ABSTRACTS OF PAPERS

(Presented on December 27, 1938, at the Detroit meeting of the Institute)

Generalizations of the Laplace-Liapounoff Theorem. W. G. MADOW, Milbank Management Corporation, New York.

The Laplace-Liapounoff Theorem states conditions under which a linear function of chance variables has a normal limiting distribution.

In dealing with limiting distributions arising in the analysis of variance, regression analysis, etc., there occurred problems which required for their solution the derivation of the joint limiting distribution of several linear functions of chance variables and the joint limiting distribution of functions which were linear in one set of chance variables for fixed values of other sets of chance variables.

These problems were solved by a matrix formulation of the Laplace-Liapounoff Theorem and by the introduction of a function whose convergence to zero in probability provided a sufficient condition for the existence of normal limiting distributions.

Various generalizations with a view towards applications in multi-variate statistical analyses are discussed. The theorems provide a rigorous and complete basis for the derivation of limiting distributions of quadratic and bilinear forms.

The Standard Errors of the Geometric and Harmonic Means. NILAN NORRIS, Hunter College.

Although certain properties of the geometric and harmonic means have been investigated extensively, there seems to have been no derivation of expressions for their variances in cases where they are used as estimates of parameters of parent populations.

Application of the modern theory of estimation makes it possible to develop simple and useful formulae for the standard errors of these two averages for each of the respective general classes of cases in which they are most suitable.

As in other instances in which standard errors are used in tests of significance, fiducial or confidence limits may be employed to overcome certain limitations of the outmoded practice of relying solely on multiples of either probable or standard errors to determine whether or not a result exists merely because of sampling fluctuations.

Note on an Integral Equation in Population Analysis. ALFRED J. LOTKA, Metropolitan Life Insurance Company, New York.

In a population in which immigration and emigration are negligible, the number $N(t)$ of the population at time t is connected with the annual births $B(t)$ and the probability $p(a)$ of surviving from birth to age a, by the obvious relation

$$N(t) = \int_0^\infty B(t - a)p(a)\, da. \tag{1}$$

If $B(t)$ and $p(a)$ are given, $N(t)$ follows at once by direct integration. The inverse problem, given $N(t)$, to find $B(t)$, requires separate treatment. The case that $N(t)$ is given or can be expressed as a sum of exponential functions has been discussed by the

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author on a former occasion. In the present communication it is shown how the function $B(t)$ can be expressed as a series proceeding in ascending derivatives of $N(t)$.

A second solution is also offered in which

$$\frac{B(t)}{N(t)} = b(t),$$

the birth rate per head is obtained as a series, the first and dominating term of which is

$$b(t) = \frac{1}{\int_0^\infty e^{-r_t a} p(a)\, da} \tag{2}$$

where r_t is the rate of natural increase at time t. This development is of interest because it corresponds to the expression for b in a population with constant birth rate, death rate, and rate of natural increase; that is

$$b = \frac{1}{\int_0^\infty e^{-ra} p(a)\, da} \tag{3}$$

so that the new expression represents $b(t)$ as the corresponding value of b in a Malthusian population, plus a series of correcting terms.

Optimum Fiducial Regions for Simultaneous Estimation of Several Population Parameters from Large Samples. S. S. Wilks, Princeton University.

If a population has a distribution law $f(x, \theta)$ where x is the variate and θ is a parameter, it is known (*Annals of Mathematical Statistics*, Vol. IX (1938) pp. 166–175) that under rather general conditions, confidence intervals, for a given confidence coefficient α, which are shortest on the average, can be obtained from large samples of n items by solving the equations

$$\frac{\frac{\partial L}{\partial \theta}}{\sqrt{n}\sqrt{E\left[\frac{\partial \log f}{\partial \theta}\right]^2}} = \pm d_\alpha \tag{1}$$

for θ, where d_α is the normal deviate given by $\frac{1}{\sqrt{2\pi}} \int_{-d_\alpha}^{+d_\alpha} e^{-\frac{1}{2}t^2}\, dt = \alpha$. L is the logarithm of the likelihood, i.e. $L = \sum_{i=1}^{n} \log f(x_i, \theta)$, where E denotes mean value with respect to the probability law $f(x, \theta)$.

The present paper is an extension of the foregoing results to the case of several parameters. It is shown under fairly general conditions that if the distribution law of x if a function $f(x, \theta_1, \cdots, \theta_h)$ depending on h parameters, then for a confidence coefficient α the fiducial region of the θ's which is smallest in size *on the average* is given by the region in the space of the θ's for which

$$\frac{1}{n} \sum_{i,j=1}^{h} a_{ij} \left(\frac{\partial L}{\partial \theta_i}\right)\left(\frac{\partial L}{\partial \theta_j}\right) \leq \chi_\alpha^2 \tag{2}$$

where χ_α^2 is such that $P(\chi^2 \leq \chi_\alpha^2) = \alpha$, where the probability is calculated from a χ^2 distribution with h degrees of freedom. The matrix $\| a_{ij} \|$ is the inverse of the matrix whose general element is

$$E\left[\frac{\partial \log f}{\partial \theta_i} \cdot \frac{\partial \log f}{\partial \theta_j}\right]$$

Similar results hold when f is a function of several random variables as well as the parameters $\theta_1, \theta_2, \cdots, \theta_h$.

A Mathematical Contribution to Immigration Assessment. CHURCHILL EISENHART, University of Wisconsin.

A certain problem is assessing the size of an immigration can be stated mathematically as follows: A sample of size N is drawn at random from a population in which the probability of A is p. Let "not-A" be denoted by B. Then the sample will contain a frequency, say x, of A and $N - x$ of B, x being a random variable. This sample is now mixed together with a very much larger sample in which the elements are all B's, and the B's belonging to the original sample lost sight of. The problem is to estimate N from the observed frequency of A, namely x, in the composite sample, p being assumed known. The maximum likelihood estimate of N is x/p. For large values of x, and a fortiori of N, confidence intervals for N take the form $N_1 \leq N \leq N_2$ where

$$N_1 = \frac{\{\sqrt{q^2t^2 + 4q(x - \frac{1}{2})} - qt\}^2}{4pq},$$

$$N_2 = \frac{\{\sqrt{q^2t^2 + 4q(x + \frac{1}{2})} + qt\}^2}{4pq},$$

$$q = 1 - p,$$

and the confidence coefficient is .95 if t is set equal to 1.96, and is .99 if t is set equal to 2.58. For small values of x the solution is more difficult but charts are being prepared from which the confidence intervals can be read off.

A Contribution to the Theory of Statistical Estimation. A. WALD, Columbia University.

Let us denote by $f(x, \theta)$ a probability density function, where θ is a parameter. Denote by Ω the set of all possible values of θ. The assumption that θ belongs to a subset ω of Ω is called a hypothesis. Let us consider a system S of subsets of Ω. Denote the hypothesis corresponding to an element ω of S by H_ω and the system of all hypotheses corresponding to the elements of S by H_S. Denote by E a sample point in the n-dimensional sample space drawn from a population with the probability density function $f(x, \theta)$, where the value of θ is unknown. We have to decide by means of the sample point E which hypothesis of the system H_S should be accepted. That is to say, for each hypothesis H_ω we have to choose a region of acceptance M_ω in the sample space. The hypothesis H_ω will be accepted if and only if the sample point E falls in the region M_ω. Denote by M_S the system of all regions M_ω. The statistical problem to be solved is the question of *how the system of regions M_S should be chosen?*

In order to answer this question, a non-negative weight function $w(\theta, \omega)$ is introduced, which is defined for all values θ and for all elements ω of S. The weight $w(\theta, \omega)$ expresses the loss caused by accepting H_ω if θ is true. The probability of accepting H_ω multiplied by the weight $w(\theta, \omega)$ is called the risk of accepting H_ω if θ is true. Denote this risk by

$r(\theta, H_\omega, M_S)$ (the risk depends obviously also on the system of regions M_S). The total risk of accepting a false hypothesis if θ is true, is given by

$$r(\theta, M_S) = \sum_{\omega} r(\theta, H_\omega, M_S)$$

where the summation is to be taken over all elements ω of S which do not contain θ.

Denote by $r(M_S)$ the maximum of $r(\theta, M_S)$ with respect to θ. The system M_S of regions for which $r(M_S)$ becomes a minimum is called the "best" system of regions relative to the weight function $w(\theta, \omega)$. Some properties of the best system of regions have been studied and the problem of its calculation has been treated.

On the Hypotheses underlying the Applications of Statistical Methods to Routine Laboratory Analyses. J. Neyman, University of California.

The problem considered is that of estimating the proportion, p, of certain designated elements of the population sampled, the estimate to be based on a random sample of n, drawn by some mechanical device, such as is sometimes used in industry and in laboratory work. Examples: (1) to estimate the proportion of defective manufactured products in mass production; (2) to estimate the proportion of seeds which are able to germinate in given conditions. One would expect that the sample proportion, say q, will be distributed in repeated samples according to the Binomial Law and that, consequently, in order to obtain the confidence limits for p, one should use the Clopper-Pearson graphs. However, the evidence obtained from special analyses on seed germination, made in the Seed Testing Station of Warsaw, Poland, shows that this assumption may not be true. The sampling there was carried out by means of a machine and involved a certain amount of mixing. As a result q was more stable than it was expected. It did not follow the Binomial Law at all, but a Normal one about p, with a standard deviation, σ, which could be well estimated from the sum of squares of deviations from respective means. For a considerable period of time (18 months) σ retained a constant value (a characteristic of the action of the sampling machine) which was rather smaller than $(n^{-1}q(1 - q))^{\frac{1}{2}}$.

Consequently, to have a preassigned frequency of correct statements concerning p, it was necessary to calculate the confidence intervals according to the formulae of the Normal Theory

$$q - t\sigma < p < q + t\sigma$$

with an appropriate value of t. Probably similar situations are rather common.

Remarks on Two Methods of Sampling Inspection. E. G. Olds, Carnegie Institute of Technology.

When the instructions for inspecting lots of size m specify that samples of size n be taken and the lot be passed without detailed inspection if no defectives are found, then, on the average, the maximum number of defectives are passed when the number of defects per lot is $\frac{m+1}{n+1}$ or $\frac{m+1}{n+1} - 1$.

If the quality of a lot is to be checked by drawing pieces until a fixed number of defective pieces are found, it is important to know that the expected number n_i necessary to obtain i defects is $i\frac{m+1}{p+1}$, where there are p defects in the lot. If $\frac{n_i}{i(m+1)}$ is used as an estimate of $\frac{1}{p+1}$, it is convenient to observe that the variance of $\frac{n_i}{i(m+1)}$ is

$$\frac{1}{p+2}\left[\frac{1}{p+1} - \frac{1}{m+1}\right]\left[\frac{1}{i} - \frac{1}{p+1}\right].$$

Commodity Transformations and Matrices. HAROLD HOTELLING, Columbia University.

If we regard the prices and quantities of n commodities as vectors we may apply the theory of linear transformations in various ways having economic and statistical significance. An example is the mixing of grains to produce results conforming to new specifications, as in international trade. Another kind of example arises in problems of multivariate statistical analysis such as those treated in my paper on "Relations between Two Sets of Variates" (*Biometrika*, 1936), concerned with properties invariant under internal linear transformations of the variates of each set. Prices transform contragrediently to quantities in all cases. Hence, if the prices and quantities of the same set of commodities are the two sets of variates, the allowable transformations are restricted. Consequently there are invariants in this case additional to those discussed in the paper mentioned. Another problem is the reduction of sets of linear demand functions to normal form and determination of invariants when transformations of prices must be contragredient to those of commodities. The question whether the demand functions are symmetrical is here of paramount importance, since symmetry is preserved by such transformations, and since there are known theoretical reasons to expect symmetry. For a non-singular set of linear symmetrical demand *or* supply functions there are no invariants under arbitrary sets of contragredient transformations; but for pairs of such sets of demand *and* supply functions there are invariants, namely the elementary divisors of the pair of matrices of coefficients. A set of demand or supply functions alone has invariants if its matrix B is not symmetrical. If B' denote the transverse or conjugate of B, the elementary divisors of $B + \lambda B'$ are such invariants.

The Future of Statistics in Mass Production.[1] WALTER A. SHEWHART, Bell Telephone Laboratories, New York.

Much has been written about the application of statistical theory and technique in studying, discovering, and measuring the effects of an existing system of unknown or chance causes. Much remains to be written about the application of statistical theory and technique in finding out how to tinker with and modify an existing chance cause system until it behaves as we would have it do. In research, we use statistical theory in helping to predict the future effects of some existing cause system. The statistician knows that his predictions will be valid if certain assumptions about the cause system are justified. Perhaps the most important assumption of this type is that the particular effects of a chance cause system under study are random. In mass production, however, the statistician has learned by experience that chance cause systems producing random effects don't just happen even under what we customarily consider to be the best regulated laboratory conditions. If the industrial statistician chooses to ignore this fact and makes predictions as if he were dealing with random cause systems, he may expect many of his predictions to fall far short of the truth: what is more, he knows that this fact will be discovered and his work discredited because in a continuing mass production process predictions are sure to be checked. Hence the industrial statistician in mass production must start not where the research statistician leaves off but, as it were, before the research statistician begins: that is, he must start by developing techniques for determining when we are justified in assuming that the underlying cause system is random. We thus arrive at a good starting point from which to consider the future of statistics in mass production.

Experience in the control of quality has provided a practical technique for detecting

[1] Summary of an address delivered at a luncheon meeting of the Institute of Mathematical Statistics, Detroit, Michigan, December 27, 1938.

and eliminating assignable causes of variability in the production process until a state of statistical control is reached where predictions based upon the assumption of randomness are likely to prove valid. It has also been shown elsewhere that by elimination of assignable causes of variability, we may make the most efficient use of raw materials, maximize the assurance of maintaining standard quality of manufactured goods, minimize the cost of inspection, and minimize the cost of rejections. Hence we may conclude that the use of statistics in mass production can be made to pay good dividends: such use can be made to have a bright future. On what then does that future depend?

To answer this question, we must consider the following three fundamental steps in the process of mass production:

I. The specification of the quality of the thing wanted.

II. The production of things designed to meet the specification.

III. The inspection of the things produced to see if they meet the specification.

An outstanding characteristic of the first step, specification, is the necessity of setting up and living within what we term a tolerance range[2] for each specified quality characteristic. If a producer contracts to live within some specified range and in taking steps II and III fails to do so, he usually loses a lot of money. Hence he must know how to set tolerance limits that he can meet. Moreover, if he is to be able to make the most efficient use of materials in many instances, he must close up as much as he feasibly can on the specified tolerance range.

Obviously, however, one can not specify a practically attainable tolerance range out of thin air: one must be limited by what it is possible to do under commercial conditions of production in step II and this in turn is revealed by inspection in step III. We must also take into account the fact previously noted that any manufacturing process to begin with is almost certain not to be in a state of statistical control. In fact, this state can only be approached through the application of certain statistical techniques that have been found useful in detecting the presence of assignable causes that can be found and removed. A point to be stressed is that the three steps—specification, production, and inspection—in mass production, cannot be taken independently: instead, they must be coördinated. It also may be shown that maximum effectiveness in the use of statistical theory can only be attained by coördinating the applications in each of the three steps. It is significant to note that in order to attain the most efficient use of materials and processes by minimizing the tolerance range and in order to minimize the cost of production, one must make effective use of the results obtained in the course of commercial production, particularly those in the third step, inspection. In fact, the three steps might be thought of as constituting a scientific experiment in which the objective is the attainment of the most efficient use of available materials in the production of manufactured goods.

Broadly speaking, the statistician of the future has before him the opportunity of helping to develop this fundamental type of experiment in many respects like the way he is successfully helping today in so many fields of research to design experimental procedures that make the most efficient use of human effort. Certain differences, of course, exist. For example, as already noted, he must start by designing a statistical control technique for randomizing, as it were, the cause system through the elimination of assignable causes. Then he can use modern statistical techniques of research in much the same way described in the literature with reasonable assurance that resulting predictions will be found valid because he has first randomized his cause system. He must, however, go farther than indicated in the current literature of statistical research in that he must provide operationally verifiable meanings for statistical terms such as random variable, accuracy, pre-

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[2] The tolerance range is not to be confused with the fiducial range of modern statistics. The distinction between the two is set forth at some length in a forthcoming publication, *Statistical Method from the Viewpoint of Quality Control*, to be published shortly by the Graduate School of the United States Department of Agriculture.

cision, true value, probability, degree of rational belief, and the like.[3] This is particularly necessary in the steps of specification and inspection because the specification is often made the basis of a contractual agreement between producers and consumers.

There is a sense in which the statistician's problem in helping to develop the mass production process so as to make the most effective use of information yielded by the process is much more complicated than the design of experiment usually considered in the literature of statistics. Whereas the customary statistical theory of design of experiment in research is concerned with comparatively small-scale experiments carried out under controlled conditions of the laboratory by a few people, the corresponding development of the mass production process must be carried out under commercial conditions on a large scale involving large numbers of people. For example, the three steps in the mass production process are usually carried out either by different companies or by different departments of the same company. The steps may involve the coördinated effort of literally hundreds and even thousands of employees, including physicists, chemists, engineers, sales agents, purchasing agents, lawyers, and economists. Very few of these people have ever had any training in statistics or probability and yet many of them must be sold on the use of statistics if the statistician is to have an opportunity of making his full contribution to the social and economic effectiveness of the mass production process. This situation constitutes a problem not only for those now in industry but also for those responsible for training the industrial leaders of tomorrow so that they will have sufficient knowledge of statistics to help them recognize the potential contributions of statistical theory and technique.

In conclusion, then, we may say that in the future the statistician in mass production must do more than simply study, discover, and measure the effects of existing chance cause systems: he must devise means for modifying these cause systems in the best way to satisfy human wants. The statistician in mass production must not be satisfied with simply measuring demand for goods; he must help change that demand by showing, among other things, how to close up the tolerance range and improve the quality of goods. He must not be content with measuring production costs; he must help decrease production costs through the use of the techniques of statistical control.

The future contribution of statistics in mass production lies not so much in solving the problems usually put to the statistician today by those not statistically trained as in taking a hand in helping to coördinate the steps of specification, production, and inspection considered as a scientific experiment for making the most efficient use of human effort in the production of goods to satisfy human wants. The long range contribution of statistics depends perhaps not so much upon getting a lot of highly trained statisticians into industry in the immediate future as it does in creating a statistically minded new generation of those physicists, chemists, engineers, and others who will in any way have a hand in developing and directing the mass production process of tomorrow.

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[3] An initial step in this direction has been taken in my Washington lectures. Loc. cit.