CONFIDENCE LIMITS FOR CONTINUOUS DISTRIBUTION FUNCTIONS[1]

BY A. WALD[2] AND J. WOLFOWITZ

1. Introduction. The theory of confidence limits for unknown parameters of distribution functions has been considerably developed in recent years. This theory assumes that there is given a family F of systems of n stochastic variables $X_1(\theta_1, \cdots, \theta_k), \cdots, X_n(\theta_1, \cdots, \theta_k)$ depending upon k parameters $\theta_1, \cdots, \theta_k$ and such that the distribution function of every element of F is known.

For the case $k = 1$, for example, this theory proceeds as follows:

Denote by E an n-tuple $x_1, \cdots, x_n$ of observed values of the stochastic variables $X_1(\theta), \cdots, X_n(\theta)$ of which we know only that they constitute a system which is an element of F. E can be represented as the point $x_1, \cdots, x_n$ in an n-dimensional Euclidean space. Let there be given a positive number α, $0 < \alpha < 1$. Then to each pair E, α there is constructed a $\underline{\theta}$-interval, $[\underline{\theta}(E, \alpha), \bar{\theta}(E, \alpha)]$ with the following property: If we were to draw a sample from the system $X_1(\theta), \cdots, X_n(\theta)$, the probability is exactly α that we shall get a system of observations $E = x_1, \cdots, x_n$ such that the interval corresponding to E, α will include θ (i.e., that $\underline{\theta}(E, \alpha) \leq \theta \leq \bar{\theta}(E, \alpha)$).

In this paper we do not limit ourselves to a family of systems of n stochastic variables depending upon a finite number of parameters, but consider the family G of all systems of n stochastic variables $X_1, \cdots, X_n$ subject only to the condition that $X_1, \cdots, X_n$ are independently distributed with the same continuous distribution function.

Let E be the point in an n-dimensional Euclidean space which corresponds to the observed values $x_1, \cdots, x_n$ of the n stochastic variables $X_1, \cdots, X_n$ of which we know only that they constitute an element of the family G, i.e., that they are independently distributed with the same continuous distribution function. Let us denote their distribution function by $f(x)$; the probability that $X_i < x$ is $f(x)$, $i = 1, \cdots, n$. Let α be a number such that $0 < \alpha < 1$. To each pair E, α we shall construct two functions, $l_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$, with the following property: The probability is α that, if we were to draw a sample from the system $X_1, \cdots, X_n$, we would get a system of observations $E = x_1, \cdots, x_n$ such that $f(x)$ lies entirely between $l_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$ (i.e., that $\underline{l}_{E,\alpha}(x) \leq f(x) \leq l_{E,\alpha}(x)$ for all x). We shall call $l_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$ the upper and lower confidence limits, respectively, corresponding to the confidence coefficient α.

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All the stochastic variables considered hereafter in this paper are to have continuous distribution functions.

2. A theorem on continuous distribution functions. Let $f(x)$ be the continuous distribution function of a stochastic variable X whose range is from $-\infty$ to $+\infty$. Let $\delta_1(x)$ and $\delta_2(x)$ be two functions defined for $0 \leq x \leq 1$ and satisfying the following requirements:

(a) $\delta_1(x)$ and $\delta_2(x)$ are non-negative and continuous for $0 \leq x \leq 1$.

(b) $l_1(x)$ and $l_2(x)$ are monotonically non-decreasing for all x, where

$$l_1(x) \equiv f(x) + \delta_1(f(x))$$
$$l_2(x) \equiv f(x) - \delta_2(f(x)).$$

(c) There exists a number h, such that $f(h) < 1$ and $l_1(h) = 1$.

(d) There exists a number h', such that $f(h') > 0$ and $l_2(h') = 0$.

(e) $l_1(x) \leq 1$ for all x
$l_2(x) \geq 0$ for all x

(f) $\delta_1(x) + \delta_2(x) \geq \frac{1}{n}$ for all x, where n is the number of random, independent observations of the stochastic variable X.

Let $\varphi(x)$ be the distribution function of such a system of observations, i.e., the ratio, to n, of the number of observations $<x$ is $\phi(x)$. $\varphi(x)$ is, of course, a multiple of $\frac{1}{n}$ for all x.

We shall consider the following problem:

What is the probability P that

$$l_2(x) \leq \varphi(x) \leq l_1(x) \tag{1}$$

for all x?

The reasons for restrictions (b), (c), (d), (e), and (f) on $\delta_1(x)$ and $\delta_2(x)$ are now apparent. If there exist two numbers $q_1 < q_2$, such that, for $q_1 < x < q_2$, $l_1(x) > l_1(q_2)$ and $l_1(q_1) = l_1(q_2)$, then, if we change $l_1(x)$ so that $l_1(x) = l_1(q_2)$ for $q_1 \leq x \leq q_2$, P will remain unchanged. An analogous process leads to a similar conclusion for $l_2(x)$. Hence $l_1(x)$ and $l_2(x)$ are to be monotonically non-decreasing. If there did not exist a number h or h', P would be 0. Hence requirements (c) and (d). Since $0 \leq \varphi(x) \leq 1$, there is no point to considering functions which do not satisfy (e). $\varphi(x)$ is a step-function whose saltuses are $\geq \frac{1}{n}$. If, for all x.

$$\delta_1(x) + \delta_2(x) < \frac{1}{n}$$

then $P = 0$ If there is an interval $[\beta, \gamma]$ within which $\delta_1(x) + \delta_2(x) < \frac{1}{n}$, then all samples in which one of the observed values lies in this interval are such that (1) does not hold for all x. For the sake of simplicity and because the situation described in (*f*) is the one of importance, we make the latter requirement.

It would appear that P depends upon $f(x)$, $\delta_1(x)$, $\delta_2(x)$, and n.

THEOREM: *P is independent of $f(x)$ and depends only upon $\delta_1(x)$, $\delta_2(x)$, and n.*

PROOF: Let $Y = f(X)$. Then Y is a stochastic variable distributed in the range 0 to 1 with a distribution function $\equiv x$. By this transformation $l_1(x)$ and $l_2(x)$ become respectively

$$(2) \qquad \left.\begin{aligned} l_1'(x) &= x + \delta_1(x) \\ l_2'(x) &= x - \delta_2(x) \end{aligned}\right\} \; 0 \leq x \leq 1.$$

Then P is the probability that the distribution function $\varphi(x)$ of a random sample of n of the stochastic variable Y shall be such that $l_2'(x) \leq \varphi(x) \leq l_1'(x)$ and is therefore independent of $f(x)$.

3. **Computation of** P. From the previous section it follows that, in computing P, we may confine ourselves to a stochastic variable X whose range is from 0 to 1 and whose distribution function $\equiv x$. Let $l_1(x)$ and $l_2(x)$ be the upper and lower limits, respectively, which are set for $\varphi(x)$. $l_1(x)$ and $l_2(x)$ are defined in (2), if the accents are omitted.

Consider the equations:

$$(3) \qquad l_1(x) = \frac{i}{n} \qquad (i = 1, 2, \cdots, n; 0 \leq x \leq 1).$$

If, for a certain i, the corresponding equation possesses one or more solutions in x, let a_i be the minimum of these solutions. If the first r of these equations (3) have no solutions, let

$$a_i = 0 \qquad (i = 1, \cdots, r).$$

If the i^{th}, say, of the equations

$$(4) \qquad l_2(x) = \frac{i-1}{n} \qquad (i = 1, \cdots, n; 0 \leq x \leq 1)$$

possesses one or more solutions in x, let b_i be the maximum of these. If the last $n - s$ of the equations (4) have no solutions, let

$$b_i = 1 \qquad (i = s + 1, \cdots, n).$$

Obviously

$$a_i \leq a_{i+1}, \qquad b_i \leq b_{i+1}, \qquad a_i \leq b_i .$$

From restrictions, (e) and (f) on $l_1(x)$ and $l_2(x)$, it follows that $a_1 = 0$, $b_n = 1$.

Suppose the sample $E = x_1, \cdots, x_n$ has been obtained. Arrange the x's in ascending order, thus: $x_{p_1}, x_{p_2}, \cdots, x_{p_n}$ where $x_{p_1} \leq x_{p_2} \leq \cdots \leq x_{p_n}$. Then necessary and sufficient conditions that (1) hold are:

$$a_i \leq x_{p_i} \leq b_i \qquad (i = 1, \cdots, n). \tag{5}$$

Let $P_k(t, \Delta t)$, $(k = 0, 1, \cdots, (n-1);\ a_{k+1} \leq t \leq b_{k+1})$ be the probability that a sample $E = x_1, \cdots, x_n$ shall fulfill the following conditions:

(a) $x_1 \leq x_2 \leq \cdots \leq x_{k+1}$,

(b) $x_1, \cdots, x_k$ satisfy the first k inequalities (5),

(c) $t \leq x_{k+1} \leq t + \Delta t$.

Let

$$P_k(t) = \lim_{\Delta t \to 0} \frac{P_k(t, \Delta t)}{\Delta t}.$$

Since $f(x) \equiv x$, we get easily

$$P_0(t) \equiv 1. \tag{6}$$

We shall now develop a recursion formula for $P_{k+1}(t)$. For this purpose let us consider the following composite event: The observations $x_1, \cdots, x_n$ satisfy the conditions (a), (b), and

$$t' \leq x_{k+1} \leq t' + \Delta t'$$

and

$$t \leq x_{k+2} \leq t + \Delta t.$$

If $a_{k+1} \leq t' \leq b_{k+1}$, the probability of this event is $P_k(t', \Delta t')\Delta t$. Now

$$\lim_{\substack{\Delta t' \to 0 \\ \Delta t \to 0}} \frac{P_k(t', \Delta t')\Delta t}{\Delta t' \cdot \Delta t} = P_k(t').$$

$P_k(t')$ is obviously the probability density of the bivariate distribution of t' and t. In order to obtain $P_{k+1}(t)$ we have to integrate $P_k(t')\,dt'$ over the region defined by the two inequalities

$$t' \leq t$$

$$a_{k+1} \leq t' \leq b_{k+1}.$$

Hence, omitting the now unnecessary accent, if

$$t \leq b_{k+1} \tag{7}$$

then

$$P_{k+1}(t) = \int_{a_{k+1}}^{t} P_k(t)\,dt \qquad (k = 0, 1, \cdots, (n-2)), \tag{8}$$

and if

$$t > b_{k+1} \tag{9}$$

then

$$(10) \qquad P_{k+1}(t) = \int_{a_{k+1}}^{b_{k+1}} P_k(t)\,dt \quad (k = 0, 1, 2, \cdots (n-2)).$$

Now, to obtain P, we cannot confine ourselves only to cases where $x_1 \leq x_2 \leq \cdots \leq x_n$, but have to consider all the $n!$ permutations of the n x's. Hence

$$(11) \qquad P = n! \int_{a_n}^{b_n} P_{n-1}(t)\,dt.$$

The fact that there are two forms of the recursion formula corresponding to the two possible cases (7) and (9) makes actual calculation very cumbersome for n of any considerable size. We shall therefore give an approximation formula which is considerably easier to apply to practical calculations.

4. **Computation of $\bar{P}$ and $\underline{P}$.** Let $\bar{P}$ be the probability that, for a sample of n, $l_1(x) \geq \varphi(x)$ for all x. Let $\underline{P}$ be the probability that, for a sample of n, $\varphi(x) \geq l_2(x)$ for all x.

Consider the inequalities

$$\begin{aligned} &(12) \qquad x_i \geq a_i \\ &(13) \qquad x_i \leq b_i \end{aligned} \Bigg\} \qquad (i = 1, 2, \cdots, n)$$

Let

$$\bar{P}_k(t, \Delta t), \qquad (k = 0, 1, \cdots, (n-1); t \geq a_{k+1})$$

be the probability that a sample $E = x_1, \cdots, x_n$ of the stochastic variable X should fulfill the following conditions:

(a) $x_1 \leq x_2 \leq \cdots \leq x_{k+1}$

(b) $x_1, \cdots, x_k$ satisfy the first k inequalities (12)

(c) $t \leq x_{k+1} \leq t + \Delta t$.

Let

$$\bar{P}_k(t) = \lim_{\Delta t \to 0} \frac{\bar{P}_k(t, \Delta t)}{\Delta t}.$$

Then, by an argument like that employed in the preceding section, we obtain

$$(14) \qquad \bar{P}_0(t) \equiv 1,$$

and the recursion formula

$$(15) \qquad \bar{P}_{k+1}(t) = \int_{a_{k+1}}^{t} \bar{P}_k(t)\,dt.$$

Let $\bar{P}_n(t)$ be defined formally by (15). Then, in the same way in which we obtained (11), we get

$$(16) \qquad \bar{P} = n!\,\bar{P}_n(1).$$

In the same manner we shall obtain an expression for $\underline{P}$.

Let $\underline{P}_k(t, \Delta t)$, $(k = 0, 1, \cdots (n-1); t \leq b_{n-k})$ be the probability that a sample $E = x_1, \cdots, x_n$ of the stochastic variable X should fulfill the following conditions:

(a) $x_{n-k} \leq x_{n-k+1} \leq \cdots \leq x_n$,

(b) $x_{n-k+1}, \cdots, x_n$ satisfy the last k inequalities (13),

(c) $t \leq x_{n-k} \leq t + \Delta t$.

Let

$$\underline{P}_k(t) = \lim_{\Delta t \to 0} \frac{\underline{P}_k(t, \Delta t)}{\Delta t}.$$

Then

$$\underline{P}_0(t) \equiv 1 \tag{17}$$

and by an argument very similar to that employed above,

$$\underline{P}_{k+1}(t) = \int_t^{b_{n-k}} \underline{P}_k(t)\, dt. \tag{18}$$

Let $\underline{P}_n(t)$ be defined formally by (18). Then

$$\underline{P} = n!\, \underline{P}_n(0). \tag{19}$$

The $\bar{P}_i(t)$ and $\underline{P}_i(t)$ are polynomials in t. Denote by c_j the constant term of $\bar{P}_j(t)$ and by d_j the constant term of $(-1)^j \underline{P}_j(t)$. Obviously

$$c_0 = 1 \tag{20}$$

$$d_0 = 1 \tag{21}$$

and

$$\bar{P}_i(t) = \frac{c_0}{i!} t^i + \frac{c_1}{(i-1)!} t^{i-1} + \cdots + c_{i-1} t + c_i \tag{22}$$

$$\underline{P}_i(t) = (-1)^i \left(\frac{d_0}{i!} t^i + \frac{d_1}{(i-1)!} t^{i-1} + \cdots + d_{i-1} t + d_i \right). \tag{23}$$

Since

$$\bar{P}_i(a_i) = 0, \qquad \underline{P}_i(b_{n-i+1}) = 0$$

we obtain

$$c_0 \frac{a_i^i}{i!} + c_1 \frac{a_i^{i-1}}{(i-1)!} + \cdots + c_{i-1} a_i + c_i = 0 \qquad (i = 1, 2, \cdots, n) \tag{24}$$

and

$$\frac{d_0}{i!} b_{n-i+1}^i + \frac{d_1}{(i-1)!} b_{n-i+1}^{i-1} + \cdots + d_{i-1} b_{n-i+1} + d_i = 0 \qquad (i = 1. 2. \cdots, n) \tag{25}$$

The determinant of (20) and the first j equations (24) ($j = 1, \cdots, n$) considered as equations in $c_0, c_1, \cdots, c_j$ equals 1, since all the elements of the principal diagonal are 1 and all the elements above the principal diagonal are 0. Then

$$c_i = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ a_1 & 1 & 0 & \cdots & 0 & 0 \\ \frac{a_2^2}{2!} & a_2 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{a_i^i}{i!} & \frac{a_i^{i-1}}{(i-1)!} & \frac{a_i^{i-2}}{(i-2)!} & \cdots & a_i & 0 \end{vmatrix} \tag{26}$$

$$= (-1)^i \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 \\ \frac{a_2^2}{2!} & a_2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{a_i^i}{i!} & \frac{a_i^{i-1}}{(i-1)!} & \frac{a_i^{i-2}}{(i-2)!} & \cdots & a_i \end{vmatrix}.$$

From (16) and (22) for $i = n$, we get

$$\bar{P} = c_0 + nc_1 + n(n-1)c_2 + \cdots + n(n-1)\cdots(3)(2)c_{n-1} + n!\,c_n$$

$$= \begin{vmatrix} \frac{n!}{n!} & \frac{n!}{(n-1)!} & \frac{n!}{(n-2)!} & \cdots & \frac{n!}{1!} & \frac{n!}{0!} \\ a_1 & 1 & 0 & \cdots & 0 & 0 \\ \frac{a_2^2}{2!} & a_2 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{a_n^n}{n!} & \frac{a_n^{n-1}}{(n-1)!} & \frac{a_n^{n-2}}{(n-2)!} & \cdots & a_n & 1 \end{vmatrix}. \tag{27}$$

In the same way, we obtain

$$d_i = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ b_n & 1 & 0 & \cdots & 0 & 0 \\ \frac{b_{n-1}^2}{2!} & b_{n-1} & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{b_{n-i+1}^i}{i!} & \frac{b_{n-i+1}^{i-1}}{(i-1)!} & \frac{b_{n-i+1}^{i-2}}{(i-2)!} & \cdots & b_{n-i+1} & 0 \end{vmatrix} \tag{28}$$

$$= (-1)^i \begin{vmatrix} b_n & 1 & 0 & \cdots & 0 \\ \frac{b_{n-1}^2}{2!} & b_{n-1} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{b_{n-i+1}^i}{i!} & \frac{b_{n-i+1}^{i-1}}{(i-1)!} & \frac{b_{n-i+1}^{i-2}}{(i-2)!} & \cdots & b_{n-i+1} \end{vmatrix}$$

and from (19) and (23) for $i = n$,

$$\underline{P} = (-1)^n n!\, d_n. \tag{29}$$

Perhaps if the determinants in (27) and (28) were to be simplified it might be easier to calculate $\bar{P}$ and $\underline{P}$ that way than by the recursion formulas.

5. **The approximation of P.** Let J be the probability that, for a sample of n, there exists at least one pair of numbers ω_1, ω_2, such that

$$0 \leq \omega_i \leq 1 \qquad (i = 1, 2)$$

$$\varphi(\omega_1) > l_1(\omega_1)$$
$$\varphi(\omega_2) < l_2(\omega_2).$$

Recalling the definitions of P, $\bar{P}$, and $\underline{P}$, it is obvious that

$$1 - P = (1 - \bar{P}) + (1 - \underline{P}) - J. \tag{30}$$

Now if

$$J \leq (1 - \bar{P})(1 - \underline{P}) \tag{31}$$

and $(1 - P)$ is small, the right member of (30) with J omitted furnishes an excellent approximation to $(1 - P)$. Suppose, for example, that it were desired to give upper and lower limits $l_1(x)$ and $l_2(x)$ such that $P = .95$. Choose $l_1(x)$ and $l_2(x)$ so that, for example, $\bar{P} = \underline{P} = .975$. Then P cannot differ from .95 by more than .000625. Even if

$$J \leq K(1 - \bar{P})(1 - \underline{P}) \tag{32}$$

where K is a small factor, say 10, the approximation would still be excellent. It seems very plausible that even (31) holds. However, we have not yet succeeded in obtaining a rigorous proof.

6. **The construction of confidence limits.** We now proceed to the construction of $\bar{l}_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$ which were defined in Section I of this paper.

A confidence coefficient $\alpha(0 < \alpha < 1)$ is selected to which it is desired that the confidence limits correspond. Functions $\delta_1(x)$ and $\delta_2(x)$ are chosen to be as defined in Section 2 and also to be such as to make $P = \alpha$. This can be done by application of the formulas for the evaluation of P.

The functions $\bar{l}_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$ are to be known when E and α are known.

Since α is given, $\mathbf{l}_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$ depend upon the outcome of the experiment which yields observed values of the stochastic variable X. Let $E = x_1, \cdots, x_n$ be this system of values and let $\varphi(x)$ be its distribution function. Consider the equations

$$\delta_2(\varphi(x) + \Delta_1(x)) = \Delta_1(x) \tag{33}$$

$$\delta_1(\varphi(x) - \Delta_2(x)) = \Delta_2(x). \tag{34}$$

For a fixed but arbitrary x, $-\infty < x < +\infty$, $\varphi(x)$ is known and (33) and (34) are equations in $\Delta_1(x)$ and $\Delta_2(x)$. If, for a certain x, (33) has one or more solutions, let $\epsilon_1(x)$ be the maximum of the set of solutions (for this x, of course). Similarly, if for a certain x, (34) has one or more solutions, let $\epsilon_2(x)$ be the maximum of the set of solutions.

We can now give $\mathbf{l}_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$ as follows:

For an x such that (33) has at least one solution,

$$\mathbf{l}_{E,\alpha}(x) = \varphi(x) + \epsilon_1(x). \tag{35}$$

For an x such that (33) has no solutions,

$$\mathbf{l}_{E,\alpha}(x) = 1. \tag{36}$$

For an x such that (34) has at least one solution,

$$\underline{l}_{E,\alpha}(x) = \varphi(x) - \epsilon_2(x). \tag{37}$$

For an x such that (34) has no solution,

$$\underline{l}_{E,\alpha}(x) = 0. \tag{38}$$

We recapitulate briefly the meaning of $\mathbf{l}_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$ which were defined in Section 1. These are two functions defined for $-\infty < x < +\infty$ which may be constructed as above after a confidence coefficient α has been assigned and after the outcome of the physical experiment which determines the stochastic point E is known. These functions have the following property: No matter what the distribution function $f(x)$ of each of n stochastic independent variables $X_1, \cdots, X_n$ may be, provided only that $f(x)$ is continuous and the same for each $X_1, \cdots, X_n$, the probability is exactly α that, if we were to perform the physical experiment which gives a set of values E of the stochastic system $X_1, \cdots, X_n$ and were then to construct $\mathbf{l}_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$, the inequality

$$\underline{l}_{E,\alpha}(x) \leq f(x) \leq \mathbf{l}_{E,\alpha}(x) \tag{39}$$

would hold for all x.

A less precise but more intuitive statement of the above result is as follows: If, in many experiments we were to proceed as above to construct $\mathbf{l}_{E,\alpha}(x)$ and $\underline{l}_{E,\alpha}(x)$ and then, in each instance, we were to predict that the unknown $f(x)$ (which need not be the same in all experiments) satisfies (39), the relative frequency of correct predictions would be α.

The formal proof of this result is exceedingly simple. For any continuous $f(x)$, the probability is α that

$$l_2(x) \leq \varphi(x) \leq l_1(x) \tag{40}$$

will hold for all x. This is so because of the way in which $\delta_1(x)$ and $\delta_2(x)$ were chosen. To prove the required result it would therefore be sufficient to show that, if (39) holds for all x, (40) holds for all x and conversely.

Let x be fixed but arbitrary. We shall show that

$$f(x) \leq l_{E,\alpha}(x) \tag{41}$$

implies

$$l_2(x) \leq \varphi(x) \tag{42}$$

and conversely.

If (33) has no solution, $\varphi(x) > l_2(1) \geq l_2(x)$, $l_{E,\alpha}(x) = 1$, and (41) and (42) are trivial. Assume therefore that (33) has at least one solution. For this situation, then, we have to show that

$$f(x) \leq \varphi(x) + \epsilon_1(x) \tag{43}$$

implies

$$l_2(x) \leq \varphi(x) \tag{44}$$

and conversely.

With x and hence $\varphi(x)$ and $\epsilon_1(x)$ fixed, consider the equation in x':

$$l_2(x') = \varphi(x). \tag{45}$$

Since $\varphi(x) \leq l_2(1)$, (45) has at least one solution. Let x'_m be the maximum of these solutions for a fixed x. Then from the definition of $\epsilon_1(x)$ it follows that

$$f(x'_m) - l_2(x'_m) = \epsilon_1(x), \tag{46}$$

or, on account of the definition of x'_m,

$$f(x'_m) = \varphi(x) + \epsilon_1(x). \tag{47}$$

Now, if (43) holds, $x \leq x'_m$ because of (47). Then, from the definition of x'_m and the fact that $l_2(x')$ is monotonically non-decreasing (44) follows.

If (44) holds, then $x \leq x'_m$ (by the definition of x'_m and the monotonic character of $l_2(x')$). Hence, because of (47), (43) is true. This shows the equivalence of (43) and (44).

In a similar manner, it may be shown that

$$l_{E,\alpha}(x) \leq f(x) \tag{48}$$

implies

$$\varphi(x) \leq l_1(x) \tag{49}$$

and conversely. This completes the proof.

7. **Miscellaneous remarks.** An expedient way of choosing $\delta_1(x)$ and $\delta_2(x)$ is such that, with c a constant,

$$
\begin{aligned}
x + \delta_1(x) &\equiv \min\,[x + c,\ 1] \\
x - \delta_2(x) &\equiv \max\,[x - c,\ 0].
\end{aligned}
\qquad 0 \le x \le 1 \tag{50}
$$

Tables of double entry could be constructed giving the c corresponding to specified α and n. With such tables available the construction of confidence limits would be quick and simple in practice. In this case, $\epsilon_1(x) = \epsilon_2(x) = c$.

Another expedient and plausible way of choosing $\delta_1(x)$ and $\delta_2(x)$ might be to choose them so that

$$
\begin{aligned}
x + \delta_1(x) &\equiv \min\,[px + q,\ 1] \\
x - \delta_2(x) &\equiv \max\,[p'x + q',\ 0]
\end{aligned}
\qquad 0 \le x \le 1 \tag{51}
$$

where p, p', q, and q' are constants. The actual construction of confidence limits could then be handled with dispatch if similar tables were constructed.

$$\bar{l}_{E,\alpha}(x) \quad \text{and} \quad \underline{l}_{E,\alpha}(x)$$

are, like $\varphi(x)$, step-functions. The situation may occur where, for $x = e$,

$$\lim_{(x<e),x\to e} \bar{l}_{E,\alpha}(x) < \lim_{(x>e),x\to e} \underline{l}_{E,\alpha}(x).$$

This would give a prediction, corresponding to the confidence coefficient α, that $f(x)$ is not continuous. If $f(x)$ is continuous the probability of such a situation is 0.

8. **Further problems.** Even with α fixed, the functions $\delta_1(x)$ and $\delta_2(x)$ may be chosen in many ways. Each different choice gives, in general, different confidence limits. Which is to be preferred? This very problem also arose in the theory of parameter estimation and the testing of hypotheses and gave rise to the Neyman-Pearson theory. It would be desirable to develop such a theory for the confidence limits discussed in this paper.

We have treated here only the case where $f(x)$ is continuous. A similar theory is needed for the case where $f(x)$ is not continuous.

It would be of practical value to construct tables such as those described in Section 7. The construction of tables could be greatly facilitated if the formulas for P or $\bar{P}$ and $\underline{P}$ could be simplified so as to render them more practical for calculation or else if they were to be replaced by asymptotic expansions.

9. **An example.** To illustrate the method we shall consider an example for the case of samples of size 6, i.e. $n = 6$.

Let $\delta_1(x)$ and $\delta_2(x)$ be given as follows:

$$\delta_1(x) = d \quad \text{for} \quad 0 \leq x \leq 1 - d,$$

$$\delta_1(x) = 1 - x \quad \text{for} \quad 1 - d < x \leq 1,$$

$$\delta_2(x) = x \quad \text{for} \quad 0 \leq x \leq d,$$

and

$$\delta_2(x) = d \quad \text{for} \quad d < x \leq 1.$$

Denote by $\bar{P}$ the probability that

$$\varphi(x) \leq f(x) + \delta_1[f(x)],$$

by $\underline{P}$ the probability that

$$\varphi(x) \geq f(x) - \delta_2[f(x)]$$

and by P the probability that

$$f(x) - \delta_2[f(x)] \leq \varphi(x) \leq f(x) + \delta_1[f(x)].$$

$\varphi(x)$ denotes the sample distribution and $f(x)$ denotes the population distribution.

Since $\delta_2(x) = \delta_1(1 - x)$, we obviously have

$$\bar{P} = \underline{P}.$$

Let us calculate $\bar{P} = \underline{P}$ in case $d = \frac{1}{2}$. According to (3) we have

$$a_1 = a_2 = a_3 = 0, \qquad a_4 = \tfrac{1}{6}, \qquad a_5 = \tfrac{1}{3}, \qquad a_6 = \tfrac{1}{2}.$$

According to (16)

$$\bar{P} = 6!\bar{P}_6(1)$$

where

$$\bar{P}_0(t) \equiv 1,$$

$$\bar{P}_k(t) \equiv \int_{a_k}^{t} \bar{P}_{k-1}(t)\, dt \qquad (k = 1, \cdots, 6).$$

Applying this recursion formula we get

$$\bar{P}_1(t) = t; \qquad \bar{P}_2(t) = \frac{t^2}{2}, \qquad \bar{P}_3(t) = \frac{t^3}{6},$$

$$\bar{P}_4(t) = \frac{t^4}{24} - \frac{1}{2^7 \cdot 3^5}$$

$$\bar{P}_5(t) = \frac{t^5}{120} - \frac{t}{2^7 \cdot 3^5} - \frac{11}{3^6 \cdot 2^7 \cdot 5}$$

$$\bar{P}_6(t) = \frac{t^6}{720} - \frac{t^2}{2^8 \cdot 3^5} - \frac{11t}{3^6 \cdot 2^7 \cdot 5} - \frac{11}{2^9 \cdot 3^6 \cdot 5}.$$

Hence

$$\bar{P} = \underline{P} = 6!\bar{P}_6(1) = 1 - \frac{85}{2592} = 0.967.$$

Let us now calculate $\bar{P} = \underline{P}$ in case $d = \frac{1}{3}$. We have

$$a_1 = a_2 = 0, \quad a_3 = \tfrac{1}{6}, \quad a_4 = \tfrac{1}{3}, \quad a_5 = \tfrac{1}{2} \text{ and } a_6 = \tfrac{2}{3}.$$

Applying the recursion formula we get

$$\bar{P}_0(t) = 1, \quad \bar{P}_1(t) = t, \quad \bar{P}_2(t) = \frac{t^2}{2}, \quad \bar{P}_3(t) = \frac{t^3}{6} - \frac{1}{2^4 \cdot 3^4},$$

$$\bar{P}_4(t) = \frac{t^4}{24} - \frac{t}{2^4 \cdot 3^4} - \frac{1}{2^4 \cdot 3^5},$$

$$\bar{P}_5(t) = \frac{t^5}{120} - \frac{t^2}{2^5 \cdot 3^4} - \frac{t}{2^4 \cdot 3^5} - \frac{11}{2^8 \cdot 3^5 \cdot 5},$$

$$\bar{P}_6(t) = \frac{t^6}{720} - \frac{t^3}{2^5 \cdot 3^5} - \frac{t^2}{2^5 \cdot 3^5} - \frac{11t}{2^8 \cdot 3^5 \cdot 5} - \frac{13}{2^7 \cdot 3^8 \cdot 5}.$$

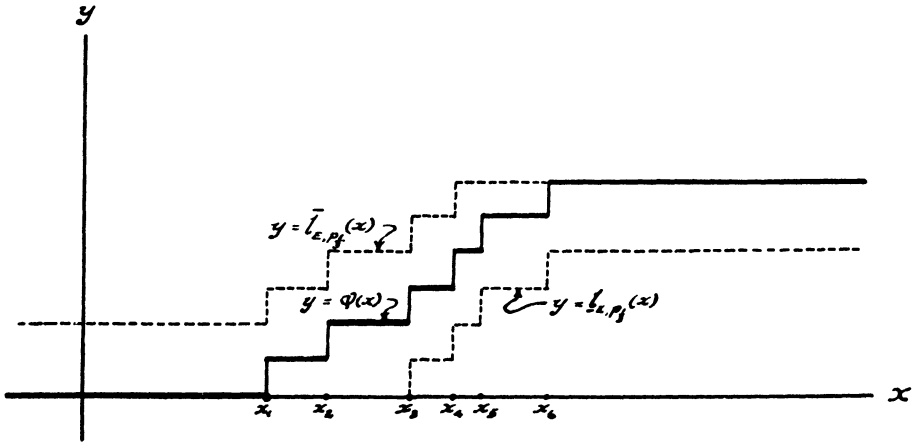

FIG. 1

Hence

$$\bar{P} = \underline{P} = 6!\bar{P}_6(1) = 1 - \frac{2483}{11664} = 0.787.$$

It is obvious that

$$1 - P = (1 - \bar{P}) + (1 - \underline{P}) - J,$$

where J denotes the probability that $\varphi(x)$ violates both limits. In case $d = \frac{1}{2}$ no $\varphi(x)$ exists which violates both limits, and therefore $J = 0$. If $d = \frac{1}{3}$,

J is not zero but so small that it can be neglected. Hence

$$P = 0.934 \quad \text{if} \quad d = \tfrac{1}{2}$$

and

$$P = 0.574 \quad \text{if} \quad d = \tfrac{1}{3}$$

P increases monotonically from 0.574 to 0.934 if d increases from $\frac{1}{3}$ to $\frac{1}{2}$. Denote by P_d the probability corresponding to d. According to (33)–(38), the confidence limits corresponding to the probability level P_d are given as follows:

$$l_{E,P_d}(x) = \varphi(x) + d \text{ if } \varphi(x) + d \leq 1,$$

$$l_{E,P_d}(x) = 1 \text{ if } \varphi(x) + d > 1,$$

$$\underline{l}_{E,P_d}(x) = \varphi(x) - d \text{ if } \varphi(x) - d \geq 0$$

and

$$\underline{l}_{E,P_d}(x) = 0 \text{ if } \varphi(x) - d < 0.$$

Substituting for d the numbers $\frac{1}{2}$ and $\frac{1}{3}$, we get the confidence limits corresponding to the probability levels 0.934 and 0.574 respectively. The upper and lower confidence limits for the *population* distribution corresponding to the probability level 0.574 are represented geometrically in Figure 1 by the upper and lower dotted broken lines for a sample of 6 having the values $x_1, x_2, \cdots x_6$. The sample distribution is represented by the solid broken line.

COLUMBIA UNIVERSITY
AND
BROOKLYN, N. Y.