ON CERTAIN DISTRIBUTIONS DERIVED FROM THE MULTINOMIAL DISTRIBUTION[1]

By Solomon Kullback

1. **Introduction.** With the multinomial distribution as a background, there may be derived a number of distributions which are of interest in certain practical applications. Several of these distributions are here presented and the theory is illustrated by specific examples.

2. **Preliminary data.** In the discussion of the distributions to be considered there are needed certain factorial sums whose values are now to be derived. In the following discussion only positive integral values (including zero) are to be considered.

There is desired the value, in terms of N, n, r, of

$$f_r(n, N) = \sum \frac{N!}{x_1!\, x_2! \cdots x_n!} \tag{2.1}$$

where the summation is for all values of $x_1, x_2, \cdots, x_n$ such that $x_1 + x_2 + \cdots + x_n = N$ and no x is equal to r.

Let us first consider the case for $r = 0$; i.e., we desire a value for the sum in (2.1) for all values of $x_1, x_2, \cdots, x_n$ such that $x_1 + x_2 + \cdots + x_n = N$ and no x is equal to zero. By the multinomial theorem, we have that[2]

$$(a_1 + a_2 + \cdots + a_n)^N = \sum \frac{N!}{x_1!\, x_2! \cdots x_n!} a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n} \tag{2.2}$$

where the summation is for all values of $x_1, x_2, \cdots, x_n$ such that $x_1 + x_2 + \cdots + x_n = N$. If $a_1 = a_2 = \cdots = a_n = 1$, then

$$n^N = \sum \frac{N!}{x_1!\, x_2! \cdots x_n!}, \qquad x_1 + x_2 + \cdots + x_n = N. \tag{2.3}$$

The sum in (2.3) may however be rearranged into the sum of a number of terms as follows:

$$(2.4)\quad \begin{cases} \sum \dfrac{N!}{x_1!\, x_2! \cdots x_n!}, & x_1 + x_2 + \cdots + x_n = N, \quad \text{no } x = 0; \\ n \sum \dfrac{N!}{x_1!\, x_2! \cdots x_{n-1}!}, & x_1 + x_2 + \cdots + x_{n-1} = N, \quad \text{no } x = 0; \\ \dfrac{n(n-1)}{2} \sum \dfrac{N!}{x_1!\, x_2! \cdots x_{n-2}!}, & x_1' + x_2 + \cdots + x_{n-2} = N, \quad \text{no } x = 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \dbinom{n}{r} \sum \dfrac{N!}{x_1!\, x_2! \cdots x_{n-r}!}, & x_1 + x_2 + \cdots + x_{n-r} = N, \quad \text{no } x = 0. \end{cases}$$

[1] Presented to the Institute of Mathematical Statistics January 2, 1936.

[2] H. S. Hall & S. R. Knight, *Higher Algebra*, MacMillan & Co., 4th Ed. (1924), Chap. 15.

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Thus we may rewrite (2.3) as

$$n^N = f_0(n, N) + nf_0(n-1, N) + \frac{n(n-1)}{2!} f_0(n-2, N) + \cdots + \binom{n}{r} f_0(n-r, N) + \cdots \tag{2.5}$$

Replacing n by $n - 1$ in (2.5) there is obtained

$$(n-1)^N = f_0(n-1, N) + (n-1) f_0(n-2, N) + \cdots + \binom{n-1}{r} f_0(n-r-1, N) + \cdots \tag{2.6}$$

Multiplying (2.6) by n and subtracting the result from (2.5), there is obtained

$$n^N - n(n-1)^N = f_0(n, N) - \frac{n(n-1)}{2!} f_0(n-2, N) - \cdots - r\binom{n}{r+1} f_0(n-r-1, N) - \cdots \tag{2.7}$$

Replacing n by $n - 2$ in (2.5) there is obtained

$$(n-2)^N = f_0(n-2, N) + (n-2) f_0(n-3, N) + \cdots + \binom{n-2}{r-1} f_0(n-r-1, N) + \cdots \tag{2.8}$$

Multiplying (2.8) by $n(n - 1)/2$ and adding the result to (2.8), there is obtained

$$n^N - n(n-1)^N + \frac{n(n-1)}{2!}(n-2)^N = f_0(n, N) + \frac{n(n-1)(n-2)}{3!} f_0(n-3, N) + \cdots + \frac{r(r-1)}{2!}\binom{n}{r+1} f_0(n-r-1, N) + \cdots \tag{2.9}$$

Continuing this process, there is finally obtained the result that

$$f_0(n, N) = n^N - n(n-1)^N + \frac{n(n-1)}{2!}(n-2)^N - \cdots \pm n \cdot 1^N \tag{2.10}$$

It may be shown[3] that the right side of (2.10) is $\Delta^n x^N$ for $x = 0$. The author has elsewhere obtained (2.10), but by a special procedure not applicable to the general case.[4]

We may readily verify (2.10) for example, for $n = 3$, $N = 5$. If $x_1 + x_2 + x_3 = 5$ and no $x = 0$, then the sets of solutions are (3,1,1), (1,3,1), (1,1,3), (2,2,1), (2,1,2), (1,2,2), and $f_0(3,5) = 3 \cdot \frac{5!}{3!1!1!} + 3 \frac{5!}{2!2!1!} = 150$. From (2.10) there is obtained $f_0(3,5) = 3^5 - 3.2^5 + 3.2/2 = 150$.

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[3] E. T. Whittaker & G. Robinson, *The Calculus of Observations*, Blackie & Son Ltd. (1924), p. 7.

[4] S. Kullback, "On the Bernoulli Distribution," *Bull. Am. Math. Soc.*, December, 1935.

For the general case, we return again to (2.3) and rearrange the right side into the sum of a number of terms as follows:

$$(2.11)\quad \begin{cases} \sum \dfrac{N!}{x_1!\,x_2!\cdots x_n!}, & x_1 + x_2 + \cdots + x_n = N, \quad \text{no } x = r; \\ \dfrac{n}{r!}\sum \dfrac{N!}{x_1!\,x_2!\cdots x_{n-1}!}, & x_1 + x_2 + \cdots + x_{n-1} = N - r, \quad \text{no } x = r; \\ \dfrac{n(n-1)}{2!\,(r!)^2}\sum \dfrac{N!}{x_1!\,x_2!\cdots x_{n-2}!}, & x_1 + x_2 + \cdots + x_{n-2} = N - 2r, \quad \text{no } x = r; \\ \cdots\cdots & \cdots\cdots \\ \dbinom{n}{k}\left(\dfrac{1}{r!}\right)^k \sum \dfrac{N!}{x_1!\,x_2!\cdots x_{n-k}}, & x_1 + x_2 + \cdots + x_{n-k} = N - kr, \quad \text{no } x = r. \end{cases}$$

Thus we may rewrite (2.3) as

$$(2.12)\quad \begin{aligned} n^N &= f_r(n, N) + \frac{nN^{(r)}}{r!} f_r(n-1, N-r) \\ &\qquad + \frac{n(n-1)N^{(2r)}}{2!\,(r!)^2} f_r(n-2, N-2r) + \cdots \end{aligned}$$

where $N^{(k)} = N(N-1)(N-2)\cdots(N-k+1)$.

Replacing n by $n - 1$ and N by $N - r$ in (2.12) there is obtained

$$(2.13)\quad \begin{aligned} (n-1)^{N-r} &= f_r(n-1, N-r) \\ &\qquad + \frac{(n-1)(N-r)^{(r)}}{r!} f_r(n-2, N-2r) + \cdots \end{aligned}$$

Multiplying (2.13) by $\dfrac{nN^{(r)}}{r!}$ and subtracting the result from (2.12), there is obtained

$$(2.14)\quad n^N - \frac{nN^{(r)}}{r!}(n-1)^{N-r} = f_r(n, N) - \frac{n(n-1)N^{(2r)}}{2!\,(r!)^2} f_r(n-2, N-2r) - \cdots$$

By continuing this process, in a manner similar to that used for the case $r = 0$ there is finally obtained

$$(2.15)\quad \begin{aligned} f_r(n, N) &= n^N - \frac{nN^{(r)}}{r!}(n-1)^{N-r} + \frac{n(n-1)N^{(2r)}}{2!\,(r!)^2}(n-2)^{N-2r} \\ &\qquad - \binom{n}{3}\frac{N^{(3r)}}{(r!)^3}(n-3)^{N-3r} + \cdots \end{aligned}$$

By setting $r = 0$ in (2.15), there is of course obtained the value already found in (2.10).

We may readily verify (2.15) for example, for $n = 3$, $N = 5$, $r = 2$. If $x_1 + x_2 + x_3 = 5$ and no $x = 2$, then the sets of solutions are (5,0,0), (0,5,0), (0,0,5), (4,1,0), (1,4,0), (1,0,4), (4,0,1), (0,1,4), (0,4,1), (3,1,1), (1,3,1), (1,1,3), and $f_2(3,5) = 3\cdot 5!/5! + 6\cdot 5!/4! + 3\cdot 5!/3! = 93$. From (2.15) there is obtained $f_2(3,5) = 3^5 - 3\cdot 5\cdot 4\cdot 2^3/2! + 3\cdot 2\cdot 5\cdot 4\cdot 3\cdot 2/2!(2!)^2 = 93$.

The same method of procedure may be applied to evaluate

$$f_{rs\cdots t}(n, N) = \sum \frac{N!}{x_1!\, x_2! \cdots x_n!}, \qquad x_1 + x_2 + \cdots + x_n = N, \quad \text{no } x = r, s, \cdots, \text{ or } t. \tag{2.16}$$

Thus, there is derived the result that

$$\begin{aligned} f_{rs}(n, N) = {} & n^N - n\left(\frac{N^{(r)}(n-1)^{N-r}}{r!} + \frac{N^{(s)}(n-1)^{N-s}}{s!}\right) \\ & + n(n-1)\left(\frac{N^{(2r)}(n-2)^{N-2r}}{2!\,(r!)^2} + \frac{N^{(r+s)}(n-2)^{N-r-s}}{(r!)\,(s!)} \right. \\ & \left. + \frac{N^{(2s)}(n-2)^{N-2s}}{2!\,(s!)^2}\right) - n(n-1)(n-2)\left(\frac{N^{(3r)}(n-3)^{N-3r}}{3!\,(r!)^3} \right. \\ & \left. + \frac{N^{(2r+s)}(n-3)^{N-2r-s}}{2!\,(r!)^2\,(s!)} + \frac{N^{(r+2s)}(n-3)^{N-r-2s}}{2!\,(r!)\,(s!)^2} + \frac{N^{(3s)}(n-3)^{N-3s}}{3!\,(s!)^3}\right) \end{aligned} \tag{2.17}$$

We may readily verify (2.17) for example, for $n = 3$, $N = 5$, $r = 0$, $s = 2$. If $x_1 + x_2 + x_3 = 5$ and no $x = 0$ or 2, then the sets of solutions are (3,1,1), (1,3,1), (1,1,3) and $f_{02}(3,5) = 3\cdot 5!/3! = 60$. From (2.17) there is obtained $f_{02}(3,5) = 3^5 - 3(2^5 + 5\cdot 4\cdot 2^3/2) + 3\cdot 2(1/2! + 5\cdot 4/2! + 5\cdot 4\cdot 3\cdot 2/(2!)^3) = 60$. It will be shown later (see section 8) that

$$\begin{aligned} f_r(n, N) = {} & f_{rs}(n, N) + \frac{nN^{(s)}}{s!} f_{rs}(n-1, N-s) \\ & + \frac{n(n-1)N^{(2s)}}{2!\,(s!)^2} f_{rs}(n-2, N-2s) + \cdots \end{aligned} \tag{2.18}$$

$$\begin{aligned} f_s(n, N) = {} & f_{rs}(n, N) + \frac{nN^{(r)}}{r!} f_{rs}(n-1, N-r) \\ & + \frac{n(n-1)N^{(2r)}}{2!\,(r!)^2} f_{rs}(n-2, N-2r) + \cdots \end{aligned} \tag{2.19}$$

From (2.18) and (2.19) there may be derived, by a method similar to that employed in deriving (2.15), that

$$\begin{aligned} f_{rs}(n, N) = {} & f_r(n, N) - \frac{nN^{(s)}}{s!} f_r(n-1, N-s) \\ & + \frac{n(n-1)N^{(2s)}}{2!\,(s!)^2} f_r(n-2, N-2s) - \cdots \end{aligned} \tag{2.20}$$

This latter result also follows from (2.17 and (2.15).

Let us now consider the following generalization of (2.1). There is desired in terms of $N, n, r, a_1, a_2, \cdots, a_n$, the value of

$$F_r(n, N, a_1, a_2, \cdots, a_n) = \sum \frac{N!}{x_1!\, x_2! \cdots x_n!} a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n} \tag{2.21}$$

where $a_1, a_2, \cdots, a_n$, are constants and the summation is for all values of $x_1, x_2, \cdots, x_n$ such that $x_1 + x_2 + \cdots + x_n = N$ and no $x = r$. The method of procedure is the same as that for the case already considered, viz when $a_1 = a_2 = \cdots = a_n = 1$.

The sum in (2.2) may be rearranged into the sum of a number of terms as follows:

$$(2.22)\quad \begin{cases} \sum \dfrac{N!}{x_1!\, x_2! \cdots x_n!} \quad a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n}, \quad x_1 + x_2 + \cdots + x_n = N, \quad \text{no } x = r; \\ \dfrac{a_1^r}{r!} \sum \dfrac{N!}{x_2! \cdots x_n!} a_2^{x_2} \cdots a_n^{x_n} + \cdots + \dfrac{a_n^r}{r!} \sum \dfrac{N!}{x_1! \cdots x_{n-1}!} a_1^{x_1} \cdots a_{n-1}^{x_{n-1}}, \\ \qquad\qquad x_1 + x_2 + \cdots + x_{n-1} = N - r, \text{ etc.}, \quad \text{no } x = r; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{a_1^r \cdots a_k^r}{(r!)^k} \sum \dfrac{N!}{x_{k+1}! \cdots x_n!} a_{k+1}^{x_{k+1}} \cdots a_n^{x_n} + \cdots \\ \qquad + \dfrac{a_{n-k+1}^r \cdots a_n^r}{(r!)^k} \sum \dfrac{N!}{x_1! \cdots x_{n-k}!} a_1^{x_1} \cdots a_{n-k}^{x_{n-k}}, \\ \qquad\qquad x_1 + x_2 + \cdots + x_{n-k} = N - kr, \text{ etc.}, \quad \text{no } x = r; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

For convenience, let us write

$$(2.23)\quad \begin{cases} A(n, N) = (a_1 + a_2 + \cdots + a_n)^N \\ A_i(n - 1, N) = (a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n)^N \\ A_{ij}(n - 2, N) = (a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_{j-1} + a_{j+1} + \cdots + a_n)^N \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ G_r(n, N) = F_r(n, N, a_1, a_2, \cdots, a_n) \\ G_r(n - 1, N, a_i) = F_r(n - 1, N, a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n) \\ G_r(n - 2, N, a_i, a_j) = F_r(n - 2, N, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

so that (2.2) may be written as

$$(2.24)\quad \begin{aligned} A(n, N) &= G_r(n, N) + \frac{N^{(r)}}{r!} \sum_{i=1}^{n} a_i^r G_r(n - 1, N - r, a_i) \\ &+ \frac{N^{(2r)}}{2!\,(r!)^2} \sum_{i,j=1}^{n} a_i^r a_j^r G_r(n - 2, N - 2r, a_i, a_j) + \cdots \qquad (i \neq j, \text{ etc.}) \end{aligned}$$

From (2.24), there are obtained n equations

$$A_i(n-1, N-r) = G_r(n-1, N-r, a_i) + \frac{(N-r)^{(r)}}{r!} \sum_{j=1}^{n} a_j^r G_r(n-2, N-2r, a_i, a_j) + \cdots \qquad (i = 1, 2, \cdots, n, j \neq 1) \tag{2.25}$$

Multiplying (2.25) by $a_i^r N^{(r)}/r!$ and subtracting the result from (2.24), there is obtained

$$A(n, N) - \sum_{i=1}^{n} \frac{a_i^r N^{(r)}}{r!} A_i(n-1, N-r) = G_r(n, N) - \frac{N^{(2r)}}{2!\,(r!)^2} \sum_{i,j=1}^{n} a_i^r a_j^r G_r(n-2, N-2r, a_i, a_j) - \cdots \qquad (i \neq j, \text{etc.}). \tag{2.26}$$

Continuing this procedure, there is finally obtained

$$G_r(n, N) = F_r(n, N, a_1, a_2, \cdots, a_n) = A(n, N) - \frac{N^{(r)}}{r!} \sum_{i=1}^{n} a_i^r A_i(n-1, N-r) + \frac{N^{(2r)}}{2!\,(r!)^2} \sum_{i,j=1}^{n} a_i^r a_j^r A_{ij}(n-2, N-2r) - \cdots \qquad (i \neq j, \text{etc.}) \tag{2.27}$$

Similar results are obtainable for

$$G_{rs\cdots t} = F_{rs\cdots t}(n, N, a_1, a_2, \cdots, a_n) = \sum \frac{N!}{x_1!\, x_2! \cdots x_n!} a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n} \tag{2.28}$$

where the summation is for all values of x_i such that $x_1 + x_2 + \cdots + x_n = N$, and no $x = r, s, \cdots$, or t.

Thus, it will be shown later (see section 8), that

$$G_r(n, N) = G_{rs}(n, N) + \frac{N^{(s)}}{s!} \sum_{i=1}^{n} a_i^s G_{rs}(n-1, N-s, a_i) + \frac{N^{(2s)}}{2!\,(s!)^2} \sum_{i,j=1}^{n} a_i^s a_j^s G_{rs}(n-2, N-2s, a_i, a_j) + \cdots \qquad (i \neq j, \text{etc.}) \tag{2.29}$$

Corresponding to the derivation of (2.27), there is obtained from (2.29) the fact that

$$G_{rs}(n, N) = G_r(n, N) - \frac{N^{(s)}}{s!} \sum_{i=1}^{n} a_i^s G_r(n-1, N-s, a_i) + \frac{N^{(2s)}}{2!\,(s!)^2} \sum_{i,j=1}^{n} a_i^s a_j^s G_r(n-2, N-2s, a_i, a_j) - \cdots \qquad (i \neq j, \text{etc.}) \tag{2.30}$$

3. The problem to be studied. Consider a trial in which one of n mutually exclusive events may occur, with the respective probabilities of occurrence

$p_1, p_2, \cdots, p_n$ where $p_1 + p_2 + \cdots + p_n = 1$. The probabilities of the various combinations of events which are possible in N trials are given by the terms of the expansion of $(p_1 + p_2 + \cdots + p_n)^N$.

In the N trials some of the possible events may not occur, others may occur one, twice, etc. It is desired to study the distribution of the number of events which do not occur; the distribution of the number of events which occur once each, etc. The simultaneous distributions of the events above described are also to be studied.

For example, the possible event may be the occurrence of a digit. A study of a sequence of random digits, in sets of ten, yielded the following three sample sets.

0	1	2	3	4	5	6	7	8	9
1	0	2	1	1	2	1	0	0	2
1	1	1	1	1	1	2	0	1	1
0	0	2	1	2	1	2	1	0	1

FIG. 1

In the first set three events do not occur, four occur once each, and three occur twice each. In the second set one event does not occur, eight events occur once each, and one event occurs twice; etc.

4. **Distribution of the number of events not occurring.** To obtain the distribution of the number of events which do not occur, there is applied to the expansion of $(p_1 + p_2 + \cdots + p_n)^N$ a procedure similar to that employed in section 2.

Thus, if π_{r0} represents the probability for r events not occurring, then

$$(4.1)\quad \begin{cases} \pi_{00} = \sum \dfrac{N!}{x_1!\,x_2! \cdots x_n!} p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n}, \quad x_1 + x_2 + \cdots + x_n = N, \\ \hfill \text{no } x = 0; \\ \pi_{10} = \sum \dfrac{N!}{x_2! \cdots x_n!} p_2^{x_2} \cdots p_n^{x_n} + \cdots + \sum \dfrac{N!}{x_1! \cdots x_{n-1}!} p_1^{x_1} \cdots p_{n-1}^{x_{n-1}}, \\ \hfill x_1 + x_2 + \cdots + x_{n-1} = N, \text{ etc.}, \qquad \text{no } x = 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \pi_{r0} = \sum \dfrac{N!}{x_{r+1}! \cdots x_n!} p_{r+1}^{x_{r+1}} \cdots p_n^{x_n} + \cdots + \sum \dfrac{N!}{x_1! \cdots x_{n-r}!} p_1^{x_1} \cdots p_{n-r}^{x_{n-r}}, \\ \hfill x_1 + x_2 + \cdots + x_{n-r} = N, \text{ etc.}, \qquad \text{no } x = 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Employing (2.21), we may write (4.1) as

$$(4.2)\quad \begin{cases} \pi_{00} = F_0(n, N, p_1, p_2, \cdots, p_n) \\ \pi_{10} = F_0(n-1, N, p_2, \cdots, p_n) + \cdots + F_0(n-1, N, p_1, p_2, \cdots, p_{n-1}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \pi_{r0} = F_0(n-r, N, p_{r+1}, \cdots, p_n) + \cdots + F_0(n-r, N, p_1, \cdots, p_{n-r}) \end{cases}$$

Since $p_1 + p_2 + \cdots + p_n = 1$ there is found from (2.27) that

$$(4.3)\quad \begin{cases} \pi_{00} = 1 - \sum_{i=1}^{n} (1 - p_i)^N + \frac{1}{2!} \sum_{i,j=1}^{n} (1 - p_i - p_j)^N \\ \qquad\qquad - \frac{1}{3!} \sum_{i,j,k=1}^{n} (1 - p_i - p_j - p_k)^N + \cdots \\ \pi_{10} = \sum_{i=1}^{n} (1 - p_i)^N - \sum_{i,j=1}^{n} (1 - p_i - p_j)^N \\ \qquad\qquad + \frac{1}{2!} \sum_{i,j,k=1}^{n} (1 - p_i - p_j - p_k)^N - \cdots \\ \pi_{20} = \frac{1}{2!}\left\{ \sum_{i,j=1}^{n} (1 - p_i - p_j)^N - \sum_{i,j,k=1}^{n} (1 - p_i - p_j - p_k)^N + \cdots \right\} \\ \pi_{30} = \frac{1}{3!}\left\{ \sum_{i,j,k=1}^{n} (1 - p_i - p_j - p_k)^N - \cdots \right\} \\ \cdots\cdots\cdots\cdots\cdots\cdots \qquad (i \neq j, \text{ etc.}) \end{cases}$$

The factorial moments[5] of the distribution given by (4.3) are easily derived. The first factorial moment is given by $\sigma_1 = \pi_{10} + 2\pi_{20} + 3\pi_{30} + \cdots + r\pi_{r0} + \cdots$ and the summation of the proper terms in (4.3) yields

$$(4.4)\qquad \sigma_1 = \sum_{i=1}^{n} (1 - p_i)^N$$

In general, the r-th factorial moment, given by $\sigma_r = \sum_{k=r}^{n} k(k-1) \cdots (k - r + 1)\pi_{k0}$ is

$$(4.5)\qquad \sigma_r = \sum_{a,b,\cdots,r=1}^{n} (1 - p_a - p_b - \cdots - p_r)^N, \qquad (a \neq b, \text{ etc.}).$$

Indeed, (4.3) illustrates the fact that, if $f(x)$ is the probability that a discontinuous variate takes the value x, then[6]

$$(4.6)\qquad f(x) = \frac{1}{x!} \sum_{k=0}^{n-x} (-1)^k \sigma_{x+k}/k!$$

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[5] J. F. Steffensen, *Interpolation* (1927), p. 101.

[6] J. F. Steffensen, "Factorial Moments and Discontinuous Frequency Functions" *Skandinavisk Aktuarietidskrift*, Vol. VI (1923), pp. 73–89.

The moments about any constant of the distribution given by (4.3) may be derived from the factorial moments by the relation[7]

$$E(x - a)^r = (1 + \sigma_1\Delta + \sigma_2\Delta^2/2! + \cdots + \sigma_r\Delta^r/r!)\cdot\xi^r \qquad (\xi = -a) \tag{4.7}$$

where Δ is the difference operator of the calculus of finite differences, and ξ is replaced by $(-a)$ after the indicated operations have been performed.

Of special interest is the case when $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, for which (4.3) becomes

$$\text{(4.8)} \quad \begin{cases} \pi_{00} = \left(\frac{1}{n}\right)^N f_0(n, N) = \left(\frac{1}{n}\right)^N \Delta^n 0^N \\ \pi_{10} = \left(\frac{1}{n}\right)^N nf_0(n-1, N) = \left(\frac{1}{n}\right)^N n\Delta^{n-1}0^N \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \pi_{r0} = \left(\frac{1}{n}\right)^N \binom{n}{r} f_0(n-r, N) = \left(\frac{1}{n}\right)^N \binom{n}{r} \Delta^{n-r}0^N \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

where $f_0(n, N)$ and $\Delta^n 0^N$ are as defined in section 2. The probabilities in (4.8) are the respective terms of the expansion of $\left(\frac{1}{n}\right)^N (1 + \Delta)^n \cdot 0^N$.

For this case the r-th factorial moment becomes

$$\sigma_r = n(n-1)\cdots(n-r+1)\,(n-r)^N/n^N \tag{4.9}$$

There is presented an example of the distribution (4.8) for the case $n = N = 10$. It is found that[8]

$$\text{(4.10)} \quad \begin{cases} \Delta 0^{10} = 1 & \Delta^6 0^{10} = 16435440 \\ \Delta^2 0^{10} = 1022 & \Delta^7 0^{10} = 29635200 \\ \Delta^3 0^{10} = 55980 & \Delta^8 0^{10} = 30240000 \\ \Delta^4 0^{10} = 818520 & \Delta^9 0^{10} = 16329600 \\ \Delta^5 0^{10} = 5103000 & \Delta^{10} 0^{10} = 3628800 \end{cases}$$

$$\text{(4.11)} \quad \begin{cases} \pi_{00} = .000362880 & \pi_{50} = .128595600 \\ \pi_{10} = .016329600 & \pi_{60} = .017188920 \\ \pi_{20} = .136080000 & \pi_{70} = .000671760 \\ \pi_{30} = .355622400 & \pi_{80} = .000004599 \\ \pi_{40} = .345144240 & \pi_{90} = .000000001 \end{cases}$$

$$\text{(4.12)} \quad \begin{cases} \sigma_1 = 3.486784401 & m = 3.486784401 \\ \sigma_2 = 9.663676416 & \sigma^2 = 0.992795358 \end{cases}$$

---

[7] This result is derived as follows: $(x - a)^r = (1 + \Delta)^x \cdot (-a)^r$; $E(x - a)^r = \sum_{x=1}^{n} (x - a)^r f(x) = \left(\sum_{x=1}^{n} (1 + \Delta)^x \cdot f(x)\right) \cdot (-a)^r = \left(\sum_{x=1}^{n} (1 + x\Delta + x(x-1)\Delta^2/2! + \cdots) f(x)\right) \cdot (-a)^r$. For a bivariate distribution it may be shown similarly that, symbolically, $E((x - a)^r (y - b)^s) = \{\exp(\sigma_{1.}\Delta_1 + \sigma_{.1}\Delta_2)\} \cdot (-a)^r (-b)^s$ where $\sigma_{1.}^m \sigma_{.1}^n = \sigma_{mn}$ and Δ_1 operates only on a and Δ_2 operates only on b. A similar result may be derived for a multivariate distribution.

[8] cf. Whittaker & Robinson, *op. cit.* p. 7.

The observed distribution was obtained by distributing 200 sets of ten digits each, the digits being found in Tippet's Random Sampling Numbers.[9] The results obtained are given in Fig. 2. Three of the 200 observed sets were illustrated in section 3.

The agreement between observed results and theoretical values is gratifying.

5. Distribution of the number of events which occur once each. Let π_{k1}, represent the probability that there are k events which occur once each. Thus, the various probabilities, obtained by rearranging the terms of the expansion of $(p_1 + p_2 + \cdots + p_n)^N$, are as follows:

$$(5.1)\quad \begin{cases} \pi_{01} = \sum \dfrac{N!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n}, \quad x_1 + x_2 + \cdots + x_n = N, \quad \text{no } x = 1; \\ \pi_{11} = p_1 \sum \dfrac{N!}{x_2! \cdots x_n!} p_2^{x_2} \cdots p_n^{x_n} + \cdots + p_n \sum \dfrac{N!}{x_1! \cdots x_{n-1}!} p_1^{x_1} \cdots p_{n-1}^{x_{n-1}}, \\ \qquad x_1 + x_2 + \cdots + x_{n-1} = N - 1, \text{ etc.}, \quad \text{no } x = 1; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \pi_{k1} = p_1 p_2 \cdots p_k \sum \dfrac{N!}{x_{k+1}! \cdots x_n!} p_{k+1}^{x_{k+1}} \cdots p_n^{x_n} + \cdots + p_{n-k+1} \cdots p_n \\ \qquad \sum \dfrac{N!}{x_1! \cdots x_{n-k}!} p_1^{x_1} \cdots p_{n-k}^{x_{n-k}}, \\ \qquad x_1 + x_2 + \cdots + x_{n-k} = N - k, \text{ etc.}, \quad \text{no } x = 1; \end{cases}$$

No. of events not occurring x	Observed frequency f	Theoretical frequency	xf	$x(x-1)f$	Observed parameters
0	0	0.08	0	0	$\bar{\sigma}_1 = 3.46$
1	8	3.26	8	0	$\bar{\sigma}_2 = 9.61$
2	22	27.22	44	44	$\bar{x} = 3.46$
3	72	71.12	216	432	$s^2 = 1.0984$
4	72	69.02	288	864	Theoretical
5	21	25.72	105	420	Parameters
6	4	3.44	24	120	$\sigma_1 = 3.49$
7	1	0.14	7	42	$\sigma_2 = 9.66$
8	0	0.00	0	0	$m = 3.49$
9	0	0.00	0	0	$\sigma^2 = 0.99$
	200	200.00	692	1922	

FIG. 2

[9] L. H. C. Tippet, Random Sampling Numbers, *Tracts for Computers*, No. XV (1927), London.

In view of (2.21) and (2.27), it is found that (5.1) becomes

$$(5.2)\quad \begin{cases} \pi_{01} = 1 - N \sum_{i=1}^{n} p_i(1-p_i)^{N-1} + \frac{N(N-1)}{2!} \sum_{i,j=1}^{n} p_i p_j (1 - p_i - p_j)^{N-2} - \cdots \\ \pi_{11} = N\left\{\sum_{i=1}^{n} p_i(1-p_i)^{N-1} - (N-1) \sum_{i,j=1}^{n} p_i p_j (1 - p_i - p_j)^{N-2} + \cdots\right\} \\ \pi_{21} = \frac{N(N-1)}{2!}\left\{\sum_{i,j=1}^{n} p_i p_j (1-p_i-p_j)^{N-2} - \cdots\right\} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases} \qquad (i \neq j, \text{ etc.})$$

From (5.2) there is readily derived the fact that

$$(5.3)\quad \sigma_r = N(N-1)\cdots(N-r+1) \sum_{a,b,\cdots,r=1}^{n} p_a p_b \cdots p_r (1 - p_a - p_b - \cdots - p_r)^{N-r}, \qquad (a \neq b, \text{ etc.})$$

For the case in which $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, the distribution in (5.2) becomes

$$(5.4)\quad \begin{cases} \pi_{01} = \left(\frac{1}{n}\right)^N f_1(n, N) \\ \pi_{11} = \left(\frac{1}{n}\right)^N nN f_1(n-1, N-1) \\ \pi_{21} = \left(\frac{1}{n}\right)^N \frac{n(n-1)N(N-1)}{2!} f_1(n-2, N-2) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \pi_{r1} = \left(\frac{1}{n}\right)^N \binom{n}{r} N^{(r)} f_1(n-r, N-r) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

where $f_1(n, N)$ and $N^{(r)}$ have been defined in section 2. For this case (5.3) becomes

$$(5.5)\quad \sigma_r = n^{(r)} N^{(r)} (n-r)^{N-r} / n^N$$

Evaluation of (5.4) and (5.5) for $n = N = 10$ yields,

$$(5.6)\quad \begin{cases} \pi_{01} = .00811639 & \pi_{41} = .27052704 & \pi_{81} = .01632960 \\ \pi_{11} = .04794633 & \pi_{51} = .15621984 & \pi_{91} = .00000000^{10} \\ \pi_{21} = .14082336 & \pi_{61} = .12700800 & \pi_{101} = .00036288 \\ \pi_{31} = .21089376 & \pi_{71} = .02177280 & \end{cases}$$

$$(5.7)\quad \begin{cases} \sigma_1 = 3.87420489 & m = 3.87420489 \\ \sigma_2 = 13.58954496 & \sigma^2 = 2.45428632 \end{cases}$$

[10] For the case $n = N = 10$ there cannot be 9 events occurring once each, since then the tenth event must also occur once.

The observed distribution, given in Fig. 3, was obtained from the 200 sets previously considered.

The agreement between the observed results and theoretical values is gratifying.

6. **Distribution of the number of events whlch occur r times each.** Let π_{kr} represent the probability that there are k events occurring r times each. Thus, the various probabilities, obtained by rearranging the terms of the expansion of $(p_1 + p_2 + \cdots + p_n)^N$, are as follows:

No. of events occurring once each x	Observed frequency f	Theoretical frequency	xf	$x(x-1)f$	Observed parameters
0	1	1.62	0	0	$\bar{\sigma}_1 = 3.905$
1	10	9.58	10	0	$\bar{\sigma}_2 = 14.000$
2	30	28.16	60	60	$\bar{x} = 3.905$
3	37	42.18	111	222	$s^2 = 2.656$
4	62	54.10	248	744	Theoretical
5	27	31.24	135	540	Parameters
6	22	25.40	132	660	$\sigma_1 = 3.874$
7	3	4.36	21	126	$\sigma_2 = 13.590$
8	8	3.26	64	448	$m = 3.874$
9	0	0.00	0	0	$\sigma^2 = 2.454$
10	0	0.08	0	0	
	200	199.98	781	2800	

FIG. 3

$$
(6.1)\quad \begin{cases}
\pi_{0r} = \sum \dfrac{N!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n}, \quad x_1 + x_2 + \cdots + x_n = N, \quad \text{no } x = r; \\[2ex]
\pi_{1r} = \dfrac{p_1^r}{r!} \sum \dfrac{N!}{x_2! \cdots x_n!} p_2^{x_2} \cdots p_n^{x_n} + \cdots + \dfrac{p_n^r}{r!} \sum \dfrac{N!}{x_1! \cdots x_{n-1}!} p_1^{x_1} \cdots p_{n-1}^{x_{n-1}}, \\[1ex]
\qquad\qquad x_1 + x_2 + \cdots + x_{n-1} = N - r, \text{ etc.}, \quad \text{no } x = r; \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\pi_{kr} = \dfrac{p_1^r p_2^r \cdots p_k^r}{(r!)^k} \sum \dfrac{N!}{x_{k+1}! \cdots x_n!} p_{k+1}^{x_{k+1}} \cdots p_n^{x_n} + \cdots \\[1ex]
\qquad\qquad + \dfrac{p_{n-k+1}^r \cdots p_n^r}{(r!)^k} \sum \dfrac{N!}{x_1! \cdots x_{n-k}!} p_1^{x_1} \cdots p_{n-k}^{x_{n-k}}, \\[1ex]
\qquad\qquad x_1 + x_2 + \cdots + x_{n-k} = N - kr, \text{ etc.}, \quad \text{no } x = r; \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

In view of (2.21) and (2.27) it is found that (6.1) becomes

$$(6.2)\quad \begin{cases} \pi_{0r} = 1 - \dfrac{N^{(r)}}{r!}\displaystyle\sum_{i=1}^{n} p_i^r(1-p_i)^{N-r} + \dfrac{N^{(2r)}}{2!\,(r!)^2}\sum_{i,j=1}^{n} p_i^r p_j^r(1-p_i-p_j)^{N-2r} - \cdots \\ \pi_{1r} = \dfrac{N^{(r)}}{r!}\left\{\displaystyle\sum_{i=1}^{n} p_i^r(1-p_i)^{N-r} - \dfrac{(N-r)^{(r)}}{r!}\sum_{i,j=1}^{n} p_i^r p_j^r(1-p_i-p_j)^{N-2r} + \cdots\right\} \\ \pi_{2r} = \dfrac{N^{(2r)}}{2!\,(r!)^2}\left\{\displaystyle\sum_{i,j=1}^{n} p_i^r p_j^r(1-p_i-p_j)^{N-2r} - \cdots\right\} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases} \qquad (i \neq j, \text{ etc.})$$

From (6.2) there is readily derived the fact that

$$(6.3)\quad \sigma_k = \frac{N^{(kr)}}{(r!)^k}\sum_{a,b,\cdots,k=1}^{n} p_a^r p_b^r \cdots p_k^r(1-p_a-p_b-\cdots-p_k)^{N-kr}, \quad (a \neq b, \text{etc.})$$

For $r = 0,1$ (6.2) and (6.3) reduce to the values previously derived.

For the case in which $p_1 = p_2 = \cdots = p_n = \dfrac{1}{n}$, the distribution in (6.2) becomes

$$(6.4)\quad \begin{cases} \pi_{0r} = \left(\dfrac{1}{n}\right)^N f_r(n, N) \\ \pi_{1r} = \left(\dfrac{1}{n}\right)^N \dfrac{nN^{(r)}}{r!} f_r(n-1, N-r) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \pi_{kr} = \left(\dfrac{1}{n}\right)^N \dbinom{n}{k} \dfrac{N^{(kr)}}{(r!)^k} f_r(n-k, N-kr) \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

where $f_r(n, N)$ has been defined in section 2. For this case (6.3) becomes

$$(6.5)\quad \sigma_k = N^{(kr)} n^{(k)} (n-k)^{N-kr}/n^N$$

7. Simultaneous distribution of the number of events not occurring, and of the number of events occurring once each. The probabilities for the simultaneous occurrence of the various combinations of the number of events not occurring, and of the number of events occurring once each, are given by rearranging the terms of the expansion of $(p_1 + p_2 + \cdots + p_n)^N$, and are given as in Fig. 4.

In Fig. 4 none of the subscripts take on equal values simultaneously, and G_{01} has been defined in section 2. Summation of the values in the k-th column of Fig. 4, yields the probability that there are $(k - 1)$ events not occurring. Comparison with (4.2) yields

$$(7.1)\quad \begin{aligned} F_0(n, N, p_1, p_2, \cdots, p_n) &= G_0(n, N) = G_{01}(n, N) + N\sum_{i=1}^{n} p_i G_{01}(n-1, N-1, p_i) \\ &\quad + \frac{N^{(2)}}{2!}\sum_{i,j=1}^{n} p_i p_j G_{01}(n-2, N-2, p_i, p_j) + \cdots, \quad (i \neq j, \text{ etc.}) \end{aligned}$$

Number of events occurring once each	Number of events not occurring: 0	1	$\cdots$	r
0	$G_{01}(n, N)$	$\sum_{i=1}^{n} G_{01}(n-1, N, p_i)$		$\cdots$
1	$N\sum_{i=1}^{n} p_i G_{01}(n-1, N-1, p_i)$	$N\sum_{i,j=1}^{n} p_i G_{01}(n-2, N-1, p_i, p_j)$		$\cdots$
2	$\frac{N^{(2)}}{2!}\sum_{i,j=1}^{n} p_i p_j G_{01}(n-2, N-2, p_i p_j)$	$\frac{N^{(2)}}{2!}\sum_{i,j,k=1}^{n} p_i p_j G_{01}(n-3, N-2, p_i, p_j, p_k)$		$\cdots$
$\vdots$				
s	$\cdots$	$\cdots$		$\frac{N^{(s)}}{r!\,s!}\sum_{a,b,\cdots,s,\alpha,\beta,\cdots,\rho=1}^{n} p_a p_b \cdots p_s G_{01}(n-r-s, N-s, p_a, \cdots, p_s, p_\alpha, \cdots, p_\rho)$

FIG. 4

Summation of the values in the k-th row of Fig. 4, yields the probability that there are $(k-1)$ events occurring once each. Comparison with (5.2) and (2.27) yields

$$
\begin{aligned}
F_1(n, N, p_1, p_2, \cdots, p_n) = G_1(n, N) &= G_{01}(n, N) + \sum_{i=1}^{n} G_{01}(n-1, N, p_i) \\
&+ \frac{1}{2!}\sum_{i,j=1}^{n} G_{01}(n-2, N, p_i, p_j) + \cdots, \quad (i \neq j, \text{ etc.})
\end{aligned} \tag{7.2}
$$

If we use x to represent the number of events not occurring, and y the number of events occurring once each, then it is found that

$$
\begin{aligned}
E(x^{(r)} y^{(s)}) = \sigma_{rs} = N^{(s)} \sum_{a,b,\cdots,s,\alpha,\beta,\cdots,\rho=1}^{n} & p_a p_b \cdots p_s (1 - p_a - \cdots - p_s \\
& - p_\alpha - \cdots - p_\rho)^{N-s}, \quad (a \neq b, \text{ etc.}).
\end{aligned} \tag{7.3}
$$

If ${}_0\bar{x}_{k1}$ represents the average number of events not occurring, when there are k events occurring once each, then from Fig. 4 there is found that

$$
{}_0\bar{x}_{01} = \frac{\sum_{i=1}^{n} G_{01}(n-1, N, p_i) + 2\sum_{i,j=1}^{n} G_{01}(n-2, N, p_i, p_j)/2! + 3\sum_{i,j,k=1}^{n} G_{01}(n-3, N, p_i, p_j, p_k)/3! + \cdots}{G_{01}(n, N) + \sum_{i=1}^{n} G_{01}(n-1, N, p_i) + \sum_{i,j=1}^{n} G_{01}(n-2, N, p_i, p_j)/2! + \cdots} \quad (i \neq j, \text{ etc.}) \tag{7.4}
$$

In view of (7.2), (7.4) reduces to

$$(7.5) \qquad {}_0\bar{x}_{01} = \left(\sum_{i=1}^{n} G_1(n, N, p_i)\right) \Big/ G_1(n, N)$$

A similar procedure, yields, in general

$$(7.6) \quad {}_0\bar{x}_{k1} = \frac{\displaystyle\sum_{a,b,\cdots,k,l=1}^{n} p_a p_b \cdots p_k G_1(n - k - 1, N - k, p_a, p_b, \cdots, p_k, p_l)}{\displaystyle\sum_{a,b,\cdots,k=1}^{n} p_a p_b \cdots p_k G_1(n - k, N - k, p_a, p_b, \cdots, p_k)} \qquad (a \neq b, \text{ etc.})$$

If ${}_1\bar{y}_{k0}$ represents the average number of events occurring once each, when there are k events not occurring, then from Fig. 4, there is found that

$$(7.7) \quad {}_1\bar{y}_{00} = \frac{N\left\{\displaystyle\sum_{i=1}^{n} p_i G_{01}(n - 1, N - 1, p_i) + 2(N - 1) \sum_{i,j=1}^{n} p_i p_j G_{01}(n - 2, N - 2, p_i, p_j)/2! + \cdots\right\}}{G_{01}(n, N) + N \displaystyle\sum_{i=1}^{n} p_i G_{01}(n - 1, N - 1, p_i) + N^{(2)} \sum_{i,j=1}^{n} p_i p_j G_{01}(n - 2, N - 2, p_i, p_j)/2!} \qquad (i \neq j, \text{ etc.})$$

In view of (7.1), (7.7) reduces to

$$(7.8) \qquad {}_1\bar{y}_{00} = \left(N \sum_{i=1}^{n} p_i G_0(n - 1, N - 1, p_i)\right) \Big/ G_0(n, N)$$

A similar procedure, yields, in general

$$(7.9) \quad {}_1\bar{y}_{k0} = \frac{N \displaystyle\sum_{a,b,\cdots,k,l=1}^{n} p_a G_0(n - k - 1, N - 1, p_a, p_b, \cdots, p_k, p_l)}{\displaystyle\sum_{a,b,\cdots,k=1}^{n} G_0(n - k, N, p_a, p_b, \cdots, p_k)} \qquad (a \neq b, \text{ etc.})$$

For the case in which $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, as may be found from Fig. 4, the probability for the simultaneous occurrence of r events not occurring, and s events occurring once each, is given by

$$(7.10) \qquad \left(\frac{1}{n}\right)^N \frac{n^{(r+s)} N^{(s)}}{r!\, s!} f_{01}(n - r - s, N - s)$$

For this case (7.1), (7.2), (7.3), (7.6), and (7.9) yield respectively

$$(7.11) \quad f_0(n, N) = f_{01}(n, N) + nN f_{01}(n - 1, N - 1) + \binom{n}{2} N^{(2)} f_{01}(n - 2, N - 2) + \cdots$$

$$(7.12) \quad f_1(n, N) = f_{01}(n, N) + n f_{01}(n - 1, N) + \binom{n}{2} f_{01}(n - 2, N) + \cdots$$

$$(7.13) \quad \sigma_{rs} = N^{(s)} n^{(r+s)} (n - r - s)^{N-s} / n^N$$

$$(7.14)\quad {}_0\bar{x}_{k1} = (n - k)f_1(n - k - 1, N - k)/f_1(n - k, N - k)$$

$$(7.15)\quad {}_1\bar{y}_{k0} = N(n - k)f_0(n - k - 1, N - 1)/f_0(n - k, N)$$

Let us consider again the case when $p_1 = p_2 = \cdots = p_n = \dfrac{1}{n}$ and $n = N = 10$. Evaluating (7.14) and (7.15) by means of (2.15) yields

$$(7.16)\quad \begin{cases} {}_0\bar{x}_{01} = 5.71 & {}_0\bar{x}_{51} = 3.02 \\ {}_0\bar{x}_{11} = 5.21 & {}_0\bar{x}_{61} = 2.10 \\ {}_0\bar{x}_{21} = 4.51 & {}_0\bar{x}_{71} = 2.00 \\ {}_0\bar{x}_{31} = 4.10 & {}_0\bar{x}_{81} = 1.00 \\ {}_0\bar{x}_{41} = 3.28 & {}_0\bar{x}_{91} = 0.00 \end{cases}$$

$$(7.17)\quad \begin{cases} {}_1\bar{y}_{00} = 10.00 & {}_1\bar{y}_{50} = 1.83 \\ {}_1\bar{y}_{10} = 8.00 & {}_1\bar{y}_{60} = 0.89 \\ {}_1\bar{y}_{20} = 6.16 & {}_1\bar{y}_{70} = 0.27 \\ {}_1\bar{y}_{30} = 4.50 & {}_1\bar{y}_{80} = 0.02 \\ {}_1\bar{y}_{00} = 3.05 & {}_1\bar{y}_{90} = 0.00 \end{cases}$$

The 200 sets of observations already considered yielded the simultaneous distribution given in Fig. 5.

Number of events occurring once each	Number of events not occurring: 0	1	2	3	4	5	6	7	8	9		$\bar{x}$
0								1			1	7.00
1					1	6	3				10	5.20
2					16	13	1				30	4.50
3					35	2					37	4.05
4				42	20						62	3.32
5				27							27	3.00
6			19	3							22	2.14
7			3								3	2.00
8		8									8	1.00
9											0	
10											0	
	0	8	22	72	72	21	4	1	0	0	200	
$\bar{y}$		8.00	6.16	4.46	3.03	1.81	1.25	0.00				

FIG. 5

The distribution in Fig. 5 yields $\bar{\sigma}_{11} = 11.89$, (7.13) yields $\sigma_{11} = 12.07959552$.

The agreement between the observed results in Fig. 5 and the theoretical values in (7.16) and (7.17) is gratifying.

8. **Simultaneous distribution of the number of events which occur r times each, and of the number of events which occur s times each.** The probabilities for the simultaneous occurrence of the various combinations of the number of events which occur r times each, and of the number of events which occur s times each, are obtained by rearranging the terms of the expansion of $(p_1 + p_2 + \cdots + p_n)^N$. If $\pi_{kr,ls}$ is the probability for the simultaneous occurrence of k events which occur r times each and l events which occur s times each, then

$$\pi_{kr,ls} = \frac{N^{(kr+ls)}}{k!\,l!\,(r!)^k\,(s!)^l} \sum_{a,b,\cdots,k,\alpha,\beta,\cdots,\lambda=1}^{n} p_a^r \cdots p_k^r p_\alpha^s \cdots p_\lambda^s G_{rs}$$
$$(n-k-l, N-kr-ls, p_a, \cdots, p_k, p_\alpha, \cdots, p_\lambda), \qquad (a \neq b, \text{etc.}) \tag{8.1}$$

where G_{rs} is defined in section 2.

From (8.1) and (6.2), there is derived, in a manner similar to the derivation of (7.1) and (7.2), the result that

$$F_r(n, N, p_1, \cdots, p_n) = G_r(n, N) = G_{rs}(n, N) + \frac{N^{(s)}}{s!} \sum_{i=1}^{n} p_i^s G_{rs}(n-1, N-s, p_i)$$
$$+ \frac{N^{(2s)}}{2!\,(s!)^2} \sum_{i,j=1}^{n} p_i^s p_j^s G_{rs}(n-2, N-2s, p_i, p_j) + \cdots, \qquad (i \neq j, \text{etc.}) \tag{8.2}$$

and a similar result by interchanging r and s in (8.2).

For the distribution given by (8.1), it is found that

$$\sigma_{kl} = \frac{N^{(kr+ls)}}{(r!)^k\,(s!)^l} \sum_{a,b,\cdots,k,\alpha,\beta,\cdots,\lambda=1}^{n} p_a^r \cdots p_k^r p_\alpha^s \cdots p_\lambda^s$$
$$(1 - p_a - \cdots - p_k - p_\alpha - \cdots - p_\lambda)^{N-kr-ls}, \qquad (a \neq b, \text{etc.}) \tag{8.3}$$

If ${}_r\bar{x}_{ls}$ represents the average number of events which occur r times each when there are l events which occur s times each, then from (8.1) and (8.2), in a manner similar to the derivation of (7.6), it is found that

$${}_r\bar{x}_{ls} = \frac{(N-ls)^{(r)} \sum_{a,\alpha,\cdots,\lambda=1}^{n} p_a^r p_\alpha^s \cdots p_\lambda^s G_s(n-1-l, N-r-ls, p_a, p_\alpha, \cdots, p_\lambda)}{r! \sum_{\alpha,\cdots,\lambda=1}^{n} p_\alpha^s \cdots p_\lambda^s G_s(n-l, N-ls, p_\alpha, \cdots, p_\lambda)} \tag{8.4}$$
$$(\alpha \neq \beta, \text{etc.})$$

If ${}_s\bar{y}_{kr}$ represents the average number of events which occur s times each when there are k events which occur r times each, then by interchanging k and l, and r and s in (8.4), there is found

$$
{}_s\bar{y}_{kr} = \frac{(N-kr)^{(s)} \sum_{a,\cdots,k,\alpha=1}^{n} p_a^r \cdots p_k^r p_\alpha^s G_r(n-k-1, N-kr-s, p_a, \cdots, p_k, p_\alpha)}{\sum_{a,b,\cdots,k=1}^{n} p_a^r \cdots p_k^r G_r(n-k, N-kr, p_a, \cdots, p_k)} \tag{8.5}
$$

$$(a \neq b, \text{ etc.})$$

For the case when $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, it is found that (8.1), (8.2), (8.3), (8.4), and (8.5) respectively yield

$$
\pi_{kr,ls} = \left(\frac{1}{n}\right)^N \frac{n^{(k+l)} N^{(kr+ls)}}{k!\, l!\, (r!)^k (s!)^l} f_{rs}(n-k-l, N-kr-ls) \tag{8.6}
$$

$$
\begin{aligned}
f_r(n, N) = f_{rs}(n, N) &+ \frac{nN^{(s)}}{s!} f_{rs}(n-1, N-s) \\
&+ \frac{n(n-1)N^{(2s)}}{2!\,(s!)^2} f_{rs}(n-2, N-2s) + \cdots
\end{aligned} \tag{8.7}
$$

$$
\sigma_{kl} = n^{(k+l)} N^{(kr+ls)} (n-k-l)^{N-kr-ls} / (r!)^k (s!)^l n^N \tag{8.8}
$$

$$
{}_r\bar{x}_{ls} = (n-l)(N-ls)^{(r)} f_s(n-1-l, N-r-ls)/r!\, f_s(n-l, N-ls) \tag{8.9}
$$

$$
{}_s\bar{y}_{kr} = (n-k)(N-kr)^{(s)} f_r(n-k-1, N-kr-s)/s!\, f_r(n-k, N-kr) \tag{8.10}
$$

For $r = 0$, $s = 1$, the results derived in this section of course reduce to those already derived in section 7.

9. **Conclusion.** It is clear that the same method of procedure may be employed to study the simultaneous distribution of the number of events which occur r, s, $\cdots$, t, times each. However we will not continue the discussion any further.

We have thus seen that the multinomial distribution serves as the background for the study of a number of distributions which have certain practical applications.

The theory discussed herein has been illustrated by several examples which yielded gratifying agreement between observed and theoretical results.

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