ON COMBINED EXPANSIONS OF PRODUCTS OF SYMMETRIC POWER SUMS AND OF SUMS OF SYMMETRIC POWER PRODUCTS WITH APPLICATIONS TO SAMPLING (Continued)

By Paul S. Dwyer

PART II. THE FUNDAMENTALS OF SAMPLING

Introduction

We consider a population of N variates in which every individual possesses a common attribute. Let the variate x_i be the measure of such an attribute for individual i. From the N variates it is possible to form $\binom{N}{n}$ different samples where each sample consists of n variates, $n \leq N$.

Each sample has its mean, variance, etc. so that there are $\binom{N}{n}$ means, $\binom{N}{n}$ variances, etc. The fundamental sampling problem, as interpreted here, is to find the relation between the moments of the $\binom{N}{n}$ means, and the moments of the $\binom{N}{n}$ variances in terms of the moments of the moments of the universe. Numerous attempts have been made to solve this problem, but each has been restricted in some way. It is the aim of Part II to indicate an approach which is broad enough to include many of the fundamental variations.

The first chapter is devoted to a listing of criteria which should be satisfied by a theoretical development which is to be considered sufficiently general. These criteria might be applied to other statistics but the theory developed here is limited to those statistics which are moments (or functions of moments) of moments. The first chapter continues with an account of the more significant papers which have contributed to a general solution of the problem. No attempt is made to indicate a complete history, but rather there is presented a brief summary of a number of the most significant contributions.

The second chapter is devoted to definitions and notation. An attempt has been made to use conventional notation whenever it is suitable.

The third chapter deals with some of the fundamental principles which are used in the general approach. It presents a crucial part of the argument as it shows how various types of sampling problems can be reduced to Carver functions.

The last three chapters deal with specific applications to some of the simpler problems. Chapter IV discusses the case of moments of the mean of the sample.

Chapter V considers the mean of the variance and the variance of the variance, while Chapter VI gives a large number of formulas, implicitly, in tabular form.

Chapter I. A Brief History of Previous Contributions

In order to assist the reader in getting a perspective with reference to previous mathematical work on the relations between the moments of the moments of the sample and the moments of the complete set of measures (universe), a list of criteria[1] is suggested below which might be applied to each contribution. These criteria group themselves naturally into two classes. The first eight questions can be answered categorically, while the remainder are less definite in nature and are not so subject to categorical answers.

1. **The Criteria.** 1. Does the method apply to one type of frequency distribution only or is it broad enough in scope to include any distribution law?

2. Is there any restriction as to the size of the sample?

3. Is there any restriction as to the size of the universe?

4. Is there any restriction as to the nature of the correlation between observations? More specifically, is the method applicable only to some particular law of formation of the sample such as "drawing with replacements," "drawing without replacements," etc., or is it broad enough in scope to allow application to other orderly replacement laws?

5. Is the application limited to one characteristic (variable) or can a large number of characteristics be treated simultaneously?

6. Is it necessary that the universe maintain the same frequency distribution during the formation of the sample or may it assume a different frequency distribution before each drawing?

7. Does the method produce exact, rather than approximate, formulas?

8. Does the method permit approximations to a required degree of accuracy?

9. Does the method enable the author to write general laws in a compact form? More specifically, can he express, in a form which is not too symbolic, any moment of a given sample moment? If not, what order of moments can be expressed?

10. Is the notation such that the general case can be turned into the more important special cases with relative ease?

11. Does the development lead logically to the introduction of new moment functions (such as the semi-invariant of Thiele [B′; 209] or the k functions of R. A. Fisher [23; 203]) which are useful in condensing the results?

12. Is a combinatorial analysis provided so that any given formula, or any part of it, can be checked for accuracy without too much effort?

2. **Review of previous results.** The articles below have been examined with the criteria in mind. No attempt is made to write specific answers to all the

---

[1] Many of these criteria have been suggested, in less explicit form by Tchouproff (15; 461–471). The "Introduction" of his Metron paper is recommended for use as a supplement to the present chapter.

criteria in each case, but rather to indicate the important features of each contribution.

The papers discussed by no means cover all the work on moments of moments, although a rather complete bibliographical background is available to the reader who desires to examine the bibliographies attached to the articles mentioned. Undoubtedly the importance of the articles written in English has been overemphasized. Since the important contributions of non-English writers (such as Thiele and Tchouproff) have eventually appeared in English, it does no serious harm to refer to the English versions even though the results may have been partially antedated by the author in some other language.

A large number of the earlier results on moments of moments were limited to a special case of the problem, usually the case in which the universe is infinite and normal. The present summary deals with those authors who, during the past four decades, have made real contributions to the problem of generalization. A detailed account of the history of moments of moments would include many valuable contributions which are not included here.

It seems expedient to start with Pearson's article "On the Probable Error of Frequency Constants" [2] which appeared at the opening of the century. Although by no means the first article in the field, it presented a rather complete set of formulas for the case of moments of moments. One advantage of these formulas is that they are relatively brief and yet this brevity results from the fact that they are approximate. The original paper dealt with the univariate case, but it was followed by a later one [6] which discussed the case of more than one variable.

These formulas have played an important rôle in that they have assisted in making it clear that the moments of moments of samples must be estimated if one is to be permitted to draw conclusions from his sampling moments and that it is possible to work out formulas which serve as the basis of those estimates.

Of great importance also was the contribution of T. N. Thiele to the sampling problem. Adapting certain ideas of Laplace, he used semi-invariants in which to express his results which he published in English in 1903 in "The Theory of Observations" [B'; 209]. He took the case of the infinite parent and any law of distribution and then worked out moments through the fourth of the variance.

An earlier contribution of the introductory period was that of Karl Pearson in 1899 [1]. This paper is significant in that it provides formulas for the four moments of the mean when sampling is from a *finite* universe. The universe is not general, but obeys a simple frequency law.

Another article of this period was that of Robert Henderson (1904) in which the first four moments of the mean were given for an infinite universe with any frequency law. This article, which was first published in *Transactions of the Actuarial Society of America* [3], was considered so important that it was republished in 1907 in the *British Journal of the Institute of Actuaries.* Henderson gave, in addition to the first four moments of the mean, first moments of m_2, m_3, m_4 although the last of these formulas is erroneous.

Another important contribution of this period was that of "Student" in 1908 [5]. He was interested in the properties of the normal distribution, but did not assume normality in his general derivation. He took an infinite population and wrote the formula for the variance of the variance. In this result he inserted the condition for normality. His further argument in the normal case implied the development of corresponding formulas for the higher moments of the variance, but he did not publish them as they were incidental to his main attack. The semi-invariant equivalent of these results had been previously given by Thiele [B'; 209–210].

The real contribution of "Student" to the general problem of moments of moments was his method, for it is his method which has been utilized by later writers. "Student's" method has the advantage that the development involves algebraic processes only. Contributions of Neyman, Church, Pepper, Carver, and the present writer are based upon it.

An important development during the next decade, 1908–1918, was the establishment of the first four moments of the mean when the samples were drawn from a finite parent without replacement. It appears that a number of men worked this problem independently. For example, one might examine the results of Pearson [4], Isserlis [7, 8], Mortara [C], Tchouproff [11], and Edgeworth [9]. Probably the best English presentations of that era were those of Isserlis [8] and Edgeworth [9] which appear in the same volume of the *Journal of the Royal Statistical Society.*

A most prolific writer on sampling during the next decade was the Russian, Tchouproff, who had been publishing in Russian and Scandinavian journals [10], [11]. His most valuable contributions were published in 1918–1923 in *Biometrika* (in English) and in *Metron* (in English).

The first series of articles was published in three different numbers of *Biometrika* in the years 1918–19 [12]. Tchouproff assumed an infinite universe and used the method of mathematical expectation. At first glance the most characteristic aspect of his work appears to be the complicated notation which he used. This notation was adopted because he undertook a much more general problem than had previously been attempted and hence needed to make new distinctions. Although he limited himself to the infinite case and one variable, he worked out the theory with the freedom that the frequency distribution of the universe might change between drawings. In the special case in which the populations are the same, he worked out the moments of the variance as far as the fourth. The chief criticism of his work concerns the complicated notation which seems to have been difficult to follow critically. A mistake in one of his formulas was not discovered for some years and then not by examination of his reasoning, but through the application of his results to an actual problem [17].

It is perhaps appropriate to insert here that in 1934 Feldman [30] rewrote the material of the second *Biometrika* article by simplifying the notation and extending the argument to the case of two (and more) variables.

Tchouproff continued to generalize his work and in the 1923 volume of *Metron*

[15] there appeared a series of articles in which there were no restrictions as to the size of the sample, no restrictions as to the type of sampling distribution (in fact the sampling distribution might vary between successive drawings), and no restrictions as to the law of replacement, or more generally as he expressed it, "no restriction as to the nature of the correlation between observations." Criterion number 5 is the only one of the first eight criteria which is not satisfied in as much as the approach is limited to that of a single variable. Also the notation was extremely complicated and, although Tchouproff gave general formulas for moments of moments, these formulas are so symbolic in form that he did not find it expedient to write out specific formulas beyond the variance of the variance for such an important special case as sampling from a finite parent without replacements.

During the same period J. Splawa-Neyman [14] had been examining the problem of sampling from a finite parent without replacements. He published his results in a Polish journal in 1923 [14] and his corrected results two years later in *Biometrika* [18]. He gave the well known formulas for the first four moments of the mean and a formula for the variance of the variance. He also gave some simple correlation formulas such as the correlation between the mean and the variance.

At this time the basic problem of moments of moments, at least as it was interpreted by Pearson and his followers, was the establishment of the first four moments of the given moment of the sample so that a Pearson curve could be fitted. A. E. R. Church, a worker in Pearson's laboratory, was assigned the task of seeing how the moments of the variance work out in actual practice. In doing this he became convinced that the formula for the fourth power of the variance, which had appeared in Tchouproff's *Biometrika* article, was incorrect. He tried to follow the argument of Tchouproff, but apparently was baffled by the complex notation and finally, at the suggestion of Pearson, decided to carry through the formula using the method of "Student." In doing this he discovered a mistake in the Tchouproff formula for the fourth power of the variance. At the same time he published [17] the formulas for the third and fourth power of the variance in the more conventional notation of that time.

It might be noted that it is particularly fitting that Church should discover this error since Tchouproff, as Pearson himself stated in an editorial [13], had pointed out a number of errors made by the Pearsonian school.

In the next volume of *Biometrika* there appears an article by Church [19] in which, among other things, formulas are derived for the third and fourth moments of the variance in the case of a finite population, sampling without replacement. Church claimed no particular credit for these formulas. His point is rather that they are almost valueless from a practical standpoint chiefly because of their length. The formula for the fourth power of the variance occupies three and one-half of the large pages of *Biometrika* and is given with the apparent aim of indicating, as Pearson said [21; 209], "the practical futility of the theoretical formulas."

Church gave full credit to Neyman for the formula for the variance of the variance and made no mention of Tchouproff's *Metron* work and of the more general presentation there given. This was particularly unfortunate because it exposed him to the charge that he ignored non-English authors. This charge was immediately made by Greenwood and Isserlis [20] who broadened it to include Neyman and, by implication, Pearson himself. They advocated the case of Tchouproff who, now dead, was unable to defend himself. They gave a survey (valuable to the cursive reader) of the pertinent contributions of the Tchouproff articles and suggested that the ignoring of Tchouproff was particularly disconcerting since it appears that Tchouproff had gone more than half way in his cooperation with English writers.

Pearson replied in an interesting article [21] which made it clear that Neyman established his results independently of Tchouproff and that the language of Neyman is much simpler than the complicated notation of Tchouproff. Pearson emphasized that Tchouproff made no attempt to give specific formulas for the third and fourth moments of the variance in the case of sampling with replacements. Pearson did not answer, at least explicitly, the claim that the Tchouproff formulas are applicable to a more general case in which there is no restriction as to the nature of the correlation between observations.

The year 1928 was marked by two important contributions. We first mention that of C. C. Craig who published his thesis in *Metron* [22]. Extending the previous results of Thiele, he was able to write the semi-invariant equivalent of the basic formulas in much less space than their previous moment formulation had demanded. He was able to write products of sample moments as well as moments of the moments themselves. His results are limited to an infinite population and one variable. The bibliography attached to his paper is commonly mentioned in later literature for its completeness. For infinite sampling it might properly be used as a supplement to the bibliography of this Part.

A most important contribution was made by R. A. Fisher [23] who was able to simplify the infinite sampling formulas greatly. He did this by introducing the sample function whose expected value is a cumulant (semi-invariant). In addition to the simplification, his ingenious attack resulted in the following contributions: (1) the recognition of the one to one correspondence between all possible independent sampling formulas and the partition of numbers, (2), that the extension of the multivariate form is accomplished by use of the partitions of multipartite numbers, (3) the tabulation of numerous new formulas, (4) the use of a general partition method by which any term in the formulas can be determined separately.

The further development of the combinatorial analysis was indicated by a paper by Fisher and Wishart which appeared in 1931 [27]. It was shown how the more involved patterns could be broken up into simpler ones.

The study of the infinite case was continued by Georgescu [28] who extended the Craig results. A feature of his work was the utilization of functions which yielded expansions of formulas in terms of successive degrees of approximation.

He applied Fisher's idea of a combinatory analysis to the conventional sample moment function.

Another paper of this series was that of Wishart [29] who gave a discriptive account of the contributions of Craig, Fisher, and Georgescu and an indication of the means of expressing the results of one writer into the language of another.

The work of Joseph Pepper which appeared in *Biometrika* in 1929 [24] should be noted. Pepper took the case of the finite parent, sampling without replacement, and two variables, and then gave an extensive list of results. He did not have a very condensed notation and was forced to assume an infinite universe for the higher moments which he studied. The important point, for historical purposes, is that Pepper combined bivariate and finite sampling. It is to be recalled that Tchouproff himself in his generalized theory gave no results for the multivariate case.

A significant advance in finite sampling was indicated by the appearance of Carver's editorial on "Fundamentals of the Theory of Sampling," which appeared in the first volume of the ANNALS OF MATHEMATICAL STATISTICS [25]. Carver took the case of a finite universe, one variable, and sampling without replacements. He presented a notation which enabled him to write the various moments of the mean through the eighth in simple form. He showed by a number of illustrations that his formula would give known results for cases both infinite and finite, when the proper restrictions were added. O'Toole [26] later generalized his results for any moment of the mean.

3. **Generalized Carver Functions and Sampling.** The use of generalized Carver functions together with the results of Part I makes possible the presentation of the general sampling theory in a compact, and yet not too symbolic, form. It is possible to write the sampling theory so that criteria 1–8 are satisfied although no attempt is made in the present paper to answer criterion 6. With reference to criteria 9–11, any affirmative answer must necessarily be tempered with qualifications as the results are far removed from that ideal solution which would permit one to determine the actual distribution of any sample moment. However the use of generalized Carver functions does permit a general concise statement of results as well as the determination of special cases. The method is also especially adapted to the introduction of new moment functions and to the use of partition analysis, although these topics are not emphasized in the present paper. In general it may be said that the use of Carver functions assists greatly in finding the theoretical sample statistics in the case of finite sampling since the Carver functions are condensed expressions of the size of the sample and the size of the parent, since they may be easily checked from symmetrical considerations, and since they are independent of the moments. They are also applicable to different replacement laws.

4. **The Use of High Moments.** Precise agreement between theoretical and practical sampling does not usually accompany the use of high moments, and

the practical statistician is apt to agree with Pearson who wrote, "I have a very firm conviction that the mathematician who uses high moments may make interesting contributions to mathematics, but he removes his work from any contact with actual statistics" [16; 117]. However since the extent of agreement between theoretical and actual results is in a sense a measure of the extent to which theoretical assumptions are actually duplicated in the experiment, it does seem sensible to discover what relations exist in the ideal theoretical case. Thiele implicitly supported the theoretical use of high moments (even in studying actual problems) when he wrote [B′; 13]:

"Therefore the general rule of the formation of good laws of presumptive errors must be:

1. In determining λ_1, and λ_2 rely almost entirely upon the actual values.
2. As to the half-invariants with high indices, say λ_6 upwards, rely as exclusively upon theoretical considerations.
3 $\cdots$."

A more explicit advocate is R. A. Fisher who wrote [23; 200], "In the present state of our knowledge any information, however incomplete, as to sampling distributions is likely to be of frequent use, irrespective of the fact that moment functions only provide statistical estimates of high efficiency for a special type of distribution."

Chapter II. Notation and Definition

The present chapter gives the fundamental definitions and appropriate notation. An attempt has been made to combine the most desirable features of the different notations of earlier writers.

5. **Ordered Sample.** An ordered sample is a sample in which distinction is made as to the order in which the variate enters the growing sample. Thus the sample found by drawing x_2 and then x_1 is the same sample as that obtained by drawing x_1 and then x_2, but it is a different ordered sample.

In some types of sampling it is possible that a given variate may appear more than once in the same sample. In general the number of ordered samples varies with the number of repeated variates. Thus the sample $x_1 + x_1$ results from but one ordered sample, while $x_1 + x_2$ results from either of two ordered samples.

6. **Power Sums.** Power sums have the same meaning as in section 11 of Part I. An adjustment of notation is necessary as we need to distinguish power sums of the sample from power sums of the universe. The a-th power sum of the universe is denoted by $(\mathbf{A})$ while the sample power sum is denoted by (a). Similarly, bold-faced numerals are used to indicate power sums of the universe, while light-faced numerals are used to indicate power sums of the sample. The symbol $(\bar{A})$ is used to indicate that the variates are deviations from the mean of the universe.

7. **Power Product Sums.** Power product sums, called power products for brevity, also have the same meaning as in section 11 of Part I. Large letters are used to represent the power products of the universe while small letters are used to indicate the power products of the sample. Thus $(Q_1Q_2 \cdots Q_s)$ represents a power product of the universe while $(q_1q_2 \cdots q_s)$ represents the corresponding power product of the sample. Power products are not used extensively except in the development of the theory of the next chapter where they play an important rôle.

8. **Expected Values.** If a given statistical function, z is formed for every possible sample, then the arithmetic mean of the z's is the expected value of z. Thus $E(z) = \frac{\sum (z)}{S}$ where the Σ holds for all possible samples and S is the number of such samples.

9. **Moments.** Moments demand precise notation since distinction must be made between moments of the universe, moments of the sample, moments of the moments of the sample, and moments about the mean for these cases. In addition we wish to indicate whether or not the universe is measured about its mean.

a. *Moments of the universe.* The conventional μ's are used to indicate the moments of the universe. In this notation $\bar{\mu}$ is used to indicate the moment about the mean of the universe. Thus

$$\mu_t = \frac{(T)}{N} = \frac{\sum x^t}{N} \qquad \text{and} \qquad \bar{\mu}_t = \frac{(\bar{T})}{N} = \frac{\sum \bar{x}^t}{N}.$$

The usual formula relating μ_t and $\bar{\mu}_t$ [22; 20] may be written

$$\bar{\mu}_t = \sum (-1)^s \binom{1^t}{(t-s)\cdot 1^s} \mu_{t-s}\mu_1^s \tag{1}$$

so that

$$\bar{\mu}_2 = \frac{(2)}{N} - \frac{(1)(1)}{N^2},$$

$$\bar{\mu}_3 = \frac{(3)}{N} - \frac{3(2)(1)}{N^2} + \frac{2(1)^3}{N^3},$$

etc.

It is to be noted that, when $(1) = 0$, $\bar{\mu}_t = \mu_t$.

b. *Moments of the sample.* We denote the moments of the sample by the letter m [23; 203].

In much statistical work deviations from the mean of the universe are used in place of the variates themselves. When the universe moments about the mean appear, we indicate them with a bar. However in denoting the moments of the samples, the moments of the mean do not appear and some other device is needed to indicate whether or not the variates are measured about the mean of the universe. The simple notations m_t and $\bar{m}_t$ are used to indicate that the variates used are deviations from the mean of the universe. A superprefix is used to indicate the case in which the variates are not measured about the mean, ${}^{|}m_t$, ${}^{|}\bar{m}_t$. The values of $\bar{m}_t$ (and ${}^{|}\bar{m}_t$) are obtained from the values of m_t (and ${}^{|}m_t$) by means of the formula

$$\bar{m}_t = \sum (-1)^s \binom{1^t}{(t-s)\cdot 1^s} m_{t-s} m_1^s. \qquad \{2\}$$

c. *Moments of the moments of a sample.* Since there are many possible samples and since a given moment can be computed for each sample, it is possible to express the expected value of this moment and the expected value of any power of it. The μ's are used for this purpose. Thus

$$\begin{aligned} \mu_r(m_t) &= E(m_t)^r \\ \mu_r({}^{|}m_t) &= E({}^{|}m_t)^r \\ \mu_r(\bar{m}_t) &= E(\bar{m}_t)^r \\ \mu_r({}^{|}\bar{m}_t) &= E({}^{|}\bar{m}_t)^r. \end{aligned} \qquad \{3\}.$$

If the first one of equations {3} represents the whole group, then the values $\bar{\mu}_r(m_t)$, $\bar{\mu}_r({}^{|}m_t)$, $\bar{\mu}_r(\bar{m}_t)$, and $\bar{\mu}_r({}^{|}\bar{m}_t)$ are indicated by

$$\bar{\mu}_r(m_t) = \sum (-1)^s \binom{1^r}{(r-s)\cdot 1^s} \mu_{r-s}(m_t)\, \mu_1^s(m_t). \qquad \{4\}.$$

d. *Moments of the product of the moments of a sample.* The term $\frac{\sum xy}{N}$ can be indicated by $E(xy) = \mu_{11}(x, y)$. Similarly the expected value of the product of m_a and m_b may be indicated by $E(m_a m_b) = \mu_{11}(m_a, m_b)$. In general

$$\mu_{r_1 r_2 \cdots r_s}(m_{a_1}, m_{a_2}, \cdots, m_{a_s}) = E(m_{a_1}^{r_1} m_{a_2}^{r_2} \cdots m_{a_s}^{r_s}) \qquad \{5\}$$

In the case of the product of sample moment functions, when the universe is not measured about its mean, it is preferable to use a single superprefix, associated with the μ instead of a number of them associated with each m function. Thus

$$\mu_{111}({}^{|}m_a, {}^{|}m_b, {}^{|}m_c) = {}^{|}\mu_{111}(m_a, m_b, m_c).$$

The usual laws for changing from moments to moments about the mean in the case of the multivariate distributions are available. Thus

$$\bar{\mu}_{11}(m_a, m_b) = \mu_{11}(m_a, m_b) - \mu_{10}(m_a, m_b)\mu_{01}(m_a, m_b). \quad \{6\}$$

$$\begin{aligned}\bar{\mu}_{111}(m_a, m_b, m_c) = {} & \mu_{111}(m_a, m_b, m_c) - \mu_{110}(m_a, m_b, m_c)\mu_{001}(m_a, m_b, m_c) \\ & - \mu_{101}(m_a, m_b, m_c)\,\mu_{010}(m_a, m_b, m_c) \\ & - \mu_{011}(m_a, m_b, m_c)\,\mu_{100}(m_a, m_b, m_c) \\ & + 2\mu_{100}(m_a, m_b, m_c)\,\mu_{010}(m_a, m_b, m_c)\,\mu_{001}(m_a, m_b, m_c) \quad \{7\}\end{aligned}$$

etc.

10. **Different Sampling Laws.** For theoretical purposes, any law may be used in the formation of samples as long as it results in functions of all possible samples which are symmetric functions of the variates. Any uniform law of replacement satisfies this condition and hence might be used in forming samples. Most statisticians who have worked on the sampling problem have been content to assume one or the other of two replacement laws. Each of these is "natural," since it has wide application in the study of actual sampling.

The two types of sampling which have received general treatment are *sampling from an infinite universe with any law of replacement* and *sampling from a finite universe with a law of no replacements.* The results of the first type are also applicable to the case of *sampling from a finite universe when replacements are made after each drawing.* These two types of sampling have been characterized by the terms "sampling from an infinite universe," or "sampling from an unlimited supply" [25; 114] and "sampling from a finite universe" [17], or "sampling from a limited supply" [25; 101].

The theory of moments of moments for the first type of sampling has been developed to a high degree by such authors as Craig [22], Fisher [23], and Georgescu [28]. This extensive development has been due in part to the fact that the assumption of an infinite universe permits application of methods which are not applicable to the study of finite variation. The probability of getting a variate remains the same no matter what the law of replacement. The assumption of an infinite universe at first appears to make the results inapplicable to all actual problems where the universe is finite. However, if the universe is large, the assumption of infinite size does not greatly alter the results, although the extent of the change can not be determined without comparison with the results of finite sampling. A justification for the use of infinite sampling in actual finite sampling problems is based on the fact that the formulas resulting from sampling from a finite parent with replacements are the same as the infinite formulas. Hence the infinite results may be used to characterize finite sampling if *sampling is done with replacement after each drawing.* This clever scheme is somewhat invalidated, in actual sampling, because of the practicability of replacing and remixing after each drawing. Until someone demonstrates a technique which is practical and effective in securing randomness, it must be said that the value of infinite sampling theory as applied to finite sampling depends upon the theoretically unsatisfactory assumption that a finite universe is infinite.

The theory of sampling from a finite universe without replacements has been developed by such authors as Isserlis [8], Tchouproff [15], Neyman [18], Church [19], Pepper [24], and Carver [25], although available results are not as extensive as those mentioned above because of the difficulty of algebraic manipulation and because of the length of the formulas. The fact is that the probability of getting a given variate varies with the different drawings. However, a "return to the bag" is not demanded.

The terms "infinite sampling" and "finite sampling" are adequate to describe the two kinds of sampling discussed above, but they are inadequate in the case of finite sampling if additional replacement laws are introduced. Hence, it seems preferable to characterize the type of sampling by the replacement law if the population is finite.

When the Carver functions represent known functions of n and N, it is possible to use them in writing moment formulas for any orderly replacement law. For example, it is shown in later sections how Carver functions can be applied to

1. Finite sampling without replacement,
2. Finite sampling with replacement after each drawing.
3. Finite sampling without replacement up to the n-th drawing before which the $n - 1$ withdrawn variates are replaced and mixed.

The Carver function can be used symbolically even in cases in which its explicit statement in terms of n and N has not been found. In some statistical formulas the Carver functions cancel, so that the results are independent of the sampling law.

11. **Variable Distribution Laws.** It is possible to generalize the theory to include the case in which the variable takes on a different frequency distribution after each drawing, i.e., the general Tchouproff formulas can be written in terms of Carver functions. This theory can also be generalized to include many variables. In this dissertation, however, it is assumed that the universe remains the same, aside from the unreplaced variates forming the sample, throughout the sampling process.

Chapter III. The Application of the Double Expansion Theorem

It is the purpose of this chapter to establish the basic theorems on which the more specific work of the later chapters is based and to show how the double expansion theorem is to be applied to the sampling problem.

12. **Formulas Concerning Ordered Samples.** a. *Sampling with replacements.* If the samples of n are taken from a universe of N variates and if the variates are replaced after each drawing, then the number of possible ordered samples is N^n since for each of the n drawings there is a choice of N.

b. *Sampling without replacement.* If the variate is not replaced after each drawing, the number of ordered samples is

$$N(N-1)\cdots(N-n+1) = N^{(n)}.$$

c. *Replacement before the last drawing only.* In case sampling is with replacement before the last drawing only, the number of ordered samples is

$$N(N-1)\cdots(N-n+2)N = N^{(n-1)}N.$$

13. **Theorem I.** *All moments of moment functions of samples can be expressed in terms of expected values of products of power sums of samples.*

By moment functions we mean rational integral isobaric moment functions [31; 22].

The theorem follows at once from the definitions of section 9. From {3}, {4}, {5}, {6}, {7} it is clear that all moments of moment functions of samples are expressible in terms of the expected values of sample moment functions. But since the sample moment functions are themselves defined in terms of power sums of the samples, the theorem follows. For example

$$\bar{\mu}_2(\bar{m}_2) = \mu_2(\bar{m}_2) - \mu_1^2(\bar{m}_2) = E\left[\frac{(2)}{n} - \frac{(1)(1)}{n^2}\right]^2 - \left[E\left\{\frac{(2)}{n} - \frac{(1)(1)}{n^2}\right\}\right]^2 \qquad \{8\}$$

and

$$\begin{aligned}\bar{\mu}_{11}(\bar{m}_2, m_1) &= \mu_{11}(\bar{m}_2, m_1) - \mu_{10}(\bar{m}_2, m_1)\mu_{01}(\bar{m}_2, m_1) \\ &= E\left[\frac{(2)(1)}{n^2} - \frac{(1)^3}{n^3}\right] - \left[E\left\{\frac{(2)}{n} - \frac{(1)(1)}{n}\right\}\right]\left[E\frac{(1)}{n}\right] \qquad \{9\}\end{aligned}$$

14. **Theorem II.** *All moments of moment functions of samples can be expressed in terms of expected values of power products of samples.*

This follows at once from the application of the multiplication theorem of Part I to the theorem of section 13. Each product of power sums is expanded by the multiplication theorem into sums of power products. Thus

$$\begin{aligned}\mu_2('\bar{m}_2) &= E\left[\frac{(2)(2)}{n^2} - \frac{2(2)(1)(1)}{n^3} + \frac{(1)^4}{n^4}\right] \\ &= \left(\frac{1}{n^2} - \frac{2}{n^3} + \frac{1}{n^4}\right)E(4) + \left(-\frac{4}{n^3} + \frac{4}{n^4}\right)E(31) + \left(\frac{1}{n^2} - \frac{2}{n^3} + \frac{3}{n^4}\right)E(22) \\ &\quad + \left(\frac{-2}{n^3} + \frac{6}{n^4}\right)E(211) + \frac{1}{n^4}E(1111). \qquad \{10\}\end{aligned}$$

15. **Theorem III.** *To every power product form* $(q_1q_2\cdots q_s)$ *there corresponds a power product form* $(Q_1Q_2\cdots Q_s)$.

The argument is simple since the terms of $(q_1q_2\cdots q_s)$ are themselves terms of $(Q_1Q_2\cdots Q_s)$. It follows at once that, if $(q_1q_2\cdots q_s)$ exists, then $(Q_1Q_2\cdots Q_s)$ exists.

As an illustration, consider the universe consisting of x_1, x_2, x_3, x_4, x_5 and the sample consisting of x_1, x_2, x_3, x_4. Then the terms of $(q_1q_2q_3) = \sum_{i_1 \neq i_2 \neq i_3}^{4} x_{i_1}^{q_1} x_{i_2}^{q_2} x_{i_3}^{q_3}$ are all contained in the terms of $(Q_1Q_2Q_3) = \sum_{i_1 \neq i_2 \neq i_3}^{5} x_{i_1}^{q_1} x_{i_2}^{q_2} x_{i_3}^{q_3}$.

16. **Theorem IV.** *If definite k's can be determined so that*

$$E(q_1q_2 \cdots q_s) = k_{p_1p_2\cdots p_s}(Q_1Q_2 \cdots Q_s), \qquad \{11\}$$

then it is possible to use the double expansion theorem and express the moments of the moments of the sample in terms of the P functions of Part I and the power sums (or moments) of the universe.

The double expansion theorem was designed to replace $(q_1q_2 \cdots q_s)$ by $k_{p_1p_2\cdots p_s}(Q_1Q_2 \cdots Q_s)$. It can be used as well to replace $E(q_1q_2 \cdots q_s)$ by $k_{p_1p_2\cdots p_s}(Q_1Q_2 \cdots Q_s)$ if the values of $k_{p_1p_2\cdots p_s}$ can be determined. The results of such a substitution in terms of the power sums of the universe are then given by the double expansion theorem. For example

$$\mu_2({}^{|}m_1) = \frac{E(1)^2}{n^2} = \frac{E(2)}{n^2} + \frac{E(11)}{n^2}$$

and if $E(2) = k_1(\mathbf{2})$ and $E(11) = k_{11}(\mathbf{11})$ then

$$\mu_2({}^{|}m_1) = (k_2 - k_{11}) \frac{(\mathbf{2})}{n^2} + \frac{k_{11}(\mathbf{1})^2}{n^2}$$

$$= \frac{K_2}{n^2}(\mathbf{2}) + \frac{K_{11}(\mathbf{1})^2}{n^2}$$

where $K_2 = k_2 - k_{11}$ and $K_{11} = k_{11}$.

It then appears that the methods and tables of Chapter I of Part I can be used in finding expressions for moments of moments, in case $k_{p_1p_2\cdots p_s}$ is known. Thus

$$\mu_2({}^{|}\bar{m}_2) = E\left[\frac{(2)}{n} - \frac{(1)(1)}{n^2}\right]^2 = E\left[\frac{(2)(2)}{n^2} - \frac{2(2)(1)(1)}{n^3} + \frac{(1)(1)(1)(1)}{n^4}\right]$$

$$= \frac{P_2(\mathbf{4}) + P_{11}(\mathbf{2})(\mathbf{2})}{n^2} - 2\left[\frac{P_3(\mathbf{4}) + 2P_{21}(\mathbf{3})(\mathbf{1}) + P_{21}(\mathbf{2})(\mathbf{2}) + P_{111}(\mathbf{2})(\mathbf{1})(\mathbf{1})}{n^3}\right]$$

$$+ \frac{P_4(\mathbf{4}) + 4P_{31}(\mathbf{3})(\mathbf{1}) + 3P_{22}(\mathbf{2})(\mathbf{2}) + 6P_{211}(\mathbf{2})(\mathbf{1})(\mathbf{1}) + P_{1111}(\mathbf{1})^4}{n^4}$$

and when $(\mathbf{1}) = 0$

$$\mu_2(\bar{m}_2) = \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4}\right)(\bar{\mathbf{4}}) + \left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4}\right)(\bar{\mathbf{2}})(\bar{\mathbf{2}}). \qquad \{13\}$$

where

$$\begin{aligned}
P_4 &= k_4 - 4k_{31} - 3k_{22} + 12k_{211} - 6k_{1111} \\
P_{31} &= \qquad k_{31} \qquad - 3k_{211} + 2k_{1111} \\
P_{22} &= \qquad\qquad k_{22} - 2k_{211} + k_{1111} \\
P_{211} &= \qquad\qquad\qquad k_{211} - k_{1111} \\
P_{1111} &= \qquad\qquad\qquad\qquad k_{1111}
\end{aligned}$$

as given by {54} of Part I.

The basic problem has thus been reduced to finding $k_{p_1 \cdots p_s}$ such that

$$E(q_1 q_2 \cdots q_s) = k_{p_1 \cdots p_s}(Q_1 Q_2 \cdots Q_s).$$

17. **Theorem V**. *The expected value of a sample power sum is always $\frac{n}{N}$ times the corresponding universe power sum no matter what the replacement law.*

The expected value of the sample power sum is always the same even though the k's take on different values for different replacement laws. We note first that the number of ordered samples, S, depends upon the replacement law. Now a given sample power sum, (a), has n terms, while the corresponding power sum of the universe, (A), has N terms. All the a-th powers of the variates in the universe appear in the ordered samples and, if we add all possible ordered samples, these terms appear the same number of times. Hence

$$\sum (a) = k_1^{|}(A) \qquad \text{and} \qquad \frac{\sum (a)}{(A)} = k_1^{|}.$$

Now the number of the a-th powers of the variate in $\sum (a)$ is Sn so that each of the N variates appears $\frac{Sn}{N}$ times. It follows that $\sum (a) = \frac{Sn}{N}(A)$ and hence that $E(a) = \frac{n}{N}(A)$. Hence

$$E(a) = k_1(A) \qquad \text{where} \qquad k_1 = \frac{n}{N} \tag{15}$$

no matter what the law of replacement.

An illustration may serve to clarify the argument. Consider a universe composed of x_1, x_2, x_3 and write the six ordered samples. Then

$$\frac{\sum (a)}{(A)} = \frac{x_1^a + x_2^a + x_1^a + x_3^a + x_2^a + x_1^a + x_2^a + x_3^a + x_3^a + x_1^a + x_3^a + x_2^a}{x_1^a + x_2^a + x_3^a} = 4$$

and

$$\frac{E(a)}{(A)} = \frac{2}{3} = \frac{n}{N}.$$

18. **Value of $k_{p_1 \cdots p_s}$ for sampling without replacement.** Consider a universe and all possible ordered samples. Form $(Q_1Q_2 \cdots Q_s)$ and $\sum (q_1q_2 \cdots q_s)$. Now $\Sigma(q_1q_2 \cdots q_s)$ is a symmetric function of the variates and consists of $N^{(n)}n^{(s)}$ products, and $(Q_1Q_2 \cdots Q_s)$ consists of $N^{(s)}$ products. Each of the $N^{(s)}$ products is repeated the same number of times in the $N^{(n)}n^{(s)}$ products of $\sum (q_1q_2 \cdots q_s)$. To find the number of times such repetition is made, it is only necessary to divide the total number of terms in $\sum (q_1q_2 \cdots q_s)$ by the number of terms in $(Q_1Q_2 \cdots Q_s)$ which gives $\frac{N^{(n)}n^{(s)}}{N^{(s)}}$. Hence

$$\sum (q_1 q_2 \cdots q_s) = \frac{N^{(n)}n^{(s)}}{N^{(s)}} (Q_1 Q_2 \cdots Q_s) \tag*{\{16\}}$$

and, dividing by the number of ordered samples, $N^{(n)}$,

$$E(q_1 q_2 \cdots q_s) = \frac{n^{(s)}}{N^{(s)}} (Q_1 Q_2 \cdots Q_s) \tag*{\{17\}}$$

so that

$$k_{p_1 \cdots p_s} = \frac{n^{(s)}}{N^{(s)}} \tag*{\{18\}}$$

as stated in section 46 of Part I.

Since $(q_1q_2 \cdots q_s) = s_1!s_2! \cdots s_p!M(q_1q_2 \cdots q_s)$

and $(Q_1Q_2 \cdots Q_s) = s_1!s_2! \cdots s_p!M(Q_1Q_2 \cdots Q_s)$

it follows that

$$EM(q_1 q_2 \cdots q_s) = \frac{n^{(s)}}{N^{(s)}} M(Q_1 Q_2 \cdots Q_s). \tag*{\{19\}}$$

Most earlier writers on finite sampling have used the idea expressed in {19} as the foundation of their work. They have found it necessary to undertake enormous algebraic manipulation to expand in terms of monomial symmetric functions and then to expand back in terms of power sums after making the coefficient adjustment. Such long derivations are not only laborious, but they are also apt to result in algebraic errors and the results obtained have not emphasized the symmetry which is inherent in the nature of the problem and which is very useful in checking calculations. It was Carver who first discovered the type of symmetric relation involved and who used it in obtaining a compact statement of the first eight moments of the sample sum in the case of a single variable. He, too, found it necessary to carry out extensive algebraic manipulations as his reference to "lavish use of symmetric functions" [25; 104] reveals. His keen insight into the essential nature of this problem led him to the conclusion that such extensive algebraic manipulation should not be necessary and that it should be possible to apply P functions to sample moments of order higher than the first. His confidence that this could be done and his encouragement in the task have contributed in a large degree to whatever merit this dissertation may have.

With $k_{p_1 \cdots p_s} = \frac{n^{(s)}}{N^{(s)}}$, it is at once possible to write the P function expansions. Following Carver, we let $\rho_1 = \frac{n}{N}$, $\rho_2 = \frac{n(n-1)}{N(N-1)}$, etc. and get, from sections 43 and 44 of Part I,

$$\begin{aligned} P_1 &= \rho_1 & P_{11} &= \rho_2 \\ P_2 &= \rho_1 - \rho_2 & P_{21} &= \rho_2 - \rho_3 \\ P_3 &= \rho_1 - 3\rho_2 + 2\rho_3 & P_{31} &= \rho_2 - 3\rho_3 + 2\rho_4 \\ P_4 &= \rho_1 - 7\rho_2 + 12\rho_3 - 6\rho_4 & P_{22} &= \rho_2 - 2\rho_3 + \rho_4 \end{aligned}$$

etc. etc.

19. **Expected Values of Products of Sample Power Sums, Sampling Without Replacement.** The tables of Chapter I of Part I are now available for use. Thus

$$\mu_3({}^1m_1) = E({}^1m_1)^3 = \frac{1}{n^3}E(1)^3 = \frac{1}{n^3}[P_3(3) + 3P_{21}(2)(1) + P_{111}(1)^3]. \quad \{20\}$$

where

$$P_3 = \frac{n}{N} - \frac{3n(n-1)}{N(N-1)} + \frac{2n(n-1)(n-2)}{N(N-1)(N-2)}$$

$$P_{21} = \frac{n(n-1)}{N(N-1)} - \frac{n(n-1)(n-2)}{N(N-1)(N-2)}$$

$$P_{111} = \frac{n(n-1)(n-2)}{N(N-1)(N-2)}.$$

Formula {20} might be written as

$$\mu_3({}^1m_1) = \frac{1}{n^3}[P_3N\mu_3 + 3P_{21}N^2\mu_2\mu_1 + P_{111}N^3\mu_1^3] \quad \{21\}$$

We note further that as $N \to \infty$

$$NP_3 \to n, \quad P_{21}N^2 \to n(n-1), \quad P_{111}N^2 \to n(n-1)(n-2)$$

so that

$$\mu_3({}^1m_1) = \frac{1}{n^3}[n\mu_3 + 3n(n-1)\mu_2\mu_1 + n(n-1)(n-2)\mu_1^3] \quad \{22\}$$

More generally

$$P_{m_1 \cdots m_r}(Q_1)(Q_2) \cdots (Q_r) = P_{m_1 \cdots m_r}N^r\mu_{q_1}\mu_{q_2} \cdots \mu_{q_r}. \quad \{23\}$$

As N approaches infinity this becomes

$$P_{m_1 \cdots m_r}(Q_1)(Q_2) \cdots (Q_r) = n^{(r)} \mu_{q_1} \mu_{q_2} \cdots \mu_{q_r} . \qquad \{24\}$$

The laws of infinite sampling may be obtained by replacing power sums by moments and $P_{m_1 \cdots m_r}$ by $n^{(r)}$. The tables given in a recent paper [31; 30–32] were obtained from the tables of P functions by this method.

20. **Sampling With Replacements.** We next consider the case of finite sampling with replacements after each drawing. This is such a simple case that the P's can be determined without finding the k's.

Consider a universe and the N^r possible ordered samples. Thus the nine ordered samples of 2 from a universe of 3 are indicated by the subscripts

11	21	31
12	22	32
13	23	33

The samples 11, 22, 33 are not repeated while the others are. The multiplication theorem can be used in grouping types of product terms as it was in Part I, but the terms themselves have different interpretation. Thus $(1)(1) = (2) + (11)$ can be written as $(1)(1) = (2) + [11]$ where the (2) indicates the sum of the n terms found by multiplying an x by itself, while the $[11]$ indicates the sum of the $n(n - 1)$ products formed by multiplying one x by another. Since some of the x's may be alike, it is possible to have squared terms in $[1 \cdot 1]$, but they are not treated as squared terms, but rather as products. For example, if $(1) = x_1 + x_1$

$$(1)(1) = x_1^2 + x_1^2 + x_1 x_1 + x_1 x_1$$

so that

$$(2) = x_1^2 + x_1^2 \text{ and } [11] = x_1 x_1 + x_1 x_1 .$$

In determining the expected value of $(1)(1)$, we note that

$$\sum (1)(1) = \sum (2) + \sum [1 \cdot 1]$$

where $\sum$ holds for the N^n possible samples. Now $\sum (2) = k_1'(2)$ and $k_1' = \frac{nN^n}{N}$ so that $E(2) = \frac{n}{N}(2)$ as indicated in Theorem V. Also $[11]$ is composed of $N^n n^{(2)}$ products of $\sum_{i,j=1}^{N} x_i x_j = \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{j=1}^{N} x_j\right)$. It follows that

$$\sum [1 \cdot 1] = \frac{N^n n^{(2)}}{N^2} (1)(1) \quad \text{and that}$$

$$E[1 \cdot 1] = \frac{n^{(2)}}{N^2} (1)(1).$$

It appears that $\frac{n^{(2)}}{N^2}$ plays the rôle of P_{11}.

$$\mu_2({}^1m_1) = \frac{1}{n^2}E[(2) + [11]]$$

$$= \frac{1}{n^2}[P_2(2) + P_{11}(1)(1)]$$

where $P_2 = \frac{n}{N}$ and $P_{11} = \frac{n^{(2)}}{N^2}$.

The corresponding argument holds for the general case. Any product of power sums can be expanded in terms of $(q_1 q_2 \cdots q_s)$. If duplicate variates are introduced, use the notation $[q_1 q_2 \cdots q_s]$. Form $[q_1 q_2 \cdots q_s]$ for all the N^n ordered samples. Now $[q_1 q_2 \cdots q_s]$ has $n^{(s)}$ terms and $\sum [q_1 q_2 \cdots q_s] = k(Q_1)(Q_2) \cdots (Q_s)$ has $n^{(s)}N^n$ terms, while $(Q_1)(Q_2) \cdots (Q_s)$ has N^s terms. It follows that $k = \frac{n^{(s)}N^n}{N^s}$, that

$$\sum [q_1 q_2 \cdots q_s] = \frac{n^{(s)}N^n}{N^s} (Q_1)(Q_2) \cdots (Q_s),$$

and that

$$E[q_1 q_2 \cdots q_s] = \frac{n^{(s)}}{N^s} (Q_1)(Q_2) \cdots (Q_s). \qquad \{25\}$$

Hence

$$P_{m_1 \cdots m_s} = \frac{n^{(s)}}{N^s}. \qquad \{26\}$$

In general

$$P_{m_1 \cdots m_s}(Q_1)(Q_2) \cdots (Q_s) = n^{(s)}\mu_{q_1}\mu_{q_2} \cdots \mu_{q_s}. \qquad \{27\}$$

Comparison with {24} shows that the same basic laws appear no matter whether sampling is carried on with replacement, or, in the infinite case, without replacement.

21. **Other Replacement Laws.** The two cases just examined represent two extremes of orderly replacement laws. It has been shown in each case how the Carver functions can be used to express relations between the moments of the moments of the sample and the moments of the universe. It is possible to show how these functions are applicable to other replacement laws. We take, as an illustration, the case in which no replacements are made after each of the first $n - 1$ drawings, but just before the last drawing the $n - 1$ variates are replaced and mixed. I do not present here the detailed argument, but simply indicate that the appropriate value of $k_{p_1 \cdots p_s}$ is

$$k_{p_1 \cdots p_s} = \frac{n^{(s)}}{N^{(s)}} + \frac{n-1}{N^{(s)}N}[(n-2)^{(s)} - n^{(s)} + (2^{p_1} + 2^{p_2} + \cdots + 2^{p_s})(n-2)^{(s-1)}] \qquad \{28\}$$

22. **Different Frequency Laws.** The distribution of variates may follow some known frequency law such as the normal, rectangular, binomial, Poisson, etc. In such a case, if the relations between the moments are known, it is possible to simplify the results.

Chapter IV. The Moments of the Mean

To illustrate the previous theory in a simple situation we consider the moments of the mean. Carver [25] has done this previously for the case of finite sampling without replacements, but he has taken the measures of the universe as deviations and has used the sample sum rather than the sample mean. O'Toole [26] has generalized Carver's work.

23. **The Moments of the Mean.** We have at once

$$\mu_1({}^1m_1) = \frac{1}{n} E(1) = \frac{1}{n} P_1(\mathbf{1}) = \mu_1$$

$$\mu_2({}^1m_1) = \frac{1}{n^2} E(1)^2 = \frac{1}{n^2} [P_2(\mathbf{2}) + P_{11}(\mathbf{1})(\mathbf{1})]$$

$$\mu_3({}^1m_1) = \frac{1}{n^3} E(1)^3 = \frac{1}{n} [P_3(\mathbf{3}) + 3P_{21}(\mathbf{2})(\mathbf{1}) + P_{111}(\mathbf{1})(\mathbf{1})(\mathbf{1})]$$

$$\mu_4({}^1m_1) = \frac{1}{n^4} E(1)^4 = \frac{1}{n^4} [P_4(\mathbf{4}) + 4P_{31}(\mathbf{3})(\mathbf{1}) + 3P_{22}(\mathbf{2})(\mathbf{2}) + 6P_{211}(\mathbf{2})(\mathbf{1})(\mathbf{1}) + P_{1111}(\mathbf{1})^4]$$

and

$$\mu_r({}^1m_1) = \frac{1}{n^r} \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} (P_1)^{\pi_1} \cdots (P_s)^{\pi_s} \qquad \{29\}$$

24. **Moments About the Mean of the Sample Mean.** Using {1}, we get

$$\bar{\mu}_2({}^1m_1) = \frac{1}{n^2} [P_2(\mathbf{2}) + (P_{11} - P_1^2)(\mathbf{1})(\mathbf{1})]$$

$$\bar{\mu}_3({}^1m_1) = \frac{1}{n^3} [P_3(\mathbf{3}) + 3(P_{21} - P_2P_1)(\mathbf{2})(\mathbf{1}) + (P_{111} - 3P_{11}P_1 + 2P_1^3)(\mathbf{1})^3]$$

$$\bar{\mu}_4({}^1m_1) = \frac{1}{n^4} [P_4(\mathbf{4}) + 4(P_{31} - P_3P_1)(\mathbf{3})(\mathbf{1}) + 3P_{22}(\mathbf{2})(\mathbf{2}) + 3(P_{211} - 2P_{21}P_1 + P_2P_1^2)(\mathbf{2})(\mathbf{1})(\mathbf{1}) + (P_{1111} - 4P_{111}P_1 + 6P_{11}P_1^2 - 3P_1^4)(\mathbf{1})^4] \qquad \{30\}$$

etc.

These formulas can be written in the notation of moments of the universe as

$$\bar{\mu}_2({}^1m_1) = \frac{1}{n^2}[P_2N\mu_2 + (P_{11} - P_1^2)N^2\mu_1^2]$$

$$\bar{\mu}_3({}^1m_1) = \frac{1}{n^3}[P_3N\mu_3 + 3(P_{21} - P_2P_1)N^2\mu_2\mu_1 + (P_{111} - 3P_{11}P_1 + 2P_1^3)N^3\mu_1^3] \quad \{31\}$$

etc.

25. **Moments of the Sample Mean When the Universe is Measured About its Mean.** When **(1)** = 0, the formulas of section {23} become

$$\mu_1(m_1) = 0$$

$$\mu_2(m_1) = \frac{1}{n^2} P_1(\bar{\mathbf{2}})$$

$$\mu_3(m_1) = \frac{1}{n^3} P_3(\bar{\mathbf{3}})$$

$$\mu_4(m_1) = \frac{1}{n^4} [P_4(\bar{\mathbf{4}}) + 3P_{22}(\bar{\mathbf{2}})^2]$$

and

$$\mu_r(m_1) = \frac{1}{n^r} \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} P_{p_1^{\pi_1}\cdots p_s^{\pi_s}}(\bar{P}_1)^{\pi_1} \cdots (\bar{P}_s)^{\pi_s} \quad \{32\}$$

where the $\sum$ holds for all partitions having no unit parts. In the language of moments {32} becomes

$$\mu_r(m_1) = \frac{1}{n^r} \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} P_{p_1^{\pi_1}\cdots p_s^{\pi_s}} N^{p_1\pi_1 \cdots p_s\pi_s}(\bar{\mu}_{p_1})^{\pi_1} \cdots (\bar{\mu}_{p_s})^{\pi_s} \quad \{33\}$$

where the $\sum$ holds for all partitions of r having no unit parts.

26. **Moments About the Mean of the Sample When the Universe is Measured From its Mean.** Similarly, when **(1)** = 0, the results of section {24} become

$$\left.\begin{aligned} \bar{\mu}_2(m_1) &= \frac{1}{n^2} P_2(\bar{\mathbf{2}}) = \frac{1}{n^2} P_2N\bar{\mu}_2 \\ \bar{\mu}_3(m_1) &= \frac{1}{n^3} P_3(\bar{\mathbf{3}}) = \frac{1}{n^3} P_3N\bar{\mu}_3 \\ \bar{\mu}_4(m_1) &= \frac{1}{n^4} [P_4(\bar{\mathbf{4}}) + 3P_{22}(\bar{\mathbf{2}})^2] \\ &= \frac{1}{n^4} [P_4N\bar{\mu}_4 + 3P_{22}N^2\bar{\mu}_2^2] \\ &\text{etc.} \end{aligned}\right\} \quad \{34\}$$

It is to be noticed that the values $\bar{\mu}_r(m_1)$ are equal to the values $\mu_r(m_1)$. This results from {4} and the fact that $\mu_1(m_1) = 0$. It should be noted also that $\bar{\mu}_r({}^1m_1) \neq \mu_r({}^1m_1)$ as $\mu_1({}^1m_1) \neq 0$.

27. **Sampling Without Replacements.** The formulas in sections 23–26 are general formulas which become more specific as given replacement laws are introduced. If the law is sampling without replacements, we recall that $P_1 = \rho_1$, $P_2 = \rho_1 - \rho_2$, $P_3 = \rho_1 - 3\rho_2 + 2\rho_3$, etc. when $\rho_s = \frac{n^{(s)}}{N^{(s)}}$. It is at once possible to write the appropriate formula. Thus

$$\bar{\mu}_3(m_1) = \mu_3(m_1) = \frac{1}{n^3} P_{3,3}$$

$$= \frac{1}{n^3}[\rho_1 - 3\rho_2 + 2\rho_3]N\mu_3 = \frac{(N-n)(N-2n)}{n^2(N-1)(N-2)}\bar{\mu}_3. \quad \{35\}$$

Now $\bar{\mu}_3 = 0$ in any symmetric universe, for example a normal or rectangular one, so $\bar{\mu}_3(m_1) = 0$.

28. **Sampling With Replacements.** In this case $P_{m_1 \cdots m_r} = \frac{n^{(r)}}{N^r}$ and we have

$$\mu_1({}^1m_1) = \mu_1$$

$$\mu_2({}^1m_1) = \frac{1}{n^2}[n\mu_2 + n(n-1)\mu_1^2]$$

$$\mu_3({}^1m_1) = \frac{1}{n^3}[n\mu_3 + 3n(n-1)\mu_2\mu_1 + n(n-1)(n-2)\mu_1^3]$$

$$\mu_4({}^1m_1) = \frac{1}{n^4}[n\mu_4 + 4n(n-1)\mu_3\mu_1 + 3n(n-1)\mu_2^2 + 6n(n-1)(n-2)\mu_2\mu_1^2 + n^{(4)}\mu_1^4]$$

and in general

$$\mu_r({}^1m_1) = \frac{1}{n^r}\sum\binom{1^r}{p_1^{\pi_1}\cdots p_s^{\pi_s}} n^{(\rho)}(\mu_{p_1})^{\pi_1}\cdots(\mu_{p_s})^{\pi_s} \quad \{36\}$$

and

$$\bar{\mu}_2({}^1m_1) = \frac{1}{n^2}[n\mu_2 - n\mu_1^2]$$

$$\bar{\mu}_3({}^1m_1) = \frac{1}{n^3}[n\mu_3 - 3n\mu_2\mu_1 + 2n\mu_1^3] \quad \{37\}$$

$$\bar{\mu}_4({}^1m_1) = \frac{1}{n^4}[n\mu_4 - 4n\mu_3\mu_1 + 3n(n-1)\mu_2^2 - 6n(n-2)\mu_2\mu_1^2 + 3n(n-2)\mu_1^4]$$

while

$$\bar{\mu}_2(m_1) = \mu_2(m_1) = \frac{\bar{\mu}_2}{n}$$

$$\bar{\mu}_3(m_1) = \mu_3(m_1) = \frac{\bar{\mu}_3}{n^2}$$

$$\bar{\mu}_4(m_1) = \mu_4(m_1) = \frac{1}{n^3}[\bar{\mu}_4 + 3(n-1)\bar{\mu}_2^2] \quad \{38\}$$

etc.

29. **Sampling With Replacements Before the Last Drawing Only.** The values of $k_{p_1 \ldots p_s}$ of section 21 determine the values of the P's. Thus $P_2 = k_2 - k_{11} = \frac{n}{N} - \frac{n(n-1)}{N(N-1)} + \frac{2(n-1)}{N(N-1)}$ and $P_{11} = \frac{n(n-1)}{N(N-1)} - \frac{2(n-1)}{N^2(N-1)}$ so

$$\bar{\mu}_2({}^1m_1) = \frac{1}{n^2}\left\{\left[n - \frac{(n-1)(n-2)}{N-1}\right]\mu_2 + \frac{(n-1)(nN-2)}{N-1}\mu_1^2\right\}. \quad \{39\}$$

$$\bar{\mu}_2(m_1) = \frac{1}{n^2}\left\{\left[n - \frac{(n-1)(n-2)}{N-1}\right]\bar{\mu}_2\right\}. \quad \{40\}$$

30. **Different Frequency Laws.** As indicated in section 22, the frequency distributions of the parent may be characterized by some moment relationship. This relationship can be inserted and the resulting formula simplified. For example, if the law of the formation of the universe is that of the hypergeometric series [25; 113]

$$\bar{\mu}_n = pq[q^{n-1} + (-1)^n p^{n-1}], \quad \{41\}$$

we have

$$\left.\begin{aligned} \bar{\mu}_2(m_1) &= \frac{P_2}{n^2} Npq \\ \bar{\mu}_3(m_1) &= \frac{P_3}{n^3} Npq(q^2 - p^2) \\ \bar{\mu}_4(m_1) &= \frac{1}{n^4}[P_4 Npq(q^3 + p^3) + 3P_{22}N^2p^2q^2] \\ &\text{etc.} \end{aligned}\right\} \quad \{42\}$$

Where the values of P_2, P_3, P_4 are to be inserted according to the replacement law which is used in forming the samples. The results for sampling without replacement agree with those given by Pearson [1].

31. **Moments of the Sample Sum.** We might use the sum of the items in the sample instead of the sample mean. For example

$$\mu_2(1) = E(1)^2 = n^2E(m_1)^2 = n^2\mu_2(m_1).$$

The results would parallel the results above except that n^r in the denominator would be eliminated. It is the sample sum which is used in Carver's article [25] and this should be noted in comparing results.

Chapter V. The Mean and Variance of the Variance

As a further illustration of the use of the Carver functions there are presented in this chapter formulas for the mean of the variance and the variance of the variance.

32. The Mean of the Variance.

$$\begin{aligned}\mu_1({}^{|}\bar{m}_2) &= E\left[\frac{(2)}{n} - \frac{(1)(1)}{n^2}\right] \\ &= \frac{P_1}{n}(\mathbf{2}) - \frac{P_2(\mathbf{2}) + P_{11}(\mathbf{1})^2}{n^2} \\ &= \frac{1}{n^2}[(nP_1 - P_2)(\mathbf{2}) - P_{11}(\mathbf{1})^2] \qquad \{43\}\end{aligned}$$

and

$$\mu_1(\bar{m}_2) = \frac{1}{n^2}(nP_1 - P_2)N\bar{\mu}_2. \qquad \{44\}$$

When sampling is with replacements $P_1 = P_2 = \frac{n}{N}$ and we get the well known

$$\mu_1(\bar{m}_2) = \frac{(n-1)}{n}\bar{\mu}_2 \qquad \{45\}$$

while when sampling is without replacements, we have the well known

$$\mu_1(\bar{m}_2) = \frac{\left(1 - \frac{1}{n}\right)\bar{\mu}_2}{1 - \frac{1}{N}}. \qquad \{46\}$$

33. The Second Moment of the Variance.

$$\mu_2({}^{|}\bar{m}_2) = E\left[\frac{(2)^2}{n^2} - \frac{2(2)(1)(1)}{n^3} + \frac{(1)^4}{n^4}\right]$$

becomes

$$\begin{aligned}\mu_2({}^{|}\bar{m}_2) = &\left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4}\right)(\mathbf{4}) - 4\left(\frac{P_{21}}{n^3} - \frac{P_{31}}{n^4}\right)(\mathbf{3})(\mathbf{1}) \\ &+ \left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4}\right)(\mathbf{2})(\mathbf{2}) - 2\left(\frac{P_{111}}{n^3} - \frac{3P_{211}}{n^4}\right)(\mathbf{2})(\mathbf{1})(\mathbf{1}) + P_{1111}(\mathbf{1})^4 \qquad \{47\}\end{aligned}$$

and

$$\mu_2(\bar{m}_2) = \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4}\right)(\bar{4}) + \left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4}\right)(\bar{2})^2. \qquad \{48\}$$

These of course can be written in terms of moments of the variance.

34. **The Variance of the Variance.** Since $\bar{\mu}_2({}^{|}\bar{m}_2) = \mu_2({}^{|}\bar{m}_2) - \mu_1^2({}^{|}\bar{m}_2)$, we have

$$\bar{\mu}_2({}^{|}\bar{m}_2) = \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4}\right)(\mathbf{4}) - 4\left(\frac{P_{21}}{n^3} - \frac{P_{31}}{n^4}\right)(\mathbf{3})(\mathbf{1})$$
$$+ \left[\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4} - \left(\frac{P_1}{n} - \frac{P_2}{n^2}\right)^2\right](\mathbf{2})(\mathbf{2})$$
$$- 2\left[\frac{P_{111}}{n^3} - \frac{3P_{211}}{n^4} - \frac{P_1P_{21}}{n^3} + \frac{P_2P_{11}}{n^4}\right)(\mathbf{2})(\mathbf{1})(\mathbf{1}) + \frac{P_{1111} - P_{11}^2}{n^4}(\mathbf{1})^4. \qquad \{49\}$$

Formula {49} may also be written as

$$\bar{\mu}_2({}^{|}\bar{m}_2) = \frac{1}{n^4}\{(n^2P_2 - 2nP_3 + P_4)N\mu_4 - 4(nP_{21} - P_{31})N^2\mu_3\mu,$$
$$+ (n^2P_{11} - 2nP_{21} + 3P_{22} - n^2P_1^2 + 2nP_1P_2 - P_2^2)N^2\mu_2^2$$
$$- 2(nP_{111} - 3P_{211} - nP_1P_{11} + P_2P_{11})N^3\mu_2\mu_1^2 + (P_{1111} - P_{11}^2)N^4\mu_1^4\}. \qquad \{50\}$$

Formulas {49} and {50} are not expressed in terms of deviations of the variates. Neither do they assume any particular replacement law nor any particular type of universe.

In case the universe is measured about its mean we can write at once, by placing $(\mathbf{1}) = 0$ in {49}

$$\bar{\mu}_2(\bar{m}_2) = \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4}\right)(\bar{4})$$
$$+ \left[\left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4}\right) - \left(\frac{P_1}{n} - \frac{P_2}{n^2}\right)^2\right](\bar{2})(\bar{2}) \qquad \{51\}$$

and

$$\bar{\mu}_2(\bar{m}_2) = \frac{1}{n^4}\{(n^2P_2 - 2nP_3 + P_4)N\bar{\mu}_4 + (n^2P_{11} - 2nP_{21} + 3P_{22} - n^2P_1^2$$
$$+ 2nP_1P_2 - P_2^2)N\bar{\mu}_2^2\}. \qquad \{52\}$$

35. **Sampling Without Replacements.** Using the P's as defined by sampling without replacements, it appears that the coefficient of the μ_4 term

$$\left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4}\right)N = \frac{N}{n^3}\frac{(N-n)(n-1)(Nn-N-n-1)}{(N-1)(N-2)(N-3)} \qquad \{53\}$$

agrees with that given by Neyman [18; 477], Tchouproff [15; 660], Pepper [24; 234], Carver [25; 270]. Also the coefficient of the μ_2^2 term

$$\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4} - \left(\frac{P_1}{n} - \frac{P_2}{n^2}\right)^2 N^2 = -\frac{N(N-n)(n-1)(N^2n - 3N^2 + 6N - 3n - 3)}{n^3(N-1)^2(N-2)(N-3)} \quad \{54\}$$

agrees with that of the above authors.

As far as the author is aware, no one has written the coefficients of $\mu_3\mu_1$, $\mu_2\mu_1^2$, and μ_1^4 in the formula for $\bar{\mu}_2({}^{|}\bar{m}_2)$.

The coefficient of $\mu_3\mu_1$ is

$$-4\left(\frac{P_{21}}{n^3} - \frac{P_{31}}{n^4}\right)N^2 = \frac{-4N(n-1)(N-n)(Nn - N - n - 1)}{n^3(N-1)(N-2)(N-3)}. \quad \{55\}$$

The coefficient of $\mu_2\mu_1^2$ is

$$-2\left(\frac{P_{111}}{n^3} - \frac{3P_{211}}{n^4} - \frac{P_1P_{11}}{n^3} + \frac{P_2P_{11}}{n^4}\right)N^3 = \frac{4N^2}{n^3}\,\frac{(n-1)(N-n)[(2n-3)N - 3(n-1)]}{(N-1)^2(N-2)(N-3)} \quad \{56\}$$

while the coefficient of μ_1^4 is

$$\left(\frac{P_{1111}}{n^4} - \frac{P_{11}P_{11}}{n^4}\right)N^4 = -\frac{2N^2}{n^3}\,\frac{(n-1)(N-n)[(2n-3)N - 3(n-1)]}{(N-1)^2(N-2)(N-3)}. \quad \{57\}$$

It is possible with some algebraic manipulation to use the P functions to express the coefficients of the moments as functions of N and n. The suggestion here is that such algebraic work is unnecessary since the left members of {53} $\cdots$ {57} are as easily handled in an actual problem as the right hand members. It is possible to compute the coefficients from the ρ's and the P's without writing explicit expansions in terms of N and n. Besides the formulas involving N and n are so lengthy that algebraic errors are apt to occur. The use of Carver functions is further advocated because the same basic formulas are applicable to all types of sampling and because the tables of Chapter I of Part I are directly applicable.

36. Sampling With Replacements. If $P_{m_1\cdots m_r} = \frac{n^{(r)}}{N^{(r)}}$, the coefficient of μ_4 is $\frac{1}{n^4}[(n(n-1)^2] = \frac{(n-1)^2}{n^3}$ while the coefficient of μ_2^2 is

$$\frac{1}{n^4}[(n^2 - 2n + 3)n(n-1) - (n^2 - 2n + 1)n^2] = \frac{(n-1)(3-n)}{n^3}.$$

Then {52} becomes

$$\bar{\mu}_2(\bar{m}_2) = \frac{1}{n^3}[(n-1)^2\bar{\mu}_4 - (n-1)(n-3)\bar{\mu}_2^2]. \qquad \{58\}$$

The formula for $\bar{\mu}_2(^{|}\bar{m}_2)$ becomes

$$\bar{\mu}_2(^{|}\bar{m}_2) = \frac{1}{n^3}[(n-1)^2\mu_4 - 4(n-1)^2\mu_3\mu_1 - (n-1)(n-3)\mu_2^2 + 4(2n-3)(n-1)\mu_2\mu_1^2 - 2(2n-3)(n-1)\mu_1^4]. \qquad \{59\}$$

Now {58} can be written in terms of semi-invariants by the use of $\bar{\mu}_4 = \lambda_4 + 3\lambda_2^2$ and $\bar{\mu}_2 = \lambda_2$ so

$$\bar{\mu}_2(\bar{m}_2) = \frac{1}{n^3}[(n-1)^2\lambda_4 + 2n(n-1)\lambda_2^2].$$

See [B′; 209], [22; 57].

37. **Different Distribution Laws.** Given frequency laws can be inserted. Thus {44} becomes

$$\mu_1'(\bar{m}_2) = \frac{1}{n^2}(nP_1 - P_2)pq \qquad \text{if the} \qquad \bar{\mu}_2 = pq$$

while {52} becomes, if $\bar{\mu}_2 = pq$ and $\bar{\mu}_4 = pq(q^3 + p^3)$

$$\bar{\mu}_2(\bar{m}_2) = \frac{N}{n^4}(n^2P_2 - 2nP_3 + P_4)pq(q^3 + p^3) + \frac{N^2}{n^4}(n^2P_{11} - 2nP_{21} + 3P_{22} - n^2P_1^2 + 2nP_2P_1 - P_2^2)p^2q^2. \qquad \{60\}$$

Other frequency laws can be inserted similarly.

Chapter VI. Tabular Presentation of Formulas.

It is the purpose of this dissertation to show how the P functions can be used in finite sampling rather than to present an exhaustive list of formulas. The specific formulas of the two previous chapters are derived, primarily, for illustrative purposes. The implication is that other formulas may be derived similarly.

However, it is possible to present, implicitly in tabular form, a number of formulas. In this chapter there are presented formulas involving moments of weight equal to or less than 6.

38. **The formulas of weight 2.**

$$\left.\begin{aligned} \mu_1(^{|}\bar{m}_2) &= \left(\frac{P_1}{n} - \frac{P_2}{n^2}\right)(\mathbf{2}) - \frac{P_{11}}{n^2}(\mathbf{1})(\mathbf{1}) \\ \mu_2(^{|}m_1) &= \left[\frac{P_2}{n^2}(\mathbf{2}) + \frac{P_{11}}{n^2}(\mathbf{1})^2\right] \end{aligned}\right\} \qquad \{61\}$$

can be written in tabular form as

	2	11		2	11
2	P_1			$\frac{1}{n}$	
11	P_2	P_{11}		$-\frac{1}{n^2}$	$\frac{1}{n^2}$

with little effort. The first entries in the top row indicate the power sums of the universe, while the columnar entries indicate the moments of the sample. Now

$$\mu_1({}'\bar{m}_2) = E\left[\frac{(2)}{n} - \frac{(1)(1)}{n^2}\right]$$

and

$$\mu_2(m_1) = E\left[\frac{(1)(1)}{n^2}\right].$$

The coefficients of the power sums in the expansion of $\bar{m}$ are entered in the right hand part of the table. Thus, under 2, there appear the entries $\frac{1}{n}$ and $-\frac{1}{n^2}$. These when multiplied by the power sums as indicated on the left, give $\bar{m}_2 = \frac{(2)}{n^2} - \frac{(1)(1)}{n^2}$. Similarly $m_1^2 = \frac{(1)(1)}{n^2}$.

Now the expected value is given by the proper P function expansion. The left hand portion of the table, which is the same as the P function table of Chapter I of Part I, gives such expansions. Thus the coefficient of (2) in $E(m_2)$ is $\frac{P_1}{n} - \frac{P_2}{n^2}$, while the coefficient of (1)(1) is $-\frac{P_{11}}{n^2}$. Hence the complete formula is

$$\mu_1({}'\bar{m}_2) = \left(\frac{P_1}{n} - \frac{P_2}{n^2}\right)(2) - \frac{P_{11}}{n^2}(1)(1)$$

as indicated above.

39. **The Formulas of Weight 3.** Similarly the table

	3	21	111		3	21	111
3	P_1				$\frac{1}{n}$		
21	P_2	P_{11}			$-\frac{3}{n^2}$	$\frac{1}{n^2}$	
111	P_3	$3P_{21}$	P_{111}		$\frac{2}{n^3}$	$-\frac{1}{n^3}$	$\frac{1}{n^3}$

can be used to give the formulas

$$\mu_1({}^{|}\bar{m}_3) = \left(\frac{P_1}{n} - \frac{3P_2}{n^2} + \frac{2P_3}{n^3}\right)(3) - 3\left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3}\right)(2)(1) + 2\frac{P_{111}}{n^3}(1)^3 \quad \{62\}$$

$$^{|}\mu_{11}(\bar{m}_2, m_1) = \left(\frac{P_2}{n^2} - \frac{P_3}{n^3}\right)(3) + \left(\frac{P_{11}}{n^2} - \frac{3P_{21}}{n^3}\right)(2)(1) - \frac{2P_{111}}{n^3}(1)^3 \quad \{63\}$$

$$\mu_3({}^{|}m_1) = \frac{P_3}{n^3}(3) + \frac{3P_{21}}{n^3}(2)(1) + \frac{P_{111}}{n^3}(1)^3. \quad \{64\}$$

In case we wish to express the results in terms of moments about the mean, $(1) = 0$, and we have

$$\mu_1(\bar{m}_3) = \left(\frac{P_1}{n} - \frac{3P_2}{n^2} + \frac{2P_3}{n^3}\right)(\bar{3}) \quad \{65\}$$

$$\mu_{11}(\bar{m}_2, m_1) = \left(\frac{P_2}{n^2} - \frac{P_3}{n^3}\right)(\bar{3}) \quad \{66\}$$

$$\mu_3(m_1) = \frac{P_3}{n^3}(\bar{3}) \quad \{67\}$$

so that

$$\mu_1(\bar{m}_3) = \left(\frac{P_1}{n} - \frac{3P_2}{n^2} + \frac{2P_3}{n^3}\right)N\bar{\mu}_3 \quad \{68\}$$

$$\mu_{11}(\bar{m}_2, m_1) = \left(\frac{P_2}{n^2} - \frac{P_3}{n^3}\right)N\bar{\mu}_3 \quad \{69\}$$

$$\mu_3(m_1) = \frac{P_3}{n^3}N\bar{\mu}_3. \quad \{70\}$$

The insertion of specific sampling laws gives the specific results of earlier authors.

40. **The Tabular Forms.** It is further evident that the power of n in the denominator is equal to the sum of the subscripts of the Carver function above it. We might utilize this knowledge and write in the right hand part of the table the numerators of the entries in the tables above. The table of weight of 3 would then appear as

	3	21	111		3	21	111
3	P_1				1		
21	P_2	P_{11}			-3	1	
111	P_3	$3P_{21}$	P_{111}		2	-1	1

and it is possible to read {62}, {63}, {64}, {65}, {66}, and {67} directly from it.

TABLE I

$w = 2$

	2	11		2	11
2	P_1			1	
11	P_2	P_{11}		−1	1

$w = 3$

	3	21	1		3	21	1
3	P_1				1		
21	P_2	P_{11}			−3	1	
1	P_3	$3P_{21}$	P_{111}		2	−1	1

$w = 4$

	4	31	22	211	1111		4	31	22	211	1111
4	P_1						1				
31	P_2	P_{11}					−4	1			
22	P_2		P_{11}						1		
211	P_3	$2P_{21}$	P_{21}	P_{111}			6	−3	−2	1	
1111	P_4	$4P_{31}$	$3P_{22}$	$6P_{211}$	P_{1111}		−3	2	1	−1	1

$w = 5$

	5	41	32	311	221	2111	1^5		5	41	32	311	221	2111	1^5
5	P_1								1						
41	P_2	P_{11}							−5	1					
32	P_2		P_{11}								1				
311	P_3	$2P_{21}$	P_{21}	P_{111}					10	−4	−1	1			
221	P_3	P_{21}	$2P_{21}$		P_{111}						−3		1		
2111	P_4	$3P_{31}$	$P_{31} + 3P_{22}$	$3P_{211}$	$3P_{211}$	P_{1111}			−10	6	5	−3	−2	1	
11111	P_5	$5P_{41}$	$10P_{32}$	$10P_{311}$	$15P_{221}$	$10P_{2111}$	P_{11111}		4	−3	−2	2	1	1	1

$w = 6$

	6	51	42	33	411	321	222	3111	2^21^2	$2\,1^4$	1^6
6	P_1										
51	P_2	P_{11}									
42	P_2		P_{11}								
33	P_2			P_{11}							
411	P_3	$2P_{21}$	P_{21}		P_{111}						
321	P_3	P_{21}	P_{21}	P_{21}		P_{111}					
222	P_3		$3P_{21}$				P_{111}				
3111	P_4	$3P_{31}$	$3P_{22}$	P_{31}	$3P_{211}$	$3P_{211}$		P_{1111}			
2211	P_4	$2P_{31}$	$P_{22} + 2P_{31}$	$2P_{22}$	P_{211}	$4P_{211}$	P_{211}		P_{1111}		
21^4	P_5	$4P_{41}$	$P_{41} + 6P_{32}$	$4P_{32}$	$6P_{311}$	$4P_{311} + 12P_{221}$	$3P_{221}$	$4P_{2111}$	$6P_{2111}$	P_{11111}	
1^6	P_6	$6P_{51}$	$15P_{42}$	$10P_{33}$	$15P_{411}$	$60P_{321}$	$15P_{222}$	$20P_{3111}$	$45P_{2211}$	$15P_{21111}$	P_{111111}

	6	51	42	33	411	321	222	3111	2^21^2	$2\,1^4$	1^6
6	1										
51	−6	1									
42			1								
33				1							
411	15	−5	−1		1						
321			−4	−6		1					
222							1				
3111	20	10	4	4	−4	−1		1			
2211			6	9		−3	−3		1		
21^4	−15	−10	−9	−12	6	5	3	−3	−2	1	
1^6	5	4	3	4	−3	−2	−1	2	1	−1	1

The tables of weight $W = 2, 3, 4, 5, 6$ are given in Table I. The right hand partitions not involving unit parts are underscored as these indicate the columns which should be used if the universe is measured about its mean. As an illustration we write from Table I the value of $\mu_2(\bar{m}_2)$. We get

$$\mu_2(\bar{m}_2) = \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4}\right) N\bar{\mu}_4 + \left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4}\right) N\bar{\mu}_2^2$$

as previously indicated.

The same tabular scheme can be used to write formulas of weight greater than 6.

41. **Moments of Other Sample Moment Functions.** It is possible to use a similar tabular scheme when we wish to find the moments of other sample moment functions. We define

$$l_2 = \frac{(2)}{n} - \frac{(1)(1)}{n^2}$$

$$l_3 = \frac{(3)}{n} - \frac{3(2)(1)}{n^2} + \frac{2(1)^3}{n^3}$$

$$l_4 = \frac{(4)}{n} - \frac{4(3)(1)}{n^2} - \frac{3(2)(2)}{n^2} + \frac{12(2)(1)(1)}{n^3} - \frac{6(1)^4}{n^4}$$

and, in general,

$$l_n = \sum (-1)^{\rho-1}(\rho - 1)! \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} \frac{(p_1)^{\pi_1} \cdots (p_s)^{\pi_s}}{n^\rho}. \qquad \{71\}$$

The formulas of weight 5 are given by Table II.

TABLE II

	5	41	*32*	311	221	2111	1^5		−5	41	32	311	221	21^3	1^5
5	P_1								1						
41	P_2	P_{11}							−5	1					
32	P_2		P_{11}						−10		1				
311	P_3	$2P_{21}$	P_{21}	P_{111}					20	−4	−1	1			
221	P_3	P_{21}	$2P_{21}$		P_{111}				30	−3	−3		1		
2111	P_4	$3P_{31}$	$P_{31} + 3P_{22}$	$3P_{211}$	$3P_{211}$	P_{1111}			−60	12	5	−3	−2	1	
11111	P_5	$5P_{41}$	$10P_{32}$	$10P_{311}$	$15P_{221}$	$10P_{2111}$	P_{11111}		24	−6	−2	2	1	−1	1

Thus for example

$$ {}^{\prime}\mu_{11}(l_4, l_1) = \left(\frac{P_2}{n^2} - \frac{7P_3}{n^3} + \frac{12P_4}{n^4} - \frac{6P_5}{n^5}\right) N\bar{\mu}_5 + \left(-\frac{10P_{21}}{n^3} + \frac{12P_{31}}{n^4} + \frac{36P_{22}}{n^4} - \frac{60P_{32}}{n^5}\right) N^2 \bar{\mu}_3 \bar{\mu}_2. \quad \{72\} $$

If all the entries in the right hand part of Table I, except the unit terms in the main diagonal, are placed equal to 0, the tables can be used to give the moment function of the m_a. Thus, when $w = 3$,

$$ \mu_1({}^{\prime}m_3) = \frac{P_1}{n}(3) \quad \{73\} $$

$$ {}^{\prime}\mu_{11}(m_2, m_1) = \frac{P_2}{n^2}(3) + \frac{P_{11}}{n^2}(2)(1) \quad \{74\} $$

$$ \mu_3({}^{\prime}m_1) = \frac{P_3}{n^3}(3) + \frac{3P_{21}}{n^3}(2)(1) + \frac{P_{111}}{n^3}(1)^3 \quad \{75\} $$

and

$$ \mu_1(m_3) = \frac{P_1}{n} N\bar{\mu}_3 \quad \{76\} $$

$$ \mu_{11}(m_2, m_1) = \frac{P_2}{n^2} N\bar{\mu}_3 \quad \{77\} $$

$$ \mu_3(m_1) = \frac{P_3}{n^3} N\bar{\mu}_3. \quad \{78\} $$

42. **Other Moment Functions.** The tables give such formulas as $\mu_r(\bar{m}_a)$, $\mu_{r_1r_2}(\bar{m}_a, \bar{m}_b)$, etc. If formulas for $\bar{\mu}_r(\bar{m}_a)$, $\bar{\mu}_{r_1r_2}(\bar{m}_a, \bar{m}_b)$ etc., are needed, it is necessary to go through the usual work of changing from moments to moments about the mean.

Let us derive a general formula for the correlation of the mean and the variance as an illustration of the use of the tabular formulas. By definition

$$ r_{11}(\bar{m}_2, m_1) = \frac{\bar{\mu}_{11}(\bar{m}_2, m_1)}{[\bar{\mu}_{20}(\bar{m}_2, m_1)\bar{\mu}_{02}(\bar{m}_2, m_1)]^{\frac{1}{2}}}. \quad \{79\} $$

Now

$$ \bar{\mu}_{11}(\bar{m}_2, m_1) = \mu_{11}(\bar{m}_2, m_1) $$

$$ \bar{\mu}_{20}(\bar{m}_2, m_1) = \mu_2(\bar{m}_2) - \mu_1^2(\bar{m}_2) $$

$$ \bar{\mu}_{02}(\bar{m}_2, m_1) = \mu_2(m_1) - \mu_1^2(m_1) = \mu_2(m_1). $$

Some of these values have appeared earlier in this paper. Without using the earlier results, we find from Table I

$$\mu_{11}(\bar{m}_2, m_1) = \left(\frac{P_2}{n^2} - \frac{P_3}{n^3}\right) N\bar{\mu}_3$$

$$\mu_2(\bar{m}_2) = \left(\frac{P_2}{n^2} - \frac{2P_3}{n^3} + \frac{P_4}{n^4}\right) N\bar{\mu}_4 + \left(\frac{P_{11}}{n^2} - \frac{2P_{21}}{n^3} + \frac{3P_{22}}{n^4}\right) N^2\bar{\mu}_2^2$$

$$\mu_1(\bar{m}_2) = \left(\frac{P_1}{n} - \frac{P_2}{n^2}\right) N\bar{\mu}_2$$

$$\mu_2(m_1) = \frac{P_2}{n^2} N\bar{\mu}_2.$$

Hence {79} becomes

$$r_{11}(\bar{m}_2, m_1) = \frac{(nP_2 - P_3)\bar{\mu}_3}{[(n^2P_2^2 - 2nP_2P_3 + P_2P_4)\bar{\mu}_4\bar{\mu}_2 - (n^2P_2P_{11} - 2nP_2P_{21} + 3P_2P_{22} - n^2P_2P_1^2 + 2nP_2^2P_1 - P_2^3)N\bar{\mu}_2^3]^{\frac{1}{2}}} \qquad \{80\}$$

Formula {80} gives the correlation between the variance and the mean no matter what the law of replacement. If the universe is symmetric, $\bar{\mu}_3 = 0$ and $r_{11}(\bar{m}_2, m_1) = 0$.

The usual special cases may be obtained. When replacements are made, {80} becomes at once

$$r_{11}(\bar{m}_2, m_1) = \frac{(n-1)\bar{\mu}_3}{[(n-1)\bar{\mu}_4\bar{\mu}_2 - (3-n)\bar{\mu}_2^3]^{\frac{1}{2}}} \qquad \{81\}$$

as indicated by Pepper [24; 246].

When no replacements are made {80} reduces to results previously given by Neyman [18; 489] and Pepper [24; 245].

43. Conclusion. The theory presented here is capable of generalization in many ways. For example, application to multivariate distributions readily follows. However an attempt has been made in this dissertation to emphasize the essence of the method. Illustrations have been chosen to indicate its inherent generality.

It should be stated, finally, that the aim of this dissertation is not primarily to provide a list of sampling formulas, but rather to provide a method by which the desired sampling formula may be derived without too much algebraic work.

In concluding this dissertation, I wish to acknowledge the guidance and encouragement of Professsor H. C. Carver. Also I wish to express my appreciation to Professor R. A. Fisher and to Professor C. C. Craig, who read the manuscript, or portions of it, and made needed suggestions for improvement. I am also indebted to Professor J. A. Nyswander and Professor T. H. Hildebrandt for valuable advice and assistance.

BIBLIOGRAPHY

(1) Pearson, K., "On Certain Properties of the Hypergeometric Series etc., *Philosophical Magazine*, 45 (1899), pp. 231-246.

(2) Pearson, K., "On the Probable Error of Frequency Constants," *Biometrika*, 2 (1902-3), pp. 273-281.

(3) Henderson, R., "Frequency Curves and Moments," *Transactions of Acturial Society of America*," 8 (1904), pp. 30-42.

(4) Pearson, K., "Note on the Significant or Non-significant Character of a Sub-sample Drawn from a Sample." *Biometrika*, 5 (1906), pp. 181-183.

(5) "Student," "The Probable Error of a Mean," *Biometrika*, 6 (1908) pp. 1-25.

(6) Pearson, K., "On the Probable Errors of Frequency Constants." *Biometrika*, 9 (1913), pp. 1-10.

(7) Isserlis, L., On the Conditions Under Which the Probable Error of Frequency Distributions Have Real Significance." *Royal Society Proceedings*, 92A (1915), pp. 23-41.

(8) Isserlis, L., "On the Value of a Mean as Calculated From a Sample," *Journal of Royal Statistical Society*, 81 (1918), pp. 75-81.

(9) Edgeworth, F., "On the Value of a Mean as Calculated from a Sample," *Journal of Royal Statistical Society*, 81 (1918), pp. 624-632.

(10) Tchouproff, A., "On the Mathematical Expectation of a Positive Integral Power of the Difference Between the Frequency and the Probability of an Event." *Proceedings of the Petrograd Polytechnic Institute*,

(11) Tchouproff, A., "Zur Theorie der Stabilitat Statisticher Reirhen," *Skandinavisk Aktuarietidskrift*, (1918).

(12) Tchouproff, A., "On the Mathematical Expectation of the Moments of Frequency Distributions." Part I *Biometrika*, 12 (1918), pp. 140-169, 185-210. Part II, *Biometrika* 13 (1920-21), pp. 283-295.

(13) Pearson, K., "Peccavimus." *Biometrika*, 12 (1918-19), pp. 259-281.

(14) Splawa-Neyman, J., "*La Revue Mensuelle de Statistique*." Tome 6 (1923) pp. 1-29.

(15) Tchouproff, A., "On the Mathematical Expectation of the Moments of Frequency Distributions in the Case of Correlated Observations," Metron 2 (1923), pp. 461-493; 646-683.

(16) Pearson, K., "Note on Professor Romanovsky's Generalization of my Frequency Curves," *Biometrika*, 16 (1924), pp. 116-117.

(17) Church, A., "On the Moments of the Distributions of Squared Deviations for Samples of N Drawn from an Indefinitely Large Population," *Biometrika*, 17 (1925), pp. 79-83.

(18) Neyman, J., "Contributions to the Theory of Small Samples Drawn From a Finite Population," *Biometrika*, 17 (1925), pp. 472-479.

(19) Church, A., "On the Means and Squared Deviations of Small Samples From Any Population," *Biometrika*, 18 (1926), pp. 321-394.

(20) Greenwood, M., and Isserlis, L., "A Historical Note on the Problems of Small Samples," *Royal Statistical Society Journal*, 90 (1927), pp. 347-352.

(21) Pearson, K., "Another Historical Note on the Problem of Small Samples," *Biometrika*, 19 (1927), pp. 207-210.

(22) Craig, C. C., "An Application of Thiele's Semi-invariants to the Sampling Problem," *Metron*, 7 (1928-29), pp. 3-74.

(23) Fisher, R. A., "Moments and Product Moments of Sampling Distributions," *Proceedings London Mathematical Society*, 2 (30) (1929), pp. 199-238.

(24) Pepper, J., "Studies in the Theory of Sampling," *Biometrika*, 21 (1929), pp. 231-258.

(25) Carver, H. C., "Fundamentals of the Theory of Sampling," *Annals of Mathematical Statistics*, 1 (1930), pp. 101-121; 260-274.

(26) O'Toole, A. L., "On Symmetric Functions and Symmetric Functions of Symmetric Functions," *Annals of Mathematical Statistics*, 2 (1931), pp. 102–149.

(27) Fisher, R. A., and Wishart, J., "The Derivation of the Pattern Formulae of Two Way Partitions From Those of Simpler Patterns," *Proceedings London Mathematical Society*, 2-33 (1931), pp. 195–208.

(28) St. Georgescu, N., "Further Contributions to the Sampling Problem." *Biometrika*, 24 (1932), pp. 65–107.

(29) Wishart, J., "A Comparison of the Semi-invariants of the Distributions of the Moment and the Semi-invariant Estimates in Sampling From an Infinite Population," *Biometrika*, 25 (1933), pp. 52-60.

(30) Feldman, H., "Mathematical Expectations of Product Moments of Samples Drawn from a Set of Infinite Populations." *Annals of Mathematical Statistics*, 6 (1935), pp. 30-52.

(31) Dwyer, P. S., "Moments of Any Rational Integral Isobaric Sample Moment Function," *Annals of Mathematical Statistics*, 8 (1937), pp. 21–65.

BOOKS

A. Whitworth, "Choice and Chance." 4th Edition (1886).

B. Thiele, T. N., "Theory of Observations," (1903).

B′. Reprinted *Annals of Mathematical Statistics*, 2 (1931), pp. 165–306.

C. Mortara, "Elementi di Statistica," Roma (1917).

D. Rietz, H. L., (Editor-in-chief), "Handbook of Mathematical Statistics" 1924.

E. Rietz, H. L., "Mathematical Statistics," (1927).