts β_μ, $(\mu = 1, \cdots, k^{(2)} - j^{(2)} + \nu^{(2)} + 1)$, being those of the i_2^{th} selection of $k^{(2)} - j^{(2)} + \nu^{(2)} + 1$ events from $k^{(2)}$ events.

The desired probability is then obtained by substituting from (49) and (50) into (48). The procedure is perfectly general, and applies directly to situations in which $p > 2$.

We shall now investigate the results obtained by requiring that the events considered satisfy a relation of implication.

Let the subsets of Ω

$$E_{1s}, E_{2s}, \cdots, E_{ks}, \quad (s = 1, \cdots, p), \tag{51}$$

be elements of Δ, and let

$$E_{is} \subset E_{it}, \quad (i = 1, \cdots, k) \tag{52}$$

if $s < t$.

It follows that

(53) $$P(E_{is}E_{it}) = P(E_{is}), \qquad (i = 1, \cdots, k), (s < t).$$

Let $j_1 \le j_2 \le \cdots \le j_t$ and let

(54) $$Q_t = \prod_{s=1}^{t} \prod_{i=1}^{j_s} E_{is}, \qquad (t = 1, 2, \cdots, p).$$

Let $j_1 \le j_2 \le \cdots \le j_t$ and let

(55) $$Q'_t = \prod_{s=1}^{t} \prod_{i=j_s+1}^{k} \bar{E}_{is}, \qquad (t = 1, 2, \cdots, p).$$

From (52) and (53), it follows that

(56) $$P(Q_t Q'_t) = P\left(\left[\prod_{s=1}^{t} \prod_{i=j_{s-1}+1}^{j_s} E_{is}\right] \left[\prod_{s=1}^{t-1} \prod_{i=j_s+1}^{j_{s+1}} \bar{E}_{is}\right] \prod_{i=j_t+1}^{k} \bar{E}_{it}\right), \quad (j_0 = 0) \quad (t = 1, 2, \cdots, p).$$

Let $j_1 \le j_2 \le \cdots \le j_p$ and for each value of s, $(s = 1, \cdots, p)$, consider a selection of $j_s + \nu_s$ events of second subscript s from (51). Let the p selections thus obtained be such that

$$j_s + \nu_s \le j_{s+1}, \quad (s = 1, 2, \cdots, p), (j_{p+1} = k),$$

and if E_{is} is one of the events of the selection of events of second subscript s then the fact that $t > s$ implies that E_{it} is one of the events of the selection of events of second subscript t.

From (52) and (53), the probability of the occurrence of all the events of the p selections thus obtained is a function of $j_p + \nu_p$ events, μ_s of which are of second subscript s, $(s = 1, \cdots, p)$ where

(57) $$\mu_1 + \mu_2 + \cdots + \mu_s = j_s + \nu_s, \qquad (s = 1, \cdots, p),$$

and for a given set of values of $j_1, j_2, \cdots, j_p$ the μ_s and ν_s determine one another uniquely, $(s = 1, \cdots, p)$.

For a definite set of values of $j_1, \cdots, j_p$ and $\mu_1, \cdots, \mu_p$ or $j_1, \cdots, j_p$ and $\nu_1, \cdots, \nu_p$ there will be

$$(j_{s+1} - j_s; \nu_s) = (j_{s+1} - j_s; j_{s+1} - \mu_1 - \cdots - \mu_s), \qquad (s = 1, \cdots, p), \; (j_{p+1} = k)$$

possible distinct selections of $j_s + \nu_s$, $(s = 1, \cdots, p)$ events of second subscript s, j_s of which are preassigned, from j_{s+1} events, $(s = 1, \cdots, p)$.

Let these selections be arranged in some order for each value of s, $s = 1, \cdots, p$, and let

(58) $$q_{i_1 i_2 \cdots i_p}(\mu_1, \mu_2, \cdots, \mu_p)$$

be the event which occurs when for all values of s, $(s = 1, \cdots, p)$, the events of the i_sth selection of $j_s + \nu_s$ events of second subscript s all occur.[13]

[13] It is understood that the j_s preassigned events of second subscript s are among the j_t preassigned events of second subscript t, $(t > s)$ in the events (58).

A typical event (58) is

$$(59) \qquad q_{1\cdots 1}(\mu_1, \cdots, \mu_p) = \prod_{s=1}^{p} \prod_{i=j_{s-1}+\nu_{s-1}+1}^{j_s+\nu_s} E_{is}, \qquad (j_0 + \nu_0 = 0).$$

There will be, for a definite j_s events of second subscript s, $(s = 1, \cdots, p)$

$$(60) \qquad \prod_{s=1}^{p} (j_{s+1} - j_s; \nu_s), \qquad (j_{p+1} = k),$$

events such as (58).

For a definite set of values of $\mu_1, \cdots, \mu_p$ there will be, for each value of s, $(s = 1, \cdots, p)$

$$(k - \mu_{s-1} - \cdots - \mu_1; \mu_s), \qquad (s = 1, 2, \cdots, p)$$

possible distinct selections of $j_s + \nu_s$ events of second subscript s, $j_{s-1} + \nu_{s-1}$ of which are preassigned from k events, $(s = 1, \cdots, p)$.

Let these selections be arranged in some order for each value of s,

$$(s = 1, \cdots, p),$$

and let

$$(61) \qquad q_{i_1 i_2 \cdots i_p}(\mu_1, \mu_2, \cdots, \mu_p)$$

be the event which occurs if and only if, for all values of s the events of the i_s^{th} set of $j_s + \nu_s$ events of second subscript s all occur, $(s = 1, \cdots, p)$, and the first subscripts of the events of the i_s^{th} set of events of second subscript s are among the first subscripts of the events of all the selections of events of second subscript greater than s, $(s = 1, \cdots, p)$.

There will be

$$(62) \qquad (k; \mu_1, \mu_2, \cdots, \mu_p)$$

events (61) which may thus be obtained.

Theorem VII. The probability that of the pK events (51) *the first j_s events of second subscript s occur and the remaining $k - j_s$ events do not occur, $s = 1, \cdots, p$,* is

$$(63) \qquad P(Q_p Q_p') = \sum_{\nu_1=0}^{j_2-j_1} \sum_{\nu_2=0}^{j_3-j_2} \cdots \sum_{\nu_p=0}^{k-j_p} (-1)^{\nu_1+\nu_2+\cdots+\nu_p} \sum_{i_1=1}^{(j_2-j_1;\nu_1)} \sum_{i_2=1}^{(j_3-j_2;\nu_2)} \cdots \sum_{i_p=0}^{(k-j_p;\nu_p)} P[q_{i_1 i_2 \cdots i_p}(\mu_1, \mu_2, \cdots, \mu_p)],$$

where the event Q_t determines the $j_s - j_{s-1} - \nu_{s-1}$ events of second subscript s, $(s = 1, \cdots, p)$, which have as first subscripts all numbers $1, 2, \cdots, j_s$ which are not among the $j_{s-1} + \nu_{s-1}$ numbers determined by the events of lower second subscript than s which are contained in $q_{i_1 \cdots i_p}(\mu_1, \cdots, \mu_p)$.

Proof. Expand (56) by means of Theorem IV.

COROLLARY. If, for each fixed set of values of $\mu_1, \mu_2, \cdots, \mu_p$ the terms (58), in number (60), are all equal, then

$$P(Q_pQ_p') = \sum_{\nu_1=0}^{j_2-j_1} \sum_{\nu_2=0}^{j_3-j_2} \cdots \sum_{\nu_p=0}^{k-j_p} (-1)^{\nu_1+\nu_2+\cdots+\nu_p} \prod_{s=1}^{p} (j_{s+1} - j_s; \nu_s) P[q_{1\cdots1}(\mu_1, \mu_2, \cdots, \mu_p)] \qquad (j_{p+1} = k). \tag{64}$$

Let

$$T(\mu_1, \mu_2, \cdots, \mu_p) = \sum_{i_1=1}^{(k;\mu_1)} \sum_{i_2=1}^{(k-\mu_1;\mu_2)} \cdots \sum_{i_p=1}^{(k-\mu_1-\cdots-\mu_{p-1};\mu_p)} P[\dot{q}_{i_1i_2\cdots i_p}(\mu_1, \mu_2, \cdots, \mu_p)]. \tag{65}$$

If all the terms of (65) are equal, then

$$T(\mu_1, \cdots, \mu_p) = (k; \mu_1, \mu_2, \cdots, \mu_p) P[\dot{q}_{1\cdots1}(\mu_1, \cdots, \mu_p)]. \tag{66}$$

Theorem VIII. The probability that of the pK events (51) *exactly j_s events of second subscript s, $s = 1, \cdots, p$ occur, is*

$$P_{(j_1\cdots j_p)} = \sum_{\nu_1=0}^{j_2-j_1} \sum_{\nu_2=0}^{j_3-j_2} \cdots \sum_{\nu_p=0}^{k-j_p} (-1)^{\nu_1+\nu_2+\cdots+\nu_p} \prod_{s=1}^{p} (\mu_s; j_s - \mu_1 - \cdots - \mu_{s-1}) T(\mu_1, \mu_2, \cdots, \mu_p). \tag{67}$$

Proof. If $A_{(j_1, \ldots, j_p)}$ is the subset of Ω determined by the requirement that exactly j_s of the events (51) occur $(s = 1, \cdots, p)$, then $A_{(j_1, \ldots, j_p)}$ is the sum of

$$(k; j_1, j_2 - j_1, j_3 - j_2, \cdots, j_p - j_{p-1})$$

disjunct sets which may be obtained by replacing P by A in (56) and forming (56) for all selections of $j_s - j_{s-1}$ occurring events from $k - j_{s-1}$ events, $(s = 1, \cdots, p)$. By Axiom V, $P_{(j_1, \ldots, j_p)}$ is the sum of the probabilities of these disjunct sets.

Substituting from (63), it is noted that all terms (61) which depend on the same μ_s, $(s = 1, \cdots, p)$, have the same sign and that all $T(\mu_1, \mu_2, \cdots, \mu_p)$ for which

$$0 \leq \nu_s \leq j_{s+1} - j_s, \qquad (s = 1, \cdots, p),$$

appear and only those appear. Furthermore any particular term (61) will occur in those of the terms (63) the $j_s' - j_{s-1}$ occurring events of second subscript s, $(s = 1, \cdots, p)$, of which contain a fixed ν_{s-1} events, the remaining $j_s - j_{s-1} - \nu_{s-1}$ events being a subset of the μ_s events of second subscript s, $(s = 1, \cdots, p)$, that actually appear in the particular term (63). Hence the coefficient of $T(\mu_1, \cdots, \mu_p)$ is

$$(-1)^{\nu_1+\cdots+\nu_p} \prod_{s=1}^{p} (\mu_s; j_s - \mu_1 - \cdots - \mu_{s-1}), \qquad (\mu_0 = 0).$$

COROLLARY. If (66) is true for all sets of possible values of $\mu_1, \mu_2, \cdots, \mu_p$ then

$$P_{(j_1,\cdots,j_p)} = \sum_{\nu_1=0}^{j_2-j_1} \sum_{\nu_2=0}^{j_3-j_2} \cdots \sum_{\nu_p=0}^{k-j_p} (-1)^{\nu_1+\nu_2+\cdots+\nu_p} \tag{68}$$
$$(k; j_1, \nu_1, j_2 - j_1 - \nu_1; \nu_2, \cdots, j_p - j_{p-1} - \nu_{p-1}, \nu_p)$$
$$P[\dot{q}_{1,\cdots 1}(\mu_1, \mu_2, \cdots, \mu_p)].$$

Theorem IX. The probability that of the pk events (51) *at least j_s, but not more than j_{g+1}, events of second subscript s occur,* ($s = 1, \cdots, g$), *and exactly j_s events of second subscript s occur,* ($s = g + 1, \cdots, p$) *is*

$$P^{(j_1,\cdots,j_g)}_{(j_{g+1},\cdots,j_p)} = \sum_{\theta_2=0}^{1} \sum_{\theta_3=0}^{1} \cdots \sum_{\theta_g=0}^{1} R_{(j_{g+1}\cdots j_p)}(1, \theta_2, \cdots, \theta_g), \tag{69}$$

where, if a 1 in the i^{th} position is denoted by δ_i, ($i = 2, \cdots, g$),

$$R_{(j_{g+1},\cdots,j_p)}(1, \delta_1, \cdots, \delta_{\gamma_1}, 0, \cdots, 0, \delta_{\gamma_2+1}, \cdots, \delta_{\gamma_3}, 0, \cdots, 0, \cdots, \delta_{\gamma_h+1}, \cdots, \delta_g)$$
$$= \sum_{\nu_p=0}^{k-j_p} \cdots \sum_{\nu_{g+1}=0}^{j_{g+2}-j_{g+1}} \sum_{\nu_g=0}^{j_{g+1}-j_g} \cdots \sum_{\nu_{\gamma_3}=j_{\gamma_4}-j_{\gamma_3}}^{j_{\gamma_4+1}-j_{\gamma_3}-1} \sum_{\nu_{\gamma_3-1}=0}^{j_{\gamma_3}-j_{\gamma_3-1}-1} \cdots \sum_{\nu_1=0}^{j_2-j_1-1} (-1)^{\nu_1+\nu_2+\cdots+\nu_p}$$
$$(j_1 + \nu_1 - 1; \nu_1) \cdots (j_{\gamma_3} + \nu_{\gamma_3} - j_{\gamma_3-1} - \nu_{\gamma_3-1} - 1; \nu_{\gamma_3}) \tag{70}$$
$$(j_{\gamma_4} + \nu_{\gamma_4} - j_{\gamma_3} - \nu_{\gamma_3} - 1; \nu_{\gamma_4}) \cdots (j_p + \nu_p - j_{p-1} - \nu_{p-1}; \nu_p)$$
$$T(j_1 + \nu_1, \cdots, j_{\gamma_3} + \nu_{\gamma_3} - j_{\gamma_3-1} - \nu_{\gamma_3-1}, 0, \cdots, 0,$$
$$j_{\gamma_4} + \nu_{\gamma_4} - j_{\gamma_3} - \nu_{\gamma_3}, \cdots, j_p + \nu_p - j_{p-1} - \nu_{p-1}).$$

Proof. We note first that there are 2^{g-1} terms in (69). Since

$$P^{(j_1,\cdots,j_g)}_{(j_{g+1},\cdots,j_p)} = \sum_{\lambda_g=j_g}^{j_{g+1}} \cdots \sum_{\lambda_2=j_2}^{\lambda_3} \sum_{\lambda_1=j_1}^{\lambda_2} P_{(\lambda_1\cdots\lambda_g j_{g+1}\cdots j_p)}, \tag{71}$$

the theorem may be proved by a process of repeated summation. From (67) and (71)

$$P^{(j_1)}_{(\lambda_2\cdots\lambda_g j_{g+1}\cdots j_p)} = \sum_{\lambda_1=j_1}^{\lambda_2} \sum_{\nu_1=0}^{\lambda_2-\lambda_1} \sum_{\nu_2=0}^{\lambda_3-\lambda_2} \cdots \sum_{\nu_p=0}^{k-j_p} (-1)^{\nu_1+\nu_2+\cdots+\nu_p} \tag{72}$$
$$(\lambda_1 + \nu_1; \nu_1)(\lambda_2 + \nu_2 - \lambda_1 - \nu_1; \nu_2) \cdots (j_p + \nu_p - j_{p-1} - \nu_{p-1}; \nu_p)$$
$$T(\lambda_1 + \nu_1, \lambda_2 + \nu_2 - \lambda_1 - \nu_1, \cdots, j_p + \nu_p - j_{p-1} - \nu_{p-1}).$$

For fixed values of $\lambda_2, \lambda_3, \cdots, \lambda_g$ there will occur in (72) all terms

$$T(j_1 + \beta_1, \lambda_2 + \nu_2 - j_1 - \beta_1, \cdots, j_p + \nu_p - j_{p-1} - \nu_{p-1}), \tag{73}$$
$$(\beta_1 = 0, \cdots, \lambda_2 - j_1), \quad (0 \leq \nu_s \leq \lambda_{s+1} - \lambda_s), \quad (s = 2, \cdots, p),$$
$$(\lambda_{g+s} = j_{g+s}\ s = 1, \cdots, p - g),$$

and any definite term (73) will occur in all

$$P_{(j_1+\alpha,\lambda_2,\cdots,j_p)} \tag{74}$$

for which

$$0 \leq \alpha \leq \beta_1 .$$

In (74), the definite term (73) will have coefficient

$$(75) \qquad \begin{aligned} &(-1)^{\beta_1-\alpha+\nu_2+\cdots+\nu_p}(j_1+\beta_1\,;j_1+\alpha)(\lambda_2+\nu_2-j_1-\beta_1\,;\nu_2)\\ &\qquad\cdots(j_p+\nu_p-j_{p-1}-\nu_{p-1}\,,\nu_p), \qquad (\alpha = 0, 1, \cdots, \beta_1),\\ &\qquad\qquad\qquad\qquad\qquad\qquad (\beta_1 = 0, \cdots, \lambda_2 - j_1). \end{aligned}$$

Hence, in (72) the definite term (73), will have coefficient

$$\begin{aligned} &(-1)^{\beta_1+\nu_2+\cdots+\nu_p}(j_1+\beta_1-1;\beta_1)(\lambda_2+\nu_2-j_1-\beta_1\,;\nu_2)\\ &\qquad\qquad\cdots(j_p+\nu_p-j_{p-1}-\nu_{p-1}\,;\nu_p), \end{aligned}$$

and

$$(76) \qquad P^{(j_1)}_{(\lambda_2,\cdots,j_p)} = R_{(\lambda_2\cdots j_p)}(1).$$

We now evaluate

$$(77) \qquad P^{(j_1j_2)}_{(\lambda_3\cdots j_p)} = \sum_{\lambda_2=j_2}^{\lambda_3} P^{(j_1)}_{(\lambda_2\cdots j_p)} .$$

For any fixed values of $\lambda_3, \cdots, \lambda_g$, there will occur in (77) all terms

$$(78) \qquad \begin{aligned} &T(j_1+\beta_1\,, j_2+\beta_2-j_1-\beta_1\,, \lambda_3+\nu_3-j_2-\beta_2\,,\\ &\qquad\qquad \cdots, j_p+\nu_p-j_{p-1}-\nu_{p-1}), \end{aligned}$$

for which either $0 \leq \beta_2 \leq \lambda_3 - j_2\,; 0 \leq \beta_1 \leq j_2 - j_1 - 1$ or $\beta_1 = j_2 - j_1 + \gamma$, $0 \leq \gamma \leq \lambda_3 - j_2\,; 0 \leq \beta_2 \leq \lambda_3 - j_2 - \gamma$.

Let $0 \leq \beta_1 \leq j_2 - j_1 - 1; 0 \leq \beta_2 \leq \lambda_3 - j_2$. Then the term (78) will occur in all

$$(79) \qquad P^{(j_1)}_{(j_2+\alpha,\lambda_3,\cdots,j_p)},$$

such that

$$0 \leq \alpha \leq \beta_2 .$$

In (79), (78) will have coefficient

$$(80) \qquad \begin{aligned} &(-1)^{\beta_1+\beta_2-\alpha+\nu_3+\cdots+\nu_p}(j_1+\beta_1-1;\beta_1)(j_2+\beta_2-j_1-\beta_1-1;\beta_2-\alpha)\\ &\qquad(\lambda_3+\nu_3-j_2-\beta_2\,;\nu_3)\cdots(j_p+\nu_p-j_{p-1}-\nu_{p-1}\,;\nu_p). \end{aligned}$$

Hence in (77), (78) will have coefficient

$$(81) \qquad \begin{aligned} &(-1)^{\beta_1+\beta_2+\nu_3+\cdots+\nu_p}(j_1+\beta_1-1;\beta_1)(j_2+\beta_2-j_1-\beta_1-1;\beta_2)\\ &(\lambda_3+\nu_3-j_2-\beta_2\,;\nu_3)\cdots(j_p+\nu_p-j_{p-1}-\nu_{p-1}\,;\nu_p),\\ &(\beta_1 = 0, \cdots, j_2 - j_1 - 1), \qquad (\beta_2 = 0, \cdots, \lambda_3 - j_2),\\ &\quad(\nu_s = 0, \cdots, \lambda_{s+1} - \lambda_s), \qquad (s = 3, \cdots, p);\\ &\qquad\qquad\qquad\qquad (\lambda_{g+s} = j_{g+s}), (s = 1, \cdots, p - g) \end{aligned}$$

Now let $\beta_1 = j_2 - j_1 + \gamma; 0 \leq \gamma \leq \lambda_3 - j_2 ; 0 \leq \beta_2 \leq \lambda_3 - j_2 - \gamma$. Then the term (78) will occur in all terms (79) such that

$$\gamma \leq \alpha \leq \beta_2 ,$$

and in (79), (78) will have coefficient (80). Summing for α, $(\alpha = \gamma, \cdots, \beta_2)$, we obtain as the coefficient of (78) in (77)

$$0, \qquad \text{if } \beta_2 > \gamma,$$

and

$$(-1)^{\beta_1+\nu_3+\cdots+\nu_p}(j_1 + \beta_1 - 1; \beta_1)(\lambda_3 + \nu_3 - j_1 - \beta_1 ; \nu_3) \cdots (j_p + \nu_p - j_{p-1} - \nu_{p-1} ; \nu_p), \qquad \text{if } \beta_2 = \gamma.$$

Hence

$$P^{(j_1 j_2)}_{(\lambda_3 \cdots j_p)} = R_{(\lambda_3 \cdots j_p)}(1, 1) + R_{(\lambda_3 \cdots j_p)}(1, 0). \tag{82}$$

If we examine (82), we note that the result of summing with respect to λ_2 has been the replacement of (76) by two sums which are similar to (76) in that the next summation index, in this case λ_3, occurs in exactly two limits of summation. If it can be shown that the two sums which occur in (82) each result in a pair of sums after summation with respect to λ_3, or more exactly if

$$\sum_{\lambda_{s+1}=j_{s+1}}^{\lambda_{s+2}} R_{(\lambda_{s+1}, \cdots, j_p)}(1, \theta_2, \cdots, \theta_s) = R_{(\lambda_{s+2} \cdots j_p)}(1, \theta_2, \cdots, \theta_s, 1) + R_{(\lambda_{s+2}, \cdots, j_p)}(1, \theta_2, \cdots, \theta_s, 0) \tag{83}$$

then the proof will be completed.

Since the truth of (83) may be demonstrated in exactly the same way in which (82) has been shown to be true, the theorem is proved.

COROLLARY. If (66) is true for all sets of possible values of $\mu_1, \mu_2, \cdots, \mu_p$ then

$$\begin{aligned} &R_{(j_{g+1}, \cdots, j_p)}(1, \delta_1, \cdots \delta_{\gamma_1}, 0, \cdots, 0, \delta_{\gamma_2+1}, \cdots, \delta_{\gamma_3}, 0, \cdots 0, \cdots, \delta_{\gamma_h+1}, \cdots, \delta_g) \\ &= \sum_{\nu_p=0}^{k-j_p} \cdots \sum_{\nu_{g+1}=0}^{j_{g+2}-j_{g+1}} \sum_{\nu_g=0}^{j_{g+1}-j_g} \cdots \sum_{\nu_{\gamma_3}=j_{\gamma_4}-j_{\gamma_3}}^{j_{\gamma_4+1}-j_{\gamma_3-1}} \cdots \sum_{\nu_1=0}^{j_2-j_1-1} (-1)^{\nu_1+\cdots+\nu_p} \\ &(j_1 + \nu_1 - 1; \nu_1) \cdots (j_{\gamma_3} + \nu_{\gamma_3} - j_{\gamma_3-1} - \nu_{\gamma_3-1} - 1; \nu_{\gamma_3}) \\ &(j_{\gamma_4} + \nu_{\gamma_4} - j_{\gamma_3} - \nu_{\gamma_3} - 1; \nu_{\gamma_4}) \cdots (j_p + \nu_p - j_{p-1} - \nu_{p-1} ; \nu_p) \\ &(k; j_1 + \nu_1, \cdots, j_{\gamma_3} + \nu_{\gamma_3} - j_{\gamma_3-1} - \nu_{\gamma_3-1}, j_{\gamma_4} \\ &\qquad + \nu_{\gamma_4} - j_{\gamma_3} - \nu_{\gamma_3}, \cdots, j_p + \nu_p - j_{p-1} - \nu_{p-1}) \\ &P[q_1 \cdots {}_1(j_1 + \nu_1, \cdots, j_{\gamma_3} + \nu_{\gamma_3} - j_{\gamma_3-1} - \nu_{\gamma_3-1}, 0, \cdots, 0, \\ &\qquad j_{\gamma_4} + \nu_{\gamma_4} - j_{\gamma_3} - \nu_{\gamma_3}, \cdots, j_p + \nu_p - j_{p-1} - \nu_{p-1})]. \end{aligned} \tag{84}$$

Let us again consider the experiments (19) and let $E^{(i)}$ have as possible results

$$O_{js}^{(i)} \qquad (j = 1, \cdots, k), (s = 1, 2) \ (i = 1, 2, \cdots, r).$$

Let

$$O_{j1}^{(i)} \supset O_{j2}^{(i)} \qquad \begin{array}{l}(i = 1, \cdots, r),\\ (j = 1, \cdots, k),\end{array}$$

i.e. $O_{j1}^{(i)}$ occurs whenever $O_{j2}^{(i)}$ occurs. Furthermore let the outcomes

$$O_{11}^{(i)}, O_{21}^{(i)}, \cdots, O_{k1}^{(i)}$$

be mutually exclusive.

Let

$$\bar{O}_{js}, \qquad (s = 1, 2),$$

occur if and only if none of

$$O_{js}^{(1)}, O_{js}^{(2)}, \cdots, O_{js}^{(k)}$$

occur.

We may wish to know the probability that at least j_1 of $\bar{O}_{11}, \cdots, \bar{O}_{k1}$ and at least j_2, $j_2 \geq j_1$, of $\bar{O}_{12}, \bar{O}_{22}, \cdots, \bar{O}_{k2}$ occur.

From Theorem IX this probability is equal to

$$P^{(j_1, j_2)} = R(1, 1) + R(1, 0), \tag{85}$$

where

$$R(1, 1) = \sum_{\nu_2=0}^{k-j_2} \sum_{\nu_1=0}^{j_2-j_1-1} (-1)^{\nu_1+\nu_2}(j_1 + \nu_1 - 1; \nu_1)$$
$$(j_2 + \nu_2 - j_1 - \nu_1 - 1; \nu_2)T(j_1 + \nu_1, j_2 + \nu_2 - j_1 - \nu_1),$$

and

$$R(1, 0) = \sum_{\nu_1=j_2-j_1}^{k-j_1} (-1)^{\nu_1}(j_1 + \nu_1 - 1; \nu_1)T(j_1 + \nu_1).$$

From (63)

$$T(j_1 + \nu_1, j_2 + \nu_2 - j_1 - \nu_1) = \sum_{i_1=1}^{(k; j_1+\nu_1)} \sum_{i_2=1}^{(k-j_1-\nu_1; j_2+\nu_2-j_1-\nu_1)} P[\dot{q}_{i_1 i_2}(j_1 + \nu_1; j_2 + \nu_2 - j_1 - \nu_1)], \tag{86}$$

where, from (61)

$$\dot{q}_{i_1 i_2}(j_1 + \nu_1, j_2 + \nu_2 - j_1 - \nu_1) = \prod_{\nu=1}^{j_1+\nu_1} \bar{O}_{\alpha_\nu 1} \prod_{\nu=j_1+\nu_1+1}^{j_2+\nu_2} \bar{O}_{\alpha_\nu 2},$$

the subscripts

$$\alpha_1, \alpha_2, \cdots, \alpha_{j_1+\nu_1} \tag{87}$$

being the first subscripts of the i_1^{th} selection of $j_1 + \nu_1$ events of second subscript 1 from

$$\bar{O}_{11}, \bar{O}_{21}, \cdots, \bar{O}_{k1},$$

and the subscripts

$$\alpha_{j_1+\nu_1+1}, \alpha_{j_1+\nu_1+2}, \cdots, \alpha_{j_2+\nu_2},$$

being the first subscripts of the i_2^{th} selection of $j_2 + \nu_2$ events of second subscript 2, $j_1 + \nu_1$ of which are (87), from

$$\bar{O}_{12}, \bar{O}_{22}, \cdots, \bar{O}_{k2}.$$

It is easy to see that

$$P[\dot{q}_{i_1 i_2}(j_1 + \nu_1, j_2 + \nu_2 - j_1 - \nu_1)] = \prod_{i=1}^{r} \left\{1 - \sum_{\nu=1}^{j_1+\nu_1} P(O_{\alpha_\nu 1}^{(i)}) - \sum_{\nu=j_1+\nu_1+1}^{j_2+\nu_2} P(O_{\alpha_\nu 2}^{(i)})\right\}.$$

Furthermore

$$T(j_1 + \nu_1) = \sum_{i_1=1}^{(k;\, j_1+\nu_1)} P[\dot{q}_{i_1}(j_1 + \nu_1)], \tag{88}$$

where

$$P[\dot{q}_{i_1}(j_1 + \nu_1)] = \prod_{i=1}^{r} \left\{1 - \sum_{\mu=1}^{j_1+\nu_1} P(O_{\alpha_\mu 1}^{(i)})\right\}.$$

Substituting from (86) and (88) into (85) the desired probability is obtained.

It may be remarked that theorems which have the same relation to Theorems VII, VIII, and IX that Theorems IV, V, and VI have to Theorems I, II, and III may be obtained without much difficulty.

BIBLIOGRAPHY

(1) A. Fisher, The Mathematical Theory of Probabilities, The MacMillan Company, New York, 1930.
(2) R. A. Fisher, The Design of Experiments, Oliver and Boyd, London, 1935.
(3) M. Fréchet, Généralités sur les Probabilités, Gauthier-Villars, Paris, 1937.
(4) E. J. Gumbel, Les valeurs extrêmes des distributions statistiques, Annales de L'Institut Henri Poincaré, vol. 5, (1935), pp. 115-158.
(5) Ch. Jordan, Le théorème de probabilité de Poincaré généralisé au cas de plusieurs variables indépendantes, Acta Scientarum Mathematicarum, Szeged, vol. 7, (1934), pp. 103-111.
(6) A. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrechnung, Ergebnisse der Mathematik, vol. 2, no. 3.
(7) H. Poincaré, Calcul des Probabilités, Gauthier-Villars, Paris, 1912.

Columbia University.