THE COMPUTATION OF MOMENTS WITH THE USE OF CUMULATIVE TOTALS

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1. **Introduction.** Various authors have shown how the moments of a frequency distribution may be computed from cumulated frequencies.[1] In order to make clear to the reader the type of technique under discussion there is presented an illustration which is, essentially, that used by Hardy, [2, p. 59]. The value $\Sigma f_x = 729$ is the last entry in column 4.

We use C_1^1 to denote the entry in column 4 which is opposite the smallest variate (or class mark if the distribution is grouped). Similarly C_2^1 is the entry above C_1^1, and C_1^2 the entry to the right of C_1^1, etc. In this notation the diagonal entries, the ones underscored in Table I, are C_1^1, C_2^2, C_3^3, C_4^4, C_5^5.

The moments[2] about the smallest variate can be expressed in terms of the cumulations of Table I in different ways. One method utilizes the diagonal entries and the differences of zero. Thus

$$\sum_0^6 xf_x = C_2^2 = 2916; \qquad \sum_0^6 x^2f_x = C_2^2 + 2C_3^3 = 12336;$$

$$\sum_0^6 x^3f_x = C_2^2 + 6C_3^3 + 6C_4^4 = 57996;$$

$$\sum_0^6 x^4f_x = C_2^2 + 14C_3^3 + 36C_4^4 + 24C_5^5 = 278316, \text{ etc.}$$

A second method utilizes the entries in the next to the last row and the differences of zero. Thus

$$\sum_0^6 xf_x = C_2^2 = 2916; \qquad \sum_0^6 x^2f_x = -C_2^2 + 2C_2^3 = 12636;$$

$$\sum_0^6 x^3f_x = C_2^2 - 6C_2^3 + 6C_2 = 57996;$$

$$\sum_0^6 x^4f_x = - C_2^2 + 14C_2^3 - 36C_2^4 + 24C_2^5 = 278316, \text{ etc.}$$

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[1] The reader is referred to reference [1] . . . [15], at end of paper.

[2] It is to be noted that we are not talking about moments *per unit frequency*. We are using the term in the sense used for example by Whittaker and Robinson. See [20, p. 18].

A third method, which seems to have escaped previous attention, involves columnar entries and multipliers whose determination and properties are a chief concern of this paper. Thus

$$\sum_0^6 xf_x = C_2^2 = 2916; \qquad \sum_0^6 x^2f_x = C_2^3 + C_3^3 = 12636;$$

$$\sum_0^6 x^3f_x = C_2^4 + 4C_3^4 + C_4^4 = 57996;$$

$$\sum_0^6 x^4f_x = C_2^5 + 11C_3^5 + 11C_4^5 + C_5^5 = 278316, \text{ etc.}$$

It is possible also to obtain formulas when the cumulations are made from the smallest variate to the largest variate and, indeed, the whole theory of the present paper could be duplicated with such a theory of cumulation.

TABLE I

Successive Frequency Cumulations

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
X	x	F_x	C^1	C^2	C^3	C^4	C^5
$a+6$	6	64	64	64	64	64	64
$a+5$	5	192	256	320	384	448	512
$a+4$	4	240	496	816	1200	1648	2160
$a+3$	3	160	656	1472	2672	4320	6480
$a+2$	2	60	716	2188	4860	9180	15660
$a+1$	1	12	728	2916	7776	16956	32616
a	0	1	729	3645	11421	28377	60993

It is possible to obtain the columnar formulas from the well known diagonal formulas. From the construction of Table I it is clear that

$$C_i^i = C_{i+1}^i + C_i^{i-1} \tag{1}$$

so that

$$\begin{gathered} C_2^2 = C_2^2; C_2^2 + 2C_3^3 = C_2^3 + C_3^3; \qquad C_2^2 + 6C_3^3 + 6C_4^4 = C_2^4 + 4C_3^4 + C_4^4; \\ C_2^2 + 14C_3^3 + 36C_4^4 + 24C_5^5 = C_2^5 + 11C_3^5 + 11C_4^5 + C_5^5. \end{gathered} \tag{2}$$

Formula (1) can be used similarly in deriving columnar formulas from row formulas, diagonal formulas from row formulas, etc.

The columnar method is here recommended as a useful substitute for the usual elementary method of computing moments. The many multiplications involved in the usual process are replaced by continued addition. The chief disadvantage of the method is the continual recording, although this obstacle is surmounted with an adding machine equipped with a recording tape. The resulting moments are easily checked with an adaptation of Charlier's check, as is shown in section 8, and methods are given by which the multipliers are easily obtained. The method is also well adapted to the use of Hollerith machines.

The introduction of such columnar multipliers tends to give a different emphasis to the cumulative totals technique. The use of diagonal entries led logically to an emphasis upon factorial moments, while the columnar method tends to emphasize the more familiar power moments. The primary application here indicated is not to elaborate and specialized techniques, but rather to the simple, though often tedious, problem of the computation of power moments.

The aims of this paper are then:

(1) To show how moments may be computed from the columnar values of the successive cumulations,
(2) To discover the properties of the columnar multipliers,
(3) To present a general theory for computation of moments using cumulative totals.

2. **The Basic Cumulative Theorem.** The use of (1) is not satisfactory in getting precise formulas for the columnar multipliers so we derive the columnar cumulative theory directly from first principles. We first prove

THEOREM I. *Let x be any real number and let u_x be a real function of x which is 0 when $x < a$ and when $x > a + k$ and which is not infinite for $x = a, a + 1, a + 2, \cdots, a + k$. Let v_x be a real function of x and $\underline{v}_x$, called range v_x, a function such that $\underline{v}_x = v_x$ when $x = a, a + 1, \cdots, a + k$ and $\underline{v}_x = 0$ at all points outside the range a to $a + k$. If $\sum_x^{a+k} u_i$ is indicated by Cu_x and $\underline{v}_x - v_{x-1}$ by ∇v_x, $\underline{v}_x - \underline{v}_{x-1}$ by $\nabla \underline{v}_x$ then*

$$\sum_a^{a+k} u_x v_x = \sum_a^{a+k} u_x \underline{v}_x = \sum_a^{a+k} Cu_x \nabla \underline{v}_x. \tag{3}$$

The values u_x, v_x, Cu_x, $\nabla \underline{v}_x$ are presented in Table II.
The theorem is proved by forming

$$\sum_a^{a+k} Cu_x \nabla \underline{v}_x \equiv u_{a+k} v_{a+k} + \cdots + u_{a+i} v_{a+i} + \cdots + u_a v_a$$

$$\equiv \sum_a^{a+k} u_x v_x \equiv \sum_a^{a+k} u_x \underline{v}_x .$$

Theorem I can also be written as

$$\sum_0^k u_{a+x} v_{a+x} = \sum_0^k Cu_{a+x} \nabla \underline{v}_{a+x} \tag{4}$$

3. The Successive Cumulation Theorem.

THEOREM II. *If* $C^2u_x = C[Cu_x]$ *and* $\nabla^2\underline{v}_x = \nabla(\nabla\underline{v}_x)$, *etc., then*

$$\sum_a^{a+k} u_x v_x = \sum_a^{a+k} u_x \underline{v}_x = \sum_a^{a+k} C^{s+1}u_x \nabla^{s+1}\underline{v}_x .$$

This theorem follows readily from Theorem I. If

$$U_x = Cu_x \quad \text{and} \quad V_x = \nabla v_x ,$$

then

$$\sum_a^{a+k} u_x v_x = \sum_a^{a+k} u_x \underline{v}_x = \sum_a^{a+k} U_x \underline{V}_x = \sum_a^{a+k} CU_x \nabla \underline{V}_x = \sum_a^{a+k} C^2 u_x \nabla^2 \underline{v}_x .$$

This process can be extended as many times as desired so that

$$\sum_a^{a+k} u_x v_x = \sum_a^{a+k} u_x \underline{v}_x = \sum_a^{a+k} C^{s+1}u_x \nabla^{s+1}\underline{v}_x . \tag{5}$$

TABLE II

Values of x, u_x , v_x , Cu_x , *and* $\nabla\underline{v}_x$.

x	u_x	v_x	Cu_x	$\nabla\underline{v}_x$
$a + k$	u_{a+k}	v_{a+k}	u_{a+k}	$v_{a+k} - v_{a+k-1}$
$a + k - 1$	u_{a+k-1}	v_{a+k-1}	$u_{a+k} + u_{a+k-1}$	$v_{a+k-1} - v_{a+k-2}$
....	....	....	....	....
$a + i$	u_{a+i}	v_{a+i}	$u_{a+k} + \cdots + u_{a+i}$	$v_{a+i} - v_{a+i-1}$
....	....	....	....	....
$a + 1$	u_{a+1}	v_{a+1}	$u_{a+k} + \cdots + u_{a+i} + \cdots + u_{a+1}$	$v_{a+1} - v_a$
a	u_a	v_a	$u_{a+k} + \cdots + u_{a+i} + \cdots + u_{a+1} + u_a$	v_a

This can also be written as

$$\sum_0^k u_{a+x}v_{a+x} = \sum_0^k u_{a+x}\underline{v}_{a+x} = \sum_0^k C^{s+1}u_{a+x}\nabla^{s+1}\underline{v}_{a+x} . \tag{6}$$

In order to determine the values $\nabla^{s+1}\underline{v}_{a+x}$, $0 \leq x \leq k$, we note that

$$\nabla^{s+1}v_{a+x} = \sum_0^s (-1)^t \binom{s+1}{t} v_{a+x-t} , \tag{7}$$

so that

$$\nabla^{s+1}\underline{v}_{a+x} = \sum_0^s (-1)^t \binom{s+1}{t} \underline{v}_{a+x-t} . \tag{8}$$

We also know that, $x \leq k$

$$\left.\begin{aligned} \underline{v}_{a+x-t} &= v_{a+x-t} && \text{when } t \leq x \\ \underline{v}_{a+x-t} &= 0 && \text{when } t > x \end{aligned}\right\} \tag{9}$$

so that

$$\nabla^{s+1}\underline{v}_{a+x} = \sum_0^x (-1)^t \binom{s+1}{t} v_{a+x-t}, \qquad 0 \leq x \leq s \tag{10}$$

$$\nabla^{s+1}\underline{v}_{a+x} = \sum_0^s (-1)^t \binom{s+1}{t} v_{a+x-t} = \nabla^{s+1} v_{a+x}, \qquad s < x \leq k. \tag{11}$$

The formula (6) can then be written

$$\sum_a^{a+k} u_x v_x = \sum_0^k u_{a+x} v_{a+x} = \sum_0^s C^{s+1} u_{a+x} \nabla^{s+1} \underline{v}_{a+x} + \sum_{s+1}^k C^{s+1} u_{a+x} \nabla^{s+1} v_{a+x}. \tag{12}$$

4. **Moments from the Cumulated Frequencies.** If $u_{a+x} = f_{a+x}$ and $v_{a+x} = (a + x)^s$, then (6) gives

$$\sum_0^k (a + x)^s f_{a+x} = \sum_0^k C^{s+1} f_{a+x} \nabla^{s+1} \underline{(a + x)}^s. \tag{13}$$

A more useful formula, obtained from (12), is

$$\sum_0^k (a + x)^s f_{a+x} = \sum_0^s C^{s+1} f_{a+x} \nabla^{s+1} \underline{(a + x)}^s, \tag{14}$$

since $\nabla^{s+1} (a + x)^s = 0$. We have then

THEOREM III. *The values of the s-th moments can be obtained from the last $s + 1$ entries of the $(s + 1)$st cumulation of the frequencies. The multipliers are the values*

$$\nabla^{s+1} \underline{(a + x)}^s = \sum_0^x (-1)^t \binom{s+1}{t} (a + x - t)^s. \tag{15}$$

Cor. 1. When $a = 0$, i.e., when the moments are measured about the smallest variate, the multipliers are

$$\nabla^{s+1} \underline{x}^s = \sum_0^x (-1)^t \binom{s+1}{t} (x - t)^s. \tag{16}$$

Cor. 2. When $a = 1$, the multipliers are

$$\nabla^{s+1} \underline{(1 + x)}^s = \sum_0^x (-1)^t \binom{s+1}{t} (1 + x - t)^s. \tag{17}$$

Cor. 3. If the moments are measured about a fixed value, p, then the new smallest variate is $a - p = a'$ and the multipliers are $\nabla^{s+1} \underline{(a' + x)^s}$.

Cor. 4. If p is the mean, m, then $a' = a - m$. If in addition $a = 0$, then $a' = -m$ and the multipliers giving moments about the mean are $\nabla^{s+1}\, \underline{(x - m)}^s$. Now

$$m = \frac{\sum_0^k x f_x}{\sum_0^k f_x} = \frac{C_1^2 \nabla^2 \underline{0} + C_2^2 \nabla^2 \underline{1}}{C_1^1} = \frac{C_2^2}{C_1^1}.$$

It follows that the multipliers giving the moments about the mean are

(18) $$\nabla^{s+1} \underline{\left(x - \frac{C_2^2}{C_1^1}\right)}^s.$$

It is to be noted that the moments about different points are obtained by applying different multipliers to the same cumulated frequencies.

5. **Values of the multipliers.** The values of the multipliers may be computed from (15). Thus $\nabla^3 \underline{(a + 1)}^2 = (a + 1)^2 - 3a^2 = -2a^2 + 2a + 1$. This becomes $2ab + 1$ when $1 - a$ is set equal to b. Values of the multipliers for the most common values of s and x are presented in Table III.

TABLE III

Values of $\nabla^{s+1}\, \underline{(a + x)}^s$

x \ s	0	1	2	3	4
4					b^4
3				b^3	$4b^3a + 6b^2 + 4b + 1$
2			b^2	$3b^2a + 3b + 1$	$6a^2b^2 + 12ab + 11$
1		b	$2ab + 1$	$3a^2b + 3a + 1$	$4a^3b + 6a^2 + 4a + 1$
0	1	a	a^2	a^3	a^4

When $a = 0, b = 1$ and the multipliers are 1; 0, 1, 1; 0, 1, 4, 1; 0, 1, 11, 11, 1; etc. as indicated in section 1. When $a = 1$, $b = 0$ and the multipliers are 1; 1, 0; 1, 4, 1, 0; 1, 11, 11, 1, 0; etc. When the moments are measured about a fixed point, p, it is only necessary to compute $a' = a - p$ and to use a' for a and $b' = 1 - a'$ for b in Table III.

We illustrate the use of the multipliers by application to the problem of Table I. The moments about the smallest variate are computed in section 1. The moments, when $a = 1$ are $\sum_0^6 (x + 1)f_x = C_1^2 = 3654$; $\sum_0^6 (x + 1)^2 f_x =$

$C_1^3 + C_2^3 = 19197$; $\sum_0^6 (x+1)^3 f_x = C_2^4 + 4C_3^4 + C_4^4 = 105381$; $\sum_0^6 (x+1)^4 f_x = C_2^5 + 11C_3^5 + 11C_4^5 + C_5^5 = 598509$.

The moments about the mean are found by forming $\frac{C_2^2}{C_1^1} = \frac{2916}{729} = 4$. Then $a = -4$ and the multipliers are 1; -4, 5; 16, -39, 25; -64, 229, -284, 125; 256, -1199, 2171, -1829, 625; etc. so that $\sum_0^6 \bar{x} f_x = 0$; $\sum_0^6 \bar{x}^2 f_x = 972$; $\sum_0^6 \bar{x}^3 f_x = -324$; $\sum_0^6 \bar{x}^4 f_x = 3564$.

Since the values of $\nabla^{s+1}(x - C_2^2/C_1^1)^s$ are expressible in terms of C_1^1 and C_2^2, it follows that the values of $\sum_0^k \bar{x}^s f_x$ are expressible in terms of cumulations. For example a formula for the second moment about the mean, which is essentially one given by Whittaker and Robinson [7, p. 193] is

$$\sum_a^{a+k} \bar{x}^2 f_x = C_2^2 + 2C_3^3 - \frac{(C_2^2)^2}{C_1^1}. \tag{19}$$

However the general method described above, supplemented with the techniques of succeeding sections, is preferred to the development and use of such formulas.

6. **Recursion Property of the Multipliers.** It is not readily apparent from Table III how the multipliers of the $(s+1)$-th cumulations can be obtained from the multipliers of the s-th cumulations. It is possible to establish a recursion formula which is useful for this purpose. Now, $a \le x \le s$,

$$\nabla^{s+1}\underline{(a+x)^s} = (a+x)^s + \sum_1^x (-1)^t \binom{s+1}{t} (a+x-t)^s$$

$$(a+x)\nabla^s\underline{(a+x)^{s-1}} = (a+x)^s + \sum_1^x (-1)^t \binom{s}{t} (a+x-t)^{s-1}(a+x)$$

$$(s+1-a-x)\nabla^s\underline{(a+x-1)^{s-1}} = \sum_1^x (-1)^{t-1} \binom{s}{t-1} (a+x-t)^{s-1}(s+1-a-x)$$

and since

$$\binom{s}{t}(a+x) - \binom{s}{t-1}(s+1-a-x) = \binom{s+1}{t}(a+x-t)$$

it follows that

$$\Delta^{s+1}\underline{(a+x)^s} = (a+x)\nabla^s\underline{(a+x)^{s-1}} + (s+1-a-x)\nabla^s\underline{(a+x-1)^{s-1}}. \tag{20}$$

When $a = 0$ we have

$$\nabla^{s+1}\underline{x}^{s} = x\nabla^{s}\underline{x}^{s-1} + (s + 1 - x)\nabla^{s}\underline{(x - 1)}^{s-1}. \tag{21}$$

Formulas (20) and (21), though somewhat formidable in appearance, are easy to apply. Thus $\nabla^3\underline{(a + 2)}^2 = (a + 2)\nabla^2\underline{(a + 2)} + (1 - a)\nabla^2\underline{(a + 1)}$. The recursion formula is especially useful in building up tables of multipliers. The following form is recommended:

As successive columnar headings use the values a, $a + 1$, $a + 2$, etc. and as successive row headings use $1 - a$, $2 - a$, $3 - a$, etc. Then $\nabla\underline{a}^0 = 1$ is placed in the upper left cell, $\nabla^2\underline{a}$ directly below $\nabla\underline{a}^0$, $\nabla^2\underline{a + 1}$ to the right of $\nabla^2\underline{a}^0$, etc. The values of $\nabla^3\underline{(a + x)}^2$ are placed in the next diagonal, etc. If this process is continued the entry $\nabla^{s+1}\underline{(a + x)}^s$ will have the entry $\nabla^s\underline{(a + x)}^{s-1}$ directly above it and the entry $\nabla^s\underline{(a + x - 1)}^{s-1}$ on its left. Also the columnar heading is $a + x$ and the row heading $s + 1 - a - x$ so that any entry is obtained by adding the product of the entry above it and the columnar heading to the product of the entry to the left and the row heading. The values of $\nabla^{s+1}\underline{x}^s$ are obtained by placing $a = 0$. They are presented, in Table IV, through $s = 8$.

TABLE IV

Values of $\nabla^{s+1}\underline{x}^s$

$s+1-x$ \ x	0	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	0	1	4	11	26	57	120	247	
3	0	1	11	66	302	1191	4293		
4	0	1	26	302	2416	15619			
5	0	1	57	1191	15619				
6	0	1	120	4293					
7	0	1	247						
8	0	1							

The table is easily extended to higher values of s. If a table of values of $\nabla^{s+1}\underline{(x + 1)}^s$ is constructed, it will be found to be like Table IV with columns and rows interchanged. Hence the values of $\nabla^{s+1}\underline{(x + 1)}^s$ are obtained from Table

IV by reading the multipliers down the diagonal. Thus the values $\nabla^3\underline{(x+1)}^2$ are 1, 4, 1, 0, etc.

The ease with which the multipliers may be computed is illustrated with $a = -4$. In this case we have

TABLE V

Values of $\nabla^{s+1}\,\underline{(x+a)}^s$ *with* $a = -4$

$s+1-a-x$ \ $a+x$	-4	-3	-2	-1	0
5	1	5	25	125	625
6	-4	-39	-284	-1829	
7	16	229	2171		
8	-64	-1199			
9	256				

These values agree with those computed more laboriously in section 5.

7. **Value of** $\sum_0^k \nabla^{s+1}\underline{(x+a)}^s$. It is to be noted in Tables III, IV, V that the sum of the entries in the diagonal having $s+1$ terms is $s!$ This is generally true and results from the fact that

$$\sum_0^k \nabla^{s+1}\underline{(x+a)}^s = \sum_0^s \nabla^{s+1}\underline{(x+a)}^s = s! \tag{22}$$

In obtaining the values of $\sum_0^k \nabla^{s+2}\underline{(x+a)}^{s+1}$ from the value of $\sum_0^k \nabla^{s+1}\underline{(x+a)}^s$ it is noted that $\nabla^{s+1}\underline{(x+a)}^s$ is used but twice. Once it is multiplied by $a+x$ and once by $s+1-a-x$ so that the net result is a multiplication by $s+1$. It follows that $\sum_0^k \nabla^{s+2}\underline{(x+a)}^{s+1} = (s+1)\sum_0^k \nabla^{s+1}\underline{(x+a)}^s$ and since $\sum_0^k \nabla^2\underline{(x+1)} = 1$, $\sum_0^k \nabla^3\underline{(x+a)}^2 = 2!$ so that in general $\sum_0^k \nabla^{s+1}\underline{(x+a)}^s = s!$

This property is useful in checking the values of the computed multipliers.

8. **The adaptation of the Charlier check.** An adaptation of the Charlier check serves as an excellent check for the computed moments. It is recalled that the Charlier check gives

$$\sum_a^{a+k} (x+1)^s f_x = \sum_{t=0}^{s} \binom{s}{t} \sum_{x=a}^{a+k} x^{s-t} f_x. \tag{23}$$

The components of the right hand member are computed by cumulative totals as indicated above. The left hand member is obtained by applying different multipliers to the same cumulated frequencies. Thus $\sum_a^{a+k} (x+1)^s f_x = \sum_0^k (x + a + 1)^s f_{x+a}$ and the multipliers of the cumulated frequencies are $\nabla^{s+1}\underline{(x+a')}^s$ where $a' = a + 1$. If $a = 0$ the Charlier check multipliers are the values $\nabla^{s+1}\underline{(x+1)}^s$ which can be read from Table IV. For example $\sum_0^6 (x+1)^4 f_x = C_1^5 + 11C_2^5 + 11C_3^5 + C_4^5 = 598509$ and this checks with $\sum_0^6 x^4 f_x + 4 \sum_0^6 x^3 f_x + 6 \sum_0^6 x^2 f_x + 4 \sum_0^6 x f_x + \sum_0^6 f_x$.

9. **Application to factorial moments.** When $u_x = f_x$, $v_x = x^{(s)} = x(x-1)(x-2)\cdots(x-s+1)$

$$\sum_0^k x^{(s)} f_x = \sum_0^k C^{s+1} f_x \nabla^{s+1} \underline{x}^{(s)}$$

and since $\nabla^{s+1}\underline{x}^{(s)}$ is 0 when $s < x \le k$, is $s!$ when $s = x$, is 0 when $0 \le x < s$,

$$\sum_0^k x^{(s)} f_x = \sum_s^k x^{(s)} f_x = s!\, C_{s+1}^{s+1}. \tag{24}$$

It follows that the underscored terms of Table I, when multiplied by $s!$, give the factorial moments. Factorial moments, first used by Sheppard [4], have since come into prominence largely because of this ease of computation.

The coefficients of $(a+b)^x$ are $1, x, \frac{x(x-1)}{2!}, \cdots, \frac{x^{(s)}}{s!}, \cdots$. If we define the binomial moment by $B_s = \sum_0^k \frac{x^{(s)}}{s!} f_x$ [6, p. 278] then $B_s = \frac{1}{s!} \sum_0^k x^{(s)} f_x = C_{s+1}^{s+1}$.

It is also possible to show that the entries under the main diagonal are binomial moments. In Table I, for example, we let $a = 1$ and add the additional row $a = 0$ with 0 frequency. Then $C_1^1 = 729$, $C_2^1 = 729$, $C_1^2 = 729 + 3645 = 4374$, etc. The new diagonal terms are directly under the old diagonal terms and give $B_{s,1} = \sum_1^7 x^{(s)} f_x = \sum_0^6 (x+1)^{(s)} f_x$. In general the terms $B_{s,l}$ are given l rows below the terms B_s and the factorial moments are $s!\, B_{s,l}$. Then

$$F_{s,l} = s!\, C_{s+1-l}^{s+1}. \tag{25}$$

For example in the problem of Table I, $F_{4,3} = \sum_3^9 x^{(4)} f_x = 4! C_2^5 = 782{,}784$. The method is especially adapted to the use of Hollerith machines, for positive integral values of l, since it is only necessary to have the machine continue its cumulation.

10. The cumulations of xf_x. It is possible to use the cumulations of xf_x in securing the values of the moments. Now

$$\sum_a^{a+k} x^{s+1} f_x = \sum_0^k (x+a)^{s+1} f_{x+a} = \sum_0^k (x+a) f_{x+a} (x+a)^s \tag{26}$$

$$= \sum_0^s C^{s+1}(x+a) f_{x+a} \nabla^{s+1} \underline{(x+a)}^s.$$

When $a = 0$, (26) becomes

$$\sum_0^k x^{s+1} f_x = \sum_0^s C^{s+1} x f_x \nabla^{s+1} \underline{x}^s. \tag{27}$$

We compute the cumulations of xf for the problem of Table I. These are given in Table VI.

TABLE VI

Cumulations of xf_x

x	f_x	xf_x	C^1	C^2	C^3	C^4
6	64	384	384	384	384	384
5	192	960	1344	1728	2112	2496
4	240	960	2304	4032	6144	8640
3	160	480	2784	6816	12960	21600
2	60	120	2904	9720	22680	44280
1	12	12	2916	12636	35316	79596
0	1	1	2916	15552	50868	130464

so that

$$\sum_0^6 xf_x = 2916; \qquad \sum_0^6 x^2 f_x = 12636; \qquad \sum_0^6 x^3 f_x = 35316 + 22680 = 57996;$$

$$\sum_0^6 x^4 f_x = 79596 + 4(44280) + 21600 = 278316.$$

In getting moments about the mean from the cumulations of xf_x, the following method is recommended.

$$\sum_0^k \bar{x}^{s+1} f_x = \sum_0^k \bar{x}^s (x - m) f_x = \sum_0^k \bar{x}^s x f_x - m \sum_0^k \bar{x}^s f_x. \tag{28}$$

and

$$\sum_0^k \bar{x}^s x f_x = \sum_0^k C^{s+1}(xf_x) \nabla^{s+1} (x - m)^s. \tag{29}$$

When $s = 1$, (28) gives $\sum_0^k \bar{x}^2 f_x = \sum_0^k \bar{x} x f_x - m \sum_0^k \bar{x} f_x$ and

$$\sum_0^k \bar{x}^2 f_x = \sum_0^k \bar{x} x f_x. \tag{30}$$

In the illustrative problem $a = -4$ so that

$$\sum_0^6 \bar{x} x f_x = -4(15552) + 5(12636) = 972$$

$$\sum_0^6 \bar{x}^2 x f_x = 16(50868) - 39(35316) + 25(22680) = 3564$$

$$\sum_0^6 \bar{x}^3 x f_x = 2268$$

and

$$\sum_0^6 \bar{x}^2 f_x = 972; \qquad \sum \bar{x}^3 f_x = 3564 - 4(972) = -324; \qquad \sum \bar{x}^4 f_x = 3564.$$

Formula (30) is of note since it permits the determination of $\sum_0^k \bar{x}^2 f_x$ directly from the cumulations of xf_x.

The factorial moments are also related to the cumulations of xf_x. Thus

$$\sum_0^k x^{(s)} f_x = \sum_0^k (x-1)^{(s-1)} x f_x = \sum_0^k C^s(xf_x) \nabla^s \underline{(x-1)^{(s-1)}} \tag{31}$$

which results in $\sum_0^k x^{(s)} f_x = (s-1)! C_s^s(xf_x)$.

It follows that

$$C_s^s(xf_x) = sC_{s+1}^{s+1}(f_x).$$

For example, the underscored terms of Table VI are respectively 1, 2, 3, 4 times underscored terms of Table I.

In general the cumulations of xf_x, rather than of f_x, are recommended since $C(xf_x)$ can be computed and recorded almost as quickly as $C(f_x)$, since one less cumulation is needed to obtain a specific moment, and since the multipliers needed to get a specific moment are smaller. A technique based on the cumulations of xf_x is especially adapted to the use of Hollerith machines. Let us take x_x to represent the sum of the x's for all items in the distribution having the same value of x. Then $xf_x = x_x$ and we have

$$\sum_a^{a+k} x^s f_x = \sum_a^{a+k} x^{s-1} x_x = \sum_a^{a+k} C^s(x_x) \nabla^s \underline{(x^{s-1})}. \tag{32}$$

If the cards are sorted for x and the tabulator is wired to print cumulative totals each time x changes, the recording tape gives the successive values of $C(x_x)$. (Care must be taken that there are no blank values of x.)

If a summary punch is available, these cumulations are punched on cards as they are cumulated and these summary cards are used in getting higher cumulations.

If no summary punch is available, it is possible to obtain $\sum x^2 f_x$ by the application of Theorem I. Thus

$$\sum_a^{a+k} x^2 f_x = \sum_a^{a+k} x x_x = \sum_a^{a+k} C(x_x)\nabla(\underline{x}),$$

and since $\nabla(\underline{x}) = a$ when $x = a$ and $\nabla(\underline{x}) = 1$ when $x > a$, it follows that $\sum_a^{a+k} x^2 f_x$ can be obtained by adding the entries above the last and then adding the last entry multiplied by a. This is essentially the Mendenhall-Warren-Hollerith method of getting $\sum x^2 f_x$ [9, p. 27].

In case $a = 0$ the technique amounts simply to adding all the entries above the bottom one.

The value $\sum x^3 f_x$ can be obtained similarly from the first order cumulations. Thus

$$\sum_a^{a+k} x^3 f_x = \sum_a^{a+k} x^2 x_x = \sum_a^{a+k} C(x_x)\nabla(\underline{x^2}) \tag{33}$$

and since $\nabla(\underline{x^2}) = a^2$ when $x = a$, $\nabla(\underline{x^2}) = 2x - 1$ when $x > a$, it follows that

$$\sum_a^{a+k} x^3 f_x = a^2 C_1^1(x_x) + \sum_{a+1}^{a+k} C(x_x)(2x - 1). \tag{34}$$

When $a = 0$, (34) becomes

$$\sum_0^k x^3 f_x = \sum_1^k C(x_x)(2x - 1) \tag{35}$$

so that the multipliers are the successive odd integers. Thus from the first order cumulations of Table VI we have

$$\sum_0^6 x f_x = 2916; \qquad \sum_0^6 x^2 f_x = 12636; \qquad \sum_0^6 x^3 f_x = 57996.$$

The cumulative method can also be applied to the method of digiting [17, p. 425].

It is also possible to obtain the moments from the cumulations of $x^2 f_x$, $x^3 f_x$, etc., since

$$\sum_a^{a+k} x^{s+2} f_x = \sum_a^{a+k} x^s x^2 f_x = \sum_a^{a+k} C^{s+1}(x^2 f_x)\nabla^{s+1}(\underline{x^s})$$

$$\sum_a^{a+k} x^{s+3} f_x = \sum_a^{a+k} x^s x^3 f_x = \sum_a^{a+k} C^{s+1}(x^3 f_x)\nabla^{s+1}(\underline{x^s})$$

but the cumulations of $x f_x$ are preferable for most purposes. The Charlier check works in all cases. It should be noted that the indicated Hollerith technique demands only the customary tabulator and not the expensive, time consuming, card punching, multiplier, [16].

11. **Product Moments. Correlation.** It is possible to apply the cumulative technique in getting product moments involving two variables. If we let y_x be the sum of all the values of y having the same value of x, then

$$\sum x^s y f_{xy} = \sum_{a}^{a+k} y_x x^s = \sum_{a}^{a+s} C^{s+1}(y_x)\nabla^{s+1}(\underline{x^s}) \tag{36}$$

so that the multipliers are the same as those previously used. When Hollerith machines are used, it is only necessary to sort the cards for x and to wire the machine to give cumulations on variables x, y, z, etc. If the machine is adjusted to take totals with each change in x, the tape records simultaneously the values of $C(x_x)$, $C(y_x)$, $C(z_x)$, etc. With a summary punch it is possible to form successive cumulations easily. The values Σx^{s+1}, $\Sigma x^s y$, $\Sigma x^s z$, etc. are then obtained by applying the multipliers. When $s = 1$, (36) becomes

$$\sum xy f_{xy} = \sum_{a}^{a+k} C^2(y_x)\nabla^2(\underline{x}) \tag{37}$$

so that the multipliers are a, $1 - a$, 0, 0, etc. When $a = 0$, the multipliers are 0, 1, 0, 0, etc. and when $a = 1$, they are 1, 0, 0, etc.

When no summary punch is available, it is necessary to obtain the values of the moments from the first order cumulations. Using Theorem I

$$\sum xy f_{xy} = \sum_{a}^{a+k} C(y_x)\nabla(\underline{x}) = aC_1^1(y_x) + \sum_{a+1}^{a+s} C(y_x). \tag{38}$$

This formula serves as the basis of the Mendenhall-Warren-Hollerith Correlation Method, [9, p. 27].

It can be shown in similar fashion that

$$\sum x^2 y f_{xy} = a^2 C_1^1 + \sum_{a+1}^{a+s} C(y_x)(2x - 1) \tag{39}$$

and when $a = 0$

$$\sum x^2 y f_{xy} = \sum_{1}^{s} C(y_x)(2x - 1). \tag{40}$$

The method is also adapted to the common problem of finding correlation coefficients from grouped data when Hollerith machines are not available and this method is recommended for the determination of these coefficients.

An illustration is presented in Table VII which shows the correlation existing between college first semester average, X, and preparatory school average, Y, for 1126 students entering the College of Literature, Science and the Arts of the University of Michigan in 1928. The coded values of X and Y are indicated by x and y and are positive integers beginning with 0. The coded values are given in descending order beginning with the upper left hand corner of the chart. The values of the cumulations are placed at the right hand side and at the bottom of the chart.

TABLE VII

Correlation with cumulative totals

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
$Y \backslash X$			4.00	3.99 3.50–	3.49 3.00–	2.99 2.50–	2.49 2.00–	1.99 1.50–	1.49 1.00–	.99 .50–	.49 .00–		
	$y \backslash x$		8	7	6	5	4	3	2	1	0		
		$f_y \backslash f_x$	13	50	107	220	341	179	121	60	35	Cx_y	Cy_y
4.00	6	18	5	2	5	5	1					113	108
3.99 3.50–	5	106	2	19	29	27	20	7		1	1	673	638
3.49 3.00–	4	178	3	12	35	53	44	18	6	5	2	1503	1350
2.99 2.50–	3	270	3	10	20	55	103	33	27	11	8	2568	2160
2.49 2.00–	2	330		6	11	54	114	67	46	19	13	3714	2820
1.99 1.50–	1	173		1	5	19	45	44	34	18	7	4244	2993
1.49 1.00–	0	51			2	7	14	10	8	6	4	4399	2993
		Cy_x	61	259	661	1330	2194	2578	2809	2923	2993	12815	10069
		Cx_x	104	454	1096	2196	3560	4097	4339	4399	4399	20245	1126

The lower right hand corner has the entries

$$\left.\begin{matrix} & \sum x & \sum y \\ \sum y & \sum xy & \sum y^2 \\ \sum x & \sum x^2 & \sum f = N \end{matrix}\right\}$$

where $\sum xy$, $\sum y$, and $\sum x$ are obtained by adding the cumulations in the columns or rows involved.

The values $C(y_y)$ are easily computed from columns (2) and (3). The values of $C(x_y)$ are computed by forming the cumulated product of the row frequency and x. The values are recorded when the products contributed by a given row have been computed. The values $C(y_x)$ and $C(x_x)$ are obtained similarly.

The value of r is easily obtained from the lower right hand entries. The value $A_{x,y} = N\Sigma xy - (\Sigma x)(\Sigma y)$ is obtained from diagonal entries, $A_{x,x} = N\Sigma x^2 - (\Sigma x)^2$

is obtained from entries in the last row, $A_{y,y} = N\Sigma y^2 - (\Sigma y)^2$ is obtained from the last column, and $r = \dfrac{A_{x,y}}{\sqrt{A_{x,x}A_{y,y}}}$ is easily computed. In the above problem $r = .441$.

The values M_x, M_y, σ_x, σ_y are also easily obtained from the lower right hand entries. The successive steps are indicated by the form

	Σx	Σy			M_y
Σy	Σxy	Σy^2			
Σx	Σx^2	N	$A_{x,x}$		
		$A_{y,y}$	$A_{x,y}$	$\sqrt{A_{y,y}}$	σ_y
			$\sqrt{A_{x,x}}$	$\sqrt{A_{x,x}A_{y,y}}$	
M_x			σ_x		r

Recent methods of applying cumulative totals theory to correlation are found in references [9], [14], [17], [18], [19].

The third order moments are obtained by multiplying the entries of $C(x_y)$, $C(y_y)$, $C(x_x)$, $C(y_x)$ by 1, 3, 5, etc. as indicated by (40). Thus $\Sigma x^3 f_x = 4399 + 3(4339) +$ etc. $= 102, 103$; $\Sigma x^2 y f_{xy} = 63121$; $\Sigma xy^2 f_{xy} = 46047$; $\Sigma y^3 f_y = 38,633$. It is hence possible to compute the skewness of each marginal distribution from Table VII. See also [18, p. 657].

12. **Conclusion.** This paper presents an outline of the computation of moments with the use of cumulative totals and columnar multipliers. Basic general theorems are derived and applications are made to one variable and two variable distributions both with and without punched card equipment. The formulas assume that the distance between successive variates (or class marks) is unity, but the reader should have no trouble in adapting the formulas to more general problems.

In the interest of brevity the development is limited to the descending cumulations. It is possible to parallel the development here by deriving formulas in terms of ascending cumulations. It is also possible to work out formulas showing relations between columnar, row, and diagonal multipliers. There are other applications such as to the evaluation of $\sum_{1}^{\kappa} x^s$, which are of interest. It is possible also that applications may be found for the general theory of sections 2 and 3 which do not demand that v_x be a power function.

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