ON AN INTEGRAL EQUATION IN POPULATION ANALYSIS

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I

A fundamental equation in population analysis rests on the following considerations: Of the persons born a years ago a certain fraction $p(a)$, ascertainable for example by means of a life table, survives to age a, and forms the a-year-old contingent of the existing population. A similar remark applies to every age of life. If, therefore, we denote by $N(t)$ the number of the population at time t, and by $B(t)$ the annual rate of births at the same time, and if we are dealing with a *closed* population, that is, one exempt from immigration and emigration, then, evidently,

$$N(t) = \int_0^\infty B(t - a)p(a)\, da. \tag{1}$$

In general $p(a)$ may be a function of t also, but we shall here consider primarily the case where $p(a)$ does not contain t explicitly.

The function $p(a)$ being known (from a life table), if $B(t)$ is given as a function of t, then $N(t)$ follows by direct integration of the right hand member of (1).

If, on the contrary, $N(t)$ is given, and $B(t)$ is to be determined, a special problem arises. On a former occasion[1] I have given a solution for cases in which the function $N(t)$ is given or can be expanded in the form of a series proceeding in ascending powers of e^{rt}, where r is constant; and, more particularly, for the case in which $N(t)$ is the logistic function

$$N(t) = \frac{N_\infty}{1 + e^{-rt}} = N_\infty(e^{rt} - e^{2rt} + e^{3rt} - \cdots). \tag{2}$$

Although $N(t)$ is expanded in an exponential series in the process of obtaining the solution by this method, in the final result these terms are reunited, and only the original function $N(t)$ as such, together with its derivatives, appears. This suggests that it should be possible to obtain the result by a more direct route, retaining the function in its original form throughout the process. This is indeed the case, as will now be shown, by a method which at the same time frees us from the assumption that $N(t)$ can be represented by an exponential series in powers of e^{rt}.

This is accomplished as follows:

[1] A. J. Lotka, *Proc. Natl. Acad. Sci.*, 1929, vol. 15, p. 793; *Human Biology*, 1931, vol. 3, p. 459.

Let us put

$$N(t) = \varphi_0(t) \tag{3}$$

and assume that $B(t)$ can be expressed as a series in $\varphi_0(t)$ and its derivatives, thus

$$B(t) = c_0\varphi_0(t) - c_1\varphi_1(t) + \frac{c_2}{2!}\varphi_2(t) - \frac{c_3}{3!}\varphi_3(t) + \cdots \tag{4}$$

$$\begin{aligned} B(t-a) = {} & c_0\left\{\varphi_0(t) - a\varphi_1(t) + \frac{a^2}{2!}\varphi_2(t) - \frac{a^3}{3!}\varphi_3(t) + \cdots\right\} \\ & - c_1\left\{\varphi_1(t) - a\,\varphi_2(t) + \frac{a^2}{2!}\varphi_3(t) - \cdots\right\} \\ & + \frac{c_2}{2!}\left\{\varphi_2(t) - a\,\varphi_3(t) + \cdots\right\} \\ & - \cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{5}$$

where $\varphi_n(t)$ denotes the nth derivative of $\varphi_0(t)$.

Introducing (5) in (1), and carrying out the integration, we obtain

$$\begin{aligned} \varphi_0(t) = {} & c_0m_0\varphi_0(t) - \{c_1m_0 + c_0m_1\}\varphi_1(t) + \frac{1}{2!}\{c_2m_0 + 2c_1m_1 + c_0m_2\}\varphi_2(t) \\ & - \frac{1}{3!}\{c_3m_0 + 3c_2m_1 + 3c_1m_2 + c_0m_3\}\varphi_3(t) + \cdots \end{aligned} \tag{6}$$

where m_n denotes the nth moment of the function $p(a)$ about the origin of a, that is,

$$m_n = \int_0^\infty a^n p(a)\,da. \tag{7}$$

Equation (6) is satisfied by putting

$$\begin{cases} 1 = c_0m_0 \\ 0 = c_0m_1 + c_1m_0 \\ 0 = c_0m_2 + 2c_1m_1 + c_2m_0 \\ 0 = c_0m_3 + 3c_1m_2 + 3c_2m_1 + c_3m_0 \end{cases} \tag{8}$$

the numerical coefficients being those of the corresponding binomial expansion.

Now consider

$$\begin{aligned} \beta(r) = \frac{1}{\int_0^\infty e^{-ra}p(a)\,da} & = \frac{1}{m_0 - m_1r + \frac{m_2}{2!}r^2 - \cdots} \\ & = C_0 - C_1r + \frac{C_2}{2!}r^2 - \cdots \end{aligned} \tag{9}$$

This gives

$$(10)\qquad \begin{cases} 1 = C_0 m_0 \\ 0 = C_0 m_1 + C_1 m_0 \\ 0 = C_0 m_2 + 2C_1 m_1 + C_2 m_0 \\ 0 = C_0 m_3 + 3C_1 m_2 + 3C_2 m_1 + C_3 m_0 \end{cases}$$

from which it is seen that the coefficients c in equation (6) are identical with the coefficients C in equation (9), that is, they are the coefficients of successive powers of r in the expansion of

$$(11)\qquad \beta(r) = \frac{1}{\int_0^\infty e^{-ra} p(a)\, da}$$

as a power series in r.

These coefficients can also be conveniently expressed in terms of the Thiele seminvariants λ of the function $p(a)$, which are defined by

$$(12)\qquad \begin{aligned} \int_0^\infty e^{-ra} p(a)\, da &= m_0 e^{-\lambda_1 r + \lambda_2 \frac{r^2}{2!} - \lambda_3 \frac{r^3}{3!} + \cdots} \\ &= m_0 - m_1 r + \frac{m_2}{2!} r^2 - \frac{m_3}{3!} r^3 + \cdots \end{aligned}$$

Differentiating the right hand member of (12) we have

$$(13)\qquad \begin{aligned} m_0 \left(\lambda_1 - \lambda_2 r + \lambda_3 \frac{r^2}{2!} - \cdots\right) & e^{-\lambda_1 r + \lambda_2 \frac{r^2}{2!} - \lambda_3 \frac{r^3}{3!} + \cdots} \\ &= \left(\lambda_1 - \lambda_2 r + \lambda_3 \frac{r^2}{2!} - \cdots\right)\left(m_0 - m_1 r + m_2 \frac{r^2}{2!} - \cdots\right) \\ &= m_1 - m_2 r + m_3 \frac{r^2}{2!} - \cdots \end{aligned}$$

Hence

$$(14)\qquad \begin{cases} m_0 = m_0 \\ m_1 = \lambda_1 m_0 \\ m_2 = \lambda_1 m_1 + \lambda_2 m_0 \\ m_3 = \lambda_1 m_2 + 2\lambda_2 m_1 + \lambda_3 m_0 \\ m_4 = \lambda_1 m_3 + 3\lambda_2 m_2 + 3\lambda_3 m_1 + \lambda_4 m_0 \end{cases}$$

again with binomial coefficients.

The λ's being thus defined, we now have

$$\frac{1}{\int_0^\infty e^{-ra}p(a)\,da} = \frac{1}{m_0}e^{\lambda_1 r - \lambda_2 \frac{r^2}{2!} + \lambda_3 \frac{r^3}{3!} - \cdots} \tag{15}$$

$$= c_0 - c_1 r + c_2 \frac{r^2}{2!} - \cdots \tag{16}$$

from which it follows, as in the case of equations (12), (13), that

$$\begin{cases} c_0 = \dfrac{1}{m_0} \\ c_1 = -\lambda_1 c_0 \\ c_2 = -\lambda_1 c_1 - \lambda_2 c_0 \\ c_3 = -\lambda_1 c_2 - 2\lambda_2 c_1 - \lambda_3 c_0 \\ c_4 = -\lambda_1 c_3 - 3\lambda_2 c_2 - 3\lambda_3 c_1 - \lambda_4 c_0 \\ c_5 = -\lambda_1 c_4 - 4\lambda_2 c_3 - 6\lambda_3 c_2 - 4\lambda_4 c_1 - \lambda_5 c_0 \end{cases} \tag{17}$$

once more with binomial coefficients. The coefficients c are, in fact, related to the negative seminvariants $-\lambda$ in the same way as the moments m are related to the direct seminvariants.

Considerable simplification in the coefficients c can be effected by a change in origin of t. This is most easily accomplished by reverting to equation (1) in which we write, instead of $B(t - a)$, the equivalent expression

$$B(t - a) = B\{(t - \lambda_1) - (a - \lambda_1)\} = B(\theta - \alpha). \tag{18}$$

In place of the moments m of the function $p(a)$, taken about $a = 0$, there then appear in (6) the corresponding moments taken about the mean age λ_1, and in the equation corresponding to (17), for the new coefficients c_0', c_1', c_2', $\cdots$ the seminvariants λ are now defined in terms of these new moments. According to a well-known property of the Thiele seminvariants this leaves all the λ's except λ_1 unchanged, while reducing this latter to zero.

The coefficients c' are therefore obtained by a set of equations identical with those for the coefficients c, in which, however, the substitution $\lambda_1 = 0$ eliminates all terms containing either λ_1 or c_1, thus

$$\begin{cases} c_0' = c_0 = \dfrac{1}{m_0} \\ c_1' = 0 \\ c_2' = -\lambda_2 c_0 \\ c_3' = -\lambda_3 c_0 \\ c_4' = -(\lambda_4 - 3\lambda_2^2)c_0 \\ c_5' = -(\lambda_5 - 10\lambda_2\lambda_3)c_0 \end{cases} \tag{19}$$

With this choice of constants the solution (4) of the fundamental equation (1) finally takes the form

$$(20)\quad B(t) = \frac{1}{m_0}\left\{\varphi_0(\theta) - \frac{\lambda_2}{2!}\varphi_2(\theta) + \frac{\lambda_3}{3!}\varphi_3(\theta) - \frac{(\lambda_4 - 3\lambda_2^2)}{4!}\varphi_4(\theta) + \cdots\right\}.$$

It is thus seen that if the population, as a function of the time, is represented by $\varphi(t)$, and expansion by Taylor's theorem is applicable to $\varphi(t - a)$ within the range $0 < a < \omega$ where ω is the highest age attained by any individual, i.e., the highest age for which $p(a)$ has a value other than zero, then the annual births $B(t)$ under the régime of a constant life table can be represented by a series (20) proceeding in successive derivatives of φ. The constant coefficients of the successive members of the series are known functions[2] of the Thiele seminvariants λ of the function $p(a)$, the probability at birth of attaining age a.

II. ALTERNATIVE SOLUTION

1. In the special case of a population growing at a constant rate r under the régime of a constant life table, the constant birth rate per head is given by

$$(21)\qquad \beta(r) = \frac{1}{\int_0^\infty e^{-ra}p(a)\,da}.$$

This suggests that when r is variable we may still have as a first approximation

$$(22)\qquad \beta(t) = \frac{1}{\int_0^\infty e^{-r_t a}p(a)\,da} = \beta(r_t)$$

and that this expression may form the first term in a series expansion of some kind. Evidence of this has, indeed, been shown[3] in the case of a population growing according to the logistic curve, but the formal justification of the supposition was not fully established, nor was the law of the series expansion determined. We now proceed to establish the series for the general case, using as a starting point the result obtained in Part I.

We revert, then, to equation (4), and, dividing by $N(t)$, we have

$$(23)\qquad \frac{B(t)}{N(t)} = b(t) = c_0 - c_1\frac{\varphi_1(t)}{\varphi_0(t)} + \frac{c_2\varphi_2(t)}{2!\varphi_0(t)} - \frac{c_3\varphi_3(t)}{3!\varphi_0(t)} + \cdots$$

[2] An obvious extension of this result is that this representation of $B(t)$ may still hold approximately when the life table is variable, and the seminvariants λ are accordingly functions of t. We may expect this approximation to be serviceable when $p(a, t)$ changes but slowly with t, a condition that will usually be satisfied in practice. See A. J. Lotka, *Human Biology*, 1931, vol. 3, p. 481.

[3] A. J. Lotka, *Proceedings Natl. Acad. Sci.*, 1929, Vol. 15, p. 796.

But $\frac{\varphi_1(t)}{\varphi_0(t)}$ is the rate of increase per head at time t, which we have denoted by r_t, that is

$$\varphi_1(t) = r_t \varphi_0(t). \tag{24}$$

To systematize notation, let us write r_1 instead of r_t, and denote by $r_2, r_3 \cdots$ successive derivatives of r_1 with respect to t. With this notation the following scheme, homogeneous as regards the weight of the terms in the right hand member, results

$$\begin{cases} \varphi_0 = \varphi_0 \\ \varphi_1 = r_1\varphi_0 \\ \varphi_2 = r_1\varphi_1 + r_2\varphi_0 \\ \varphi_3 = r_1\varphi_2 + 2r_2\varphi_1 + r_3\varphi_0 \\ \varphi_4 = r_1\varphi_3 + 3r_2\varphi_2 + 3r_3\varphi_1 + r_4\varphi_0 \end{cases} \tag{25}$$

again with binomial coefficients.

Eliminating derivatives of φ from the right hand members of the set of equations (25), we find

$$\begin{cases} \frac{\varphi_1}{\varphi_0} = r_1 \\ \frac{\varphi_2}{\varphi_0} = r_1^2 + r_2 \\ \frac{\varphi_3}{\varphi_0} = r_1^3 + 3r_1r_2 + r_3 \\ \frac{\varphi_4}{\varphi_0} = r_1^4 + 6r_1^2r_2 + 4r_1r_3 + 3r_2^2 + r_4 \end{cases} \tag{26}$$

Introducing the expression (26) for $\frac{\varphi_n}{\varphi_0}$ in (23), and rearranging terms, we find

$$\begin{aligned} b(t) &= c_0 - c_1r_1 + \frac{c_2r_1^2}{2!} - \frac{c_3r_1^3}{3!} + \cdots \\ &+ \frac{r_2}{2!}\left(c_2 - c_3r_1 + \frac{c_4r_1^2}{2!} - \cdots\right) \\ &- \frac{r_3}{3!}\left(c_3 - c_4r_1 + \frac{c_5r_1^2}{2!} - \cdots\right) \\ &+ \frac{(r_4 + 3r_2^2)}{4!}\left(c_4 - c_5r_1 + \frac{c_6r_1^2}{2!} - \cdots\right) \\ &- \frac{(r_5 + 10r_2r_3)}{5!}\left(c_5 - c_6r_1 + \frac{c_7r_1^2}{2!} - \cdots\right). \end{aligned} \tag{27}$$

It will be seen that the factors r_2, r_3, $(r_4 + 3r_2^2)$, $(r_5 + 10r_2r_3)$, etc., by which, in successive terms, the power series in r_1 are multiplied, are obtained by the formal substitution $r_1 = 0$ in the corresponding expressions (26) for $\frac{\varphi_n}{\varphi_0}$, as for example,

$$(28)\quad \begin{cases} \frac{\varphi_0}{\varphi_0} = 1 & \frac{\varphi_3}{\varphi_0} = r_1^3 + 3r_1r_2 + r_3 \\ \frac{\varphi_1}{\varphi_0} = r_1 & \left[\frac{\varphi_3}{\varphi_0}\right]_0 = r_3 \\ \left[\frac{\varphi_1}{\varphi_0}\right]_0 = 0 & \frac{\varphi_4}{\varphi_0} = r_1^4 + 6r_1^2r_2 + 4r_1r_3 + 3r_2^2 + r_4 \\ \frac{\varphi_2}{\varphi_0} = r_1^2 + r_2 & \left[\frac{\varphi_4}{\varphi_0}\right]_0 = 3r_2^2 + r_4 \\ \left[\frac{\varphi_2}{\varphi_0}\right]_0 = r_2 & \end{cases}$$

With this interpretation of the symbol $\left[\frac{\varphi_n}{\varphi_0}\right]_0$, we may therefore write

$$(29)\qquad b(t) = \sum_{n=0}^{n=\infty} \frac{1}{n!}\left[\frac{\varphi_n}{\varphi_0}\right]_0 \frac{\partial^n \beta(r_1)}{\partial r_1^n}$$

with the understanding that

$$\frac{\partial^0 \beta(r_1)}{\partial r_1^0} = \beta(r_1) = \beta(r_t).$$

Furthermore, since $\frac{\varphi_0}{\varphi_0} = 1$ and $\left[\frac{\varphi_1}{\varphi_0}\right]_0 = 0$, equation (29) can also be written in the form

$$(30)\qquad b(t) = \beta(r_t) + \sum_{n=2}^{n=\infty} \frac{1}{n!}\left[\frac{\varphi_n}{\varphi_0}\right]_0 \frac{\partial^n \beta(r_1)}{\partial r_1^n}$$

which establishes the desired result, namely, that $b(t)$ is expressed in terms of a fully defined series, in which the first term is

$$(22)\qquad \beta(r_t) = \frac{1}{\int_0^\infty e^{-r_t a} p(a)\, da}.$$

It will be observed that equation (23) can be written

$$b(t) = \sum \frac{1}{n!}\left[\frac{\partial^n \beta(r)}{\partial r^n}\right]_0 \frac{\varphi_n}{\varphi_0}$$

so that, in view of (29), we have

$$\sum \frac{1}{n!}\left[\frac{\partial^n \beta(r)}{\partial r^n}\right]_0 \frac{\varphi_n}{\varphi_0} = \sum \frac{1}{n!}\frac{\partial^n \beta(r)}{\partial r^n}\left[\frac{\varphi_n}{\varphi_0}\right]_0$$

a somewhat remarkable relation.

Analytically, our problem must thus be considered solved, but for purposes of computation, as well as on account of a certain analytical interest of their own, it is desirable to examine certain properties of the various characteristics that appear in the treatment of the problem.

2. **Successive partial derivatives of $\beta(r)$.** In the application of the formulae (29) or (30), it is necessary to obtain successive partial derivatives of $\beta(r)$ with respect to r. The values of the derivatives $\frac{\partial^n \beta}{\partial r^n}$ can be computed directly from (9), but more exact values are obtained by taking advantage of certain special properties of these derivatives. With this in view, it is desirable, first of all, to consider certain properties of the moments M_n and the seminvariants Λ_n of the function

$$f(r) = e^{-ra}p(a) \tag{31}$$

We note that

$$M_n = \int_0^\infty a^n e^{-ra} p(a)\, da \tag{32}$$

$$\frac{\partial M_n}{\partial r} = -\int_0^\infty a^{n+1} e^{-ra} p(a)\, da$$

$$= -M_{n+1}. \tag{33}$$

Now the seminvariants Λ of the function $e^{-ra}p(a)$ are defined by

$$\begin{cases} M_1 = \Lambda_1 M_0 \\ M_2 = \Lambda_1 M_1 + \Lambda_2 M_0 \\ M_3 = \Lambda_1 M_2 + 2\Lambda_2 M_1 + \Lambda_3 M_0 \\ M_4 = \Lambda_1 M_3 + 3\Lambda_2 M_2 + 3\Lambda_3 M_1 + \Lambda_4 M_0. \end{cases} \tag{34}$$

On the other hand, in view of (33)

$$\begin{cases} M_1 = \Lambda_1 M_0 \\ M_2 = \Lambda_1 M_1 - \Lambda_1' M_0 \\ M_3 = \Lambda_1 M_2 - 2\Lambda_1' M_1 + \Lambda_1'' M_0. \end{cases} \tag{35}$$

where the primes denote derivatives with respect to r.

Hence

(36)
$$\begin{cases} \Lambda_2 = -\Lambda_1' \\ \Lambda_3 = -\Lambda_2' = \Lambda_1'' \end{cases}$$

and generally

(37)
$$\Lambda_{n+1} = -\frac{\partial \Lambda_n}{\partial r}$$

that is, if successive moments are successive negative partial derivatives with respect to r, the same is true of successive seminvariants.

Furthermore, we have

(38)
$$\frac{\partial \beta(r)}{\partial r} = \beta(r) \frac{\int_0^\infty a e^{-ra} p(a)\, da}{\int_0^\infty e^{-ra} p(a)\, da}$$

(39)
$$= \Lambda_1 \beta(r).$$

Hence, in this sense, and denoting successive partial derivatives of β by subscripts

(40)
$$\begin{cases} \beta_1 = \Lambda_1 \beta_0 \\ \beta_2 = \Lambda_1 \beta_1 - \Lambda_2 \beta_0 \\ \beta_3 = \Lambda_1 \beta_2 - 2\Lambda_2 \beta_1 + \Lambda_3 \beta_0 \end{cases}$$

a set of equations from which successive partial derivatives of $\beta(r)$ can be obtained if the seminvariants Λ are given.

In actual computation the seminvariants Λ are calculated according to (34) from the moments M, which themselves must first be computed as functions of r. If the entire computation is required for only one particular value of r, the moments M may be calculated directly by numerical integration of (32). But if their values are required for a series of values of r, direct computation would be very laborious. Unless r is rather small, merely expanding the exponential under the integral sign and integrating term by term is unsatisfactory as the series converges too slowly. Much more rapid convergence is secured by obtaining a series development not of the moments themselves, but of the ratio between two successive moments, thus:

(41)
$$\frac{M_{n+1}}{M_n} = \frac{m_{n+1} - r m_{n+2} + \frac{r^2}{2!} \cdot m_{n+3} - \cdots}{m_n - r m_{n+1} + \frac{r^2}{2!} \cdot m_{n+2} - \cdots}$$

$$= \lambda_{n1} - r\lambda_{n2} + \frac{r^2}{2!} \cdot \lambda_{n3} - \cdots$$

where λ_{nj} is the jth seminvariant of $a^n p(a)$, that is,

$$
(42)\qquad \begin{cases} m_n = m_n \\ m_{n+1} = \lambda_{n1} m_n \\ m_{n+2} = \lambda_{n1} m_{n+1} + \lambda_{n2} m_n \\ m_{n+3} = \lambda_{n1} m_{n+2} + 2\lambda_{n2} m_{n+1} + \lambda_{n3} m_n \end{cases}
$$

Furthermore, according to (33)

$$
(43)\qquad \begin{aligned} \frac{\partial M_n}{M_n \partial r} &= -\frac{M_{n+1}}{M_n} \\ &= -\left(\lambda_{n1} - r\lambda_{n2} + \frac{r^2}{2!}\lambda_{n3} - \cdots\right) \end{aligned}
$$

Hence

$$
(44)\qquad M_n = m_n e^{-r\lambda_{n1} + \frac{r^2}{2!}\lambda_{n2} - \cdots}
$$

a formula which enables us to compute directly the moments M of $e^{-ra}p(a)$ from the moments of $p(a)$ and seminvariants of $a^n p(a)$. The seminvariants Λ and the derivatives $\beta_1, \beta_2 \cdots$ then follow according to (34) and (40).

3. Recapitulation. By virtue of the various properties of the moments and seminvariants thus developed, the following routine may be followed in the computation of the successive derivatives β_n. By direct computation, determine the moments m_n of $p(a)$. Then obtain in succession the several characteristics as follows, the numbers over the arrows indicating the pertinent equation in the text:

$$
m_n \xrightarrow{(42)} \lambda_{nj} \xrightarrow{(44)} M_n \xrightarrow{(34)} \Lambda_n \xrightarrow{(40)} \beta_n
$$

4. Numerical example. By way of illustration the results obtained in preceding sections were applied to a logistic population for which a series expansion of the annual births $B(t)$ in terms of the logistic and its derivatives was available from a previous computation[4] carried out by a method less general than the one here presented. Of special interest in the numerical results now to be shown is the comparison between the two representations, on the one hand $B(t)$ in terms of $\varphi(t)$, the logistic in this case, and its derivatives; on the other hand $b(t)$ the birth rate per head, in terms of $\beta(r_t)$ and its partial derivatives with respect to r.

The data on which these computations are based are derived from the actual growth of the population of the United States, which from 1790 to 1930 followed rather closely the logistic function

[4] *Human Biology*, loc. cit., *Jl. Soc. Statistique, Paris*, 1933, vol. 74, pp. 336, 341.

TABLE I

Analysis of the Birth Rate Curve in a Logistic Population Subject to a Constant Mortality

Numerical Values of Life Table and Other Characteristics, Based on the Mortality in the United States* 1919–1920

Characteristic	Defining Equation	0	1	2	3	4	5
		Moments of $p(a)$ and Seminvariants of $a^n p(a)$ According to United States Life Table for 1919–1920, White Females					
m_n	7	57.518	2014.9	99,179.	56,712. $\times 10^2$	35,407. $\times 10^4$	23,455. $\times 10^6$
λ_{n1}	12, 14	35.031	497.12	3365.	−223,390.	−74,241. $\times 10^2$	
λ_{n2}	42	49.222	391.79	−1386.	−112,242.	26,338. $\times 10^2$	
λ_{n3}	42	57.182	300.28	−1984.	−34,242.		
λ_{n4}	42	62.433	237.99	−1744.			
λ_{n5}	42	66.245	195.52				
		Moments and Seminvariants of $e^{-ra}p(a)$ and the Corresponding Values of $\left[\frac{\varphi_n}{\varphi_0}\right]_0$ and of β_n for Selected Values of r					
		Calendar Year 1800 (r = .030578)					
M_n	32	24.298	538.26	20,028.	947,340.	51,178. $\times 10^3$	2,827. $\times 10^6$
Λ_n	34		22.152	333.54	5952.0	22,238.	−13,734. $\times 10^3$
$[\varphi_n/\varphi_0]_0$	28	1.000	0	−2591. $\times 10^{-8}$	−770. $\times 10^{-9}$	−196. $\times 10^{-10}$	−515. $\times 10^{-12}$
β_n	40		.91103	6.4638	−219.77	4016.6	−32,849.

		Calendar Year 1850 (r = .027727)					
M_n	32	25.922	599.35	22,946.	11,075. $\times 10^2$	60,825. $\times 10^3$	3,444. $\times 10^6$
Λ_n	34		23.121	350.60	6,045.4	8,266.3	−14,199. $\times 10^3$
$[\varphi_n/\varphi_0]_0$	28	1.000	0	−10,256. $\times 10^{-8}$	−246. $\times 10^{-8}$	−66. $\times 10^{-10}$	634. $\times 10^{-12}$
β_n	40		.89148	7.0938	−228.00	3,116.5	72,165.
		Calendar Year 1900 (r = .019138)					
M_n	32	32.066	844.67	35,154.	17,972. $\times 10^2$	10,329. $\times 10^4$	6,239. $\times 10^6$
Λ_n	34		26.341	402.43	5,966.9	−50,322.	−14,445. $\times 10^3$
$[\varphi_n/\varphi_0]_0$	28	1.000	0	−23,515. $\times 10^{-8}$	−161. $\times 10^{-8}$	265. $\times 10^{-9}$	272. $\times 10^{-11}$
β_n	40		.82155	9.0890	−235.72	−906.36	395,810.
		Calendar Year 1950 (r = .007684)					
M_n	32	44.580	1,396.7	64,491.	35,353. $\times 10^2$	21,407. $\times 10^4$	13,783. $\times 10^6$
Λ_n	34		31.329	465.12	4,837.5	−155,820.	−11,278. $\times 10^3$
$[\varphi_n/\varphi_0]_0$	28	1.000	0	−18,242. $\times 10^{-8}$	293. $\times 10^{-8}$	119. $\times 10^{-9}$	−355. $\times 10^{-11}$
β_n	40		.70274	11.583	−182.32	−8180.6	604,140.
		Calendar Year 2000 (r = .001980)					
M_n	32	53.715	1,829.2	88,617.	50,144. $\times 10^2$	31,066. $\times 10^4$	20,452. $\times 10^6$
Λ_n	34		34.054	490.09	3790.9	−208,380.	−8,334. $\times 10^3$
$[\varphi_n/\varphi_0]_0$	28	1.000	0	−583. $\times 10^{-7}$	16. $\times 10^{-7}$	−27. $\times 10^{-9}$	482. $\times 10^{-12}$
β_n	40		.63398	12.466	−126.32	−11,541.	511,440.

* The life table for white females was used, for which the requisite constants were available from the author's prior publications.

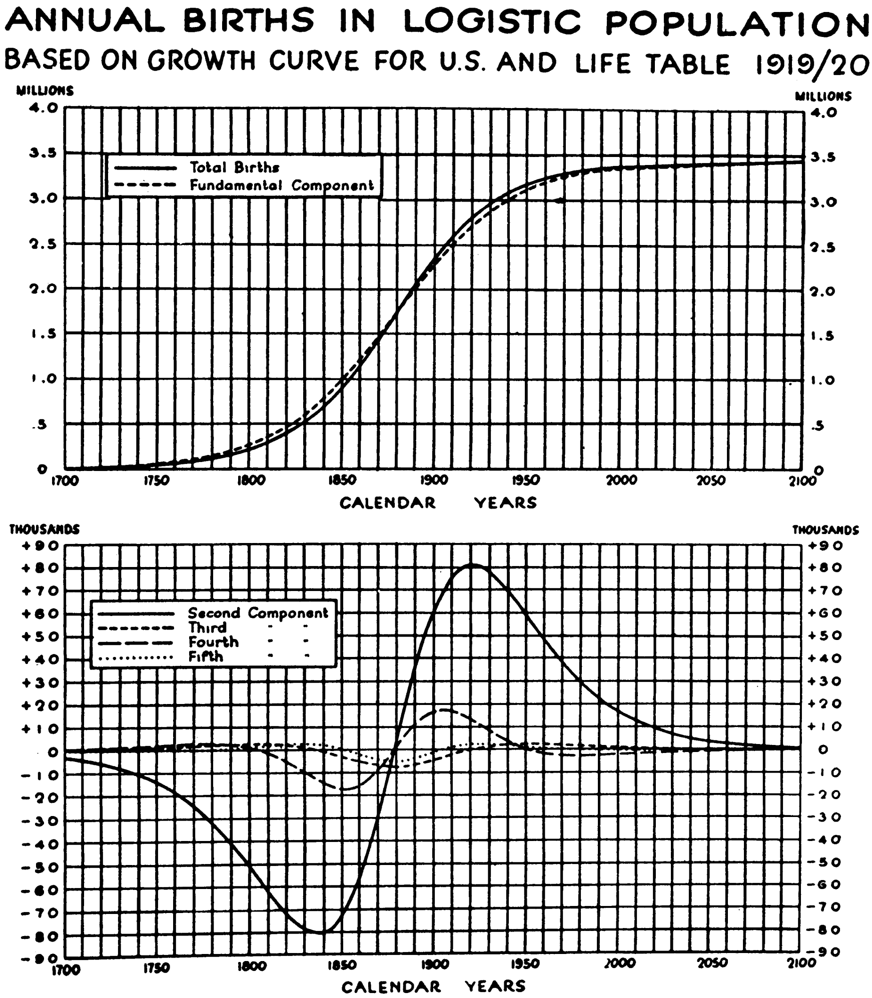

FIG. 1

$$N(t) = \frac{197{,}493{,}000}{1 + e^{-.0314(t'-1914)}} = \frac{N_\infty}{1 + e^{-.0314t}} \tag{45}$$
$$= N_\infty \Phi(t)$$

where $\Phi(t)$ is used to distinguish the special case of the logistic function, from the general case $\varphi(t)$, and where t' denotes the calendar year.

This was combined, in the computations, with the life table for white females, United States 1919–1920, supposed constant throughout the period.[5]

[5] This is, of course, an arbitrary assumption made here simply for illustrative purposes. The life table for white females was used because of related computations regarding the intrinsic rate of natural increase, which have been reported on elsewhere.

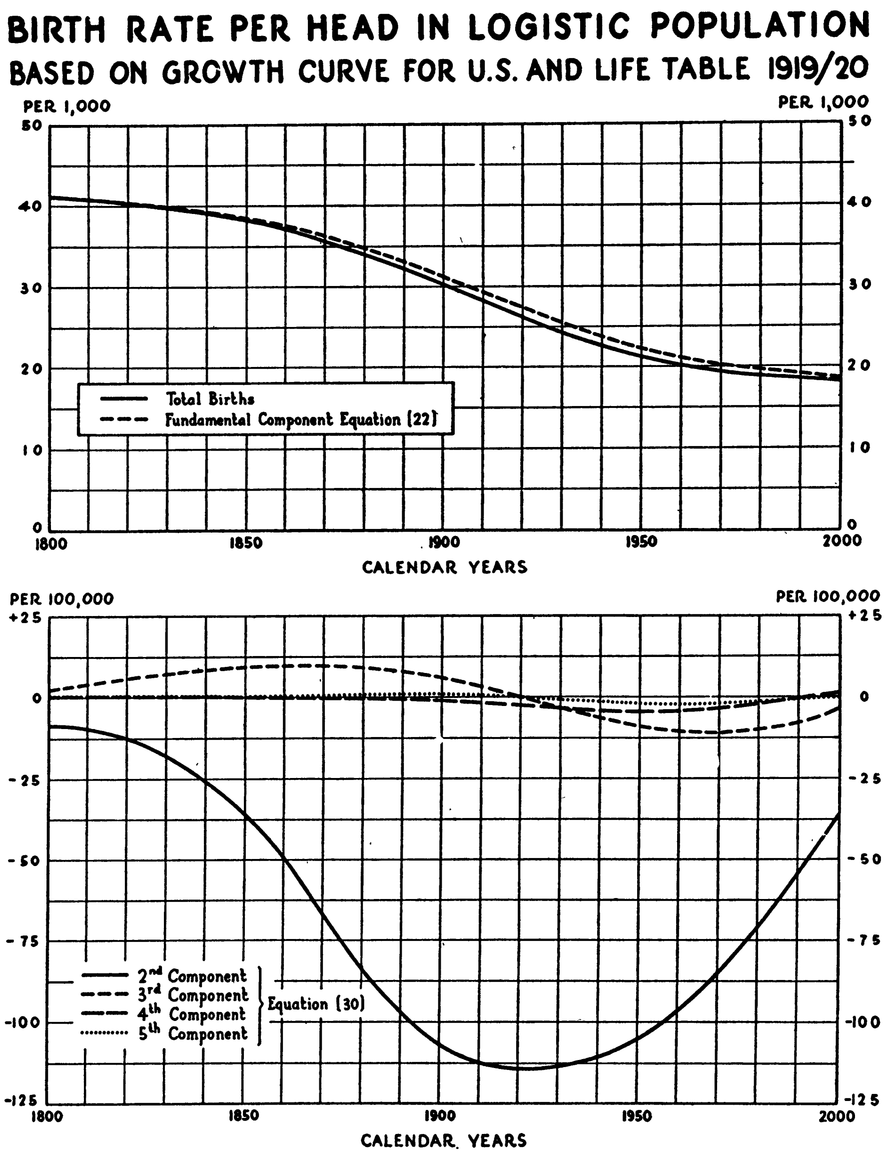

FIG. 2

With this basis, the fundamental data are as follows:

1. Quantities depending solely on the life table, namely, m_n, λ_{nj}. These are exhibited in the first section of Table I.

2. Quantities depending on the life table and also on r_t, namely, M_n, Λ_n, $\left[\frac{\varphi_n}{\varphi_0}\right]_0$, β_n. These are exhibited in the remaining sections of Table I.

5. Comparison of the representation (20) of the annual births $B(t)$ and the representation (30) of the annual birth rate $b(t)$. It is interesting to make this comparison, as applied to the case of the logistic population, for there are certain points of marked difference. The graphs Fig. 1 and Fig. 2 show this at a glance. In both cases the fundamental component alone yields a very fair approximation to the full solution, but the second component is of very different character in the two cases. In the composition of $B(t)$ it starts from a vanishing value, diminishes through negative values to a minimum, then, passing through zero at the "center," it rises to a maximum positive value, and finally approaches zero asymptotically from above.

The second component of $b(t)$, starting also from a vanishing value, forms a single downwardly convex loop, and then approaches zero asymptotically from below.

The higher components in both cases are relatively insignificant.

III. APPENDIX

1. Symbols used. It may be convenient to assemble together here certain of the symbols used in the text:

$$m_n = \int_0^\infty a^n p(a)\,da = n\text{th moment of } p(a)$$

$$M_n = \int_0^\infty a^n e^{-ra} p(a)\,da = n\text{th moment of } e^{-ra} p(a)$$

$\lambda_{nj} = j$th seminvariant of $a^n p(a)$

$\lambda_{0j} = \lambda_j = j$th seminvariant of $p(a)$

$\Lambda_j = j$th seminvariant of $e^{-ra} p(a)$

$$\beta(r) = \frac{1}{\int_0^\infty e^{-ra} p(a)\,da}$$

$$\beta_n = \frac{\partial^n \beta}{\partial r^n}$$

$$r_{1+n} = \frac{\partial^n r_1}{\partial t^n}$$

$\left[\frac{\varphi_n}{\varphi_0}\right]_0$ For definition of this, see equations (25), (26), (28)

2. Derivatives of $\varphi(t)$ and properties of the Logistic Function. In the particular case that $\varphi(t)$ is the logistic function $\Phi(t)$, the successive derivatives $\Phi_1, \Phi_2, \cdots$ may be obtained step by step by equations (25), (26), taking advantage of special properties of that function.

$$
\begin{cases}
\Phi(t) + \Phi(-t) = \dfrac{1}{1 + e^{-Kt}} + \dfrac{1}{1 + e^{Kt}} \\
\qquad = \dfrac{1}{1 + e^{-Kt}} + \dfrac{e^{-Kt}}{1 + e^{-Kt}} \\
\qquad = 1
\end{cases}
\tag{46}
$$

Hence, putting

$$\Psi(t) = \Phi(-t) \tag{47}$$

We have

$$\Phi + \Psi = 1 \tag{48}$$

$$\Psi = 1 - \Phi \tag{49}$$

Denoting the nth derivatives by the subscript n, it follows at once from (49) that

$$\Psi_n = -\Phi_n \tag{50}$$

$$
\begin{aligned}
r_1 = \frac{\Phi_1}{\Phi_0} &= \frac{Ke^{-Kt}}{(1 + e^{-Kt})^2} \Big/ \frac{1}{1 + e^{-Kt}} \\
&= \frac{K}{1 + e^{Kt}} = K\Psi_0
\end{aligned}
\tag{51}
$$

$$r_{n+1} = K\Psi_n \tag{52}$$

Hence, in the case of the logistic, the algorithm (25) takes the form

$$\Phi_1 = K\Phi_0\Psi_0 = K\Phi_0(1 - \Phi_0) \tag{53}$$

$$
\begin{aligned}
\Phi_2 &= K\{\Phi_0\Psi_1 + \Phi_1\Psi_0\} \\
&= -K\Phi_1\{\Phi_0 - (1 - \Phi_0)\} \\
&= K^2\Phi_0(1 - \Phi_0)(1 - 2\Phi_0)
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
\Phi_3 &= K\{\Phi_0\Psi_2 + 2\Phi_1\Psi_1 + \Phi_2\Psi_0\} \\
&= K^3\Phi_0(1 - \Phi_0)(1 - 6\Phi_0 + 6\Phi_0^2) \\
&= K^3\Phi_0(1 - \Phi_0)\left(\frac{1}{2} + \frac{\sqrt{3}}{6} - \Phi_0\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{6} - \Phi_0\right)
\end{aligned}
\tag{55}
$$

It is seen that all derivatives vanish at $\Phi_0 = 0$ and at $\Phi_0 = +1$, that is at $t = \pm\infty$. Furthermore, Φ_2 vanishes at $\Phi_0 = \frac{1}{2}$, that is at $t = 0$; and Φ_3 vanishes at $\Phi_0 = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{6}$, that is at $\tanh \dfrac{Kt}{2} = \dfrac{\sqrt{3}}{3}$, since

$$\Phi_0(t) = \frac{1}{2} + \frac{1}{2}\tanh\frac{Kt}{2} \tag{56}$$

Successive derivatives of Φ can thus be computed successively according to (53), (54), (55), etc. For purposes of record, however, it may be convenient to note here explicit expressions for these derivatives, and a simple algorithm by which the numerical coefficients occurring in them can be written down at sight. It is found, by carrying out the differentiation directly, that

$$
(57) \quad \begin{cases}
\Phi_1 = re^{rt} \cdot \dfrac{1}{(1 + e^{rt})^2} \\[2ex]
\Phi_2 = r^2 e^{rt} \cdot \dfrac{1 - e^{rt}}{(1 + e^{rt})^3} \\[2ex]
\Phi_3 = r^3 e^{rt} \cdot \dfrac{1 - 4e^{rt} + e^{2rt}}{(1 + e^{rt})^4} \\[2ex]
\Phi_4 = r^4 e^{rt} \cdot \dfrac{1 - 11e^{rt} + 11e^{2rt} - e^{3rt}}{(1 + e^{rt})^5} \\[2ex]
\Phi_5 = r^5 e^{rt} \cdot \dfrac{1 - 26e^{rt} + 66e^{2rt} - 26e^{3rt} + e^{4rt}}{(1 + e^{rt})^6}
\end{cases}
$$

The numerical coefficients can be obtained by the modification of the Pascal triangle shown in Fig. 3. Its use is most easily explained by an example. Thus the coefficient -4 in the third line is obtained as the sum of the two immediately adjoining figures in the line above it, each multiplied by the rank of the oblique row in which it appears. This rank is indicated by the corresponding number written above the ruled line forming the "roof" of the triangle. Thus the second coefficient in the third horizontal line from above is obtained as $(1 \times -2) + (-1 \times 2) = -4$. Similarly, the third coefficient in the last line of the diagram (which must be regarded actually as extending indefinitely) is the sum of $(-57 \times -5) + (302 \times 3) = 1191$.

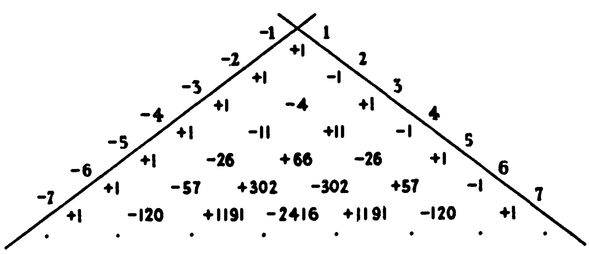

FIG. 3. Scheme for computing numerical coefficients in successive derivatives of logistic function.

2. **Construction of coefficients in (26).** The numerical coefficients appearing in equation (26) are constructed according to the following rules:

(a). The expression $\frac{\varphi_n}{\varphi_0}$ contains all possible products of the form $r_a r_b r_c \cdots$ the sum of whose subscripts is n, due regard being had to powers of r. Thus for example $\frac{\varphi_4}{\varphi_0}$ contains r_1^4, that is $r_1 r_1 r_1 r_1$; also $r_1^2 r_2$, $r_1 r_3$, r_2^2 and r_4.

(b). If a, b, c are all different, the coefficient $Q_{abc\ldots}$ of $r_a r_b r_c \cdots$ is formed according to the following pattern, in which ${}^pC_q$ denotes, in the customary notation, the binomial coefficient $\binom{p}{q}$

$$Q_{abc} = {}^nC_{n-(a+b+c)}\,{}^{a+b+c}C_{b+c}\,{}^{b+c}C_c \tag{58}$$
$$= \frac{n!}{\{n-(a+b+c)\}!\,a!\,b!\,c!}$$

If some of the subscripts are equal, that is if some of the factors occur as the sth power of r, then the formula for Q is modified by the introduction of the corresponding factorial $s!$ in the denominator, according to the pattern of the following example:

If $b = c$, so that $r_a r_b r_c = r_a r_b^2$

then the corresponding coefficient is

$$Q_{abb} = \frac{1}{2!}\cdot {}^nC_{n-(a+2b)}\,{}^{a+2b}C_{2b}\,{}^{2b}C_b \tag{59}$$

$$= \frac{n!}{2!\{n-(a+2b)\}!\,a!\,(b!)^2} \tag{60}$$

More generally, the coefficient of $r_a^u r_b^v r_c^w \cdots$ is

$$Q_{abc}^{(uvw)} = \frac{n!}{(a!)^u(b!)^v(c!)^w\cdots u!\,v!\,w!\cdots} \tag{61}$$

where

$$a > b > c \tag{62}$$

and

$$au + bv + cw + \cdots = n \tag{63}$$

Formula (59) may be found more convenient than (60) if a table of the binomial coefficients is available; for in the case here exhibited for example, formula (59) requires only 3 tabular values to be looked up, whereas formula (60) calls for 4.

It may be noted that coefficients of this form occur in certain formulae relating to seminvariants,[6] also in the theory of partitions.[7]

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[6] See for example R. Frisch, *Sur les semi-invariants et moments employés dans l'étude des distributions statistiques*, Oslo, 1926; C. C. Craig, *Metron*, 1928, vol. VII, p. 10.

[7] Dwyer, *Annals of Mathematical Statistics*, 1938, vol. 8, p. 21; vol. 9, pp. 4, 8. E. G. Olds, *Bulletin Am. Math. Soc.*, 1938, vol. 44, p. 412. H. S. Wall, *Bulletin Am. Math. Soc.*, 1938, vol. 44, p. 395; P. S. Dwyer, *Annals Math. Statistics*, 1938, vol. 9, p. 116. E. A. Cornish and R. A. Fisher, *Revue de l'Institut Internat. Statist.*, 1937, vol. 5, p. 307.