TABLE 1

City	8th grade graduates	Initial approximation	First correction term	First approximation	Second correction term	Quotas	Percent sampled
Duluth, Minn.	5,500	4,000	−.02968	3,881	−.00077	3,878	70.51
Birmingham, Ala.	9,000	5,500	+.06641	6,399−	+.00148	5,343	59.37
Denver, Colo.	12,500	6,000	−.02690	5,352	−.00164	6,409	51.27
Seattle, Wash.	15,000	6,500	+.07525	6,989	+.00257	7,007	46.71
San Francisco, Cal.	21,000	8,000	+.01425	8,114	−.00341	8,086	38.50
St. Louis, Mo.	31,000	10,000	−.07349	9,265	+.00129	9,277	29.93
Total	94,000	40,000		40,000		40,000	

simply to draw contrasts between any two strata we would seek to minimize the standard error of the difference,

$$\sigma_{\Delta_{jk}} = \sqrt{S_j'^2\left(\frac{1}{n_j} - \frac{1}{N_j}\right) + S_k'^2\left(\frac{1}{n_k} - \frac{1}{N_k}\right)}$$

subject to the condition,

$$\sum_1^m n_i = n.$$

This leads to the result

$$\frac{S_j'}{n_j} = \frac{S_k'}{n_k}.$$

Thus, the number of samplings from each stratum is, for all practical purposes, proportional to the standard deviations, irrespective of the size of the various strata.

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ON THE COEFFICIENTS OF THE EXPANSION OF $X^{(n)}$

BY J. A. JOSEPH

Let us construct the following triangular arrangement of numbers:

$$\begin{array}{cccccccccccc}
 & & & & & 1 & & & & & & \\
 & & & & 1 & & 1 & & & & & \\
 & & & 1 & & 3 & & 2 & & & & \\
 & & 1 & & 6 & & 11 & & 6 & & & \\
 & 1 & & 10 & & 35 & & 50 & & 24 & & \\
 & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot \\
 & 1 & f_1(n-1) & f_2(n-1) & \cdot & \cdot & \cdot & f_{n-2}(n-1) & f_{n-1}(n-1) & & & \\
1 & f_1(n) & f_2(n) & \cdot & \cdot & \cdot & \cdot & f_{n-1}(n) & f_n(n) & & &
\end{array}$$

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where the n-th row can be constructed from the preceding row by means of the expression

$$n \cdot f_i(n-1) + f_{i+1}(n-1) = f_{i+1}(n). \tag{1}$$

For example, the element 35 in the middle of the 4th row is obtained from the two elements immediately above it, $4 \cdot 6 + 11 = 35$. (The top element is counted as the zeroth row.)

The elements in the $(n-1)$st row are the coefficients in the expansion of $x^{(n)}$ as a function of x, using the notation of the calculus of finite differences. For example,

$$\begin{aligned} x^{(4)} &= x(x-1)(x-2)(x-3) \\ &= x^4 - 6x^3 + 11x^2 - 6x. \end{aligned}$$

Of course, the signs of the coefficients alternate.

The function $f_i(n)$ is the sum of the products of the first n integers taken i at a time, namely

$$f_i(n) = \sum^{n} \epsilon_1 \epsilon_2 \cdots \epsilon_i \tag{2}$$

the summation being a symmetric function of the integers $1, 2, 3, \cdots, n$.

Equation (1) can be written as a linear, first order difference equation,

$$\begin{aligned} \Delta f_{i+1}(n-1) \equiv f_{i+1}(n) - f_{i+1}(n-1) &= n \cdot f_i(n-1) \\ f_{i+1}(n-1) &= \Delta^{-1}[n \cdot f_i(n-1)]. \end{aligned} \tag{3}$$

Since $f_0(n) = 1$ for all values of n, we can find $f_1(n)$, and consequently $f_2(n)$, and so on. Thus

$$\begin{aligned} f_1(n-1) &= \Delta^{-1} n = \frac{n^{(2)}}{2} \\ f_2(n-1) &= \Delta^{-1}\left[n \cdot \frac{n^{(2)}}{2}\right] = \frac{3n^{(4)} + 8n^{(3)}}{24} \\ f_3(n-1) &= \Delta^{-1}\left[n\left(\frac{3n^{(4)} + 8n^{(3)}}{24}\right)\right] \\ &= \frac{n^{(6)} + 8n^{(5)} + 12n^{(4)}}{48}. \end{aligned} \tag{4}$$

The following theorems are true for the "triangle":

THEOREM 1. *The sum of the elements in any* n-th *row is equal to* $(n+1)!$, *namely,*

$$\sum_{i=0}^{i} f_i(n) = (n+1)! \tag{5}$$

THEOREM 2. *The sum of the even elements of any row is equal to the sum of the odd elements, or*

(6) $$\sum_{i=0}^{n} (-1)^i f_i(n) = 0.$$

From these coefficients we can generate the Bernoulli numbers:

$$B_0 = \frac{1}{2}$$

$$B_0 - B_1 = \frac{2!}{3}$$

$$2B_0 - 3B_1 + B_2 = \frac{3!}{4}$$

(7) $$6B_0 - 11B_1 + 6B_2 - B_3 = \frac{4!}{5}$$

$$24B_0 - 50B_1 + 35B_2 - 10B_3 + B_4 = \frac{5!}{6}$$

. . . .

$$f_n(n)B_0 - f_{n-1}(n)B_1 + f_{n-2}(n)B_2 - \cdots + (-1)^n f_0 B_n = \frac{(n+1)!}{n+2}$$

Or, as a determinant:

(8) $$| B_n | = \begin{vmatrix} \frac{1}{2} & 1 & 0 & 0 & \cdots & 0 \\ \frac{2!}{3} & 1 & 1 & 0 & \cdots & 0 \\ \frac{3!}{4} & 2 & 3 & 1 & \cdots & 0 \\ \frac{4!}{5} & 6 & 11 & 6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \frac{(n+1)!}{n+2} & f_n(n) & f_{n-1}(n) & f_{n-2}(n) & \cdots & f_1(n) \end{vmatrix}$$

giving

$$B_0 = \tfrac{1}{2}, \quad B_1 = -\tfrac{1}{6}, \quad B_2 = B_4 = B_6 = \cdots = B_{2n} = 0,$$
$$B_3 = \tfrac{1}{30}, \quad B_5 = -\tfrac{1}{42}, \cdots.$$

We may now take another "triangle":

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 3 & & 1 & & & \\ & & 1 & & 6 & & 7 & & 1 & & \\ & 1 & & 10 & & 25 & & 15 & & 1 & \\ & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot \\ & 1 & F_1(n-1) & F_2(n-1) & \cdots & & \cdots & F_{n-2}(n-1) & F_{n-1}(n-1) & & \\ 1 & F_1(n) & F_2(n) & \cdots & & \cdots & & \cdots & F_{n-1}(n) & F_n(n) & \end{array}$$

where the n-th row is obtained from the preceding row by the expression

$$(n - i)F_i(n - 1) + F_{i+1}(n - 1) = F_{i+1}(n). \tag{9}$$

For example, from the third row: 1, 6, 7, 1, we obtain the fourth row: $1, 4\cdot 1 + 6 = 10, 3\cdot 6 + 7 = 25, 2\cdot 7 + 1 = 15, 1$. The following theorem is true for the $F_i(n)$:

THEOREM 3. *The elements in the* $(n - 1)$*st row are the coefficients in the expansion of* x^n *as a function of the factorials* $x^{(i)}$.

For example:

$$x^4 = x^{(4)} + 6x^{(3)} + 7x^{(2)} + x.$$

From the generating equation (9) we can obtain, as before, the form of the functions $F_0(n)$, $F_1(n)$, $\cdots$

$$\begin{aligned} \Delta F_{i+1}(n - 1) &\equiv F_{i+1}(n) - F_{i+1}(n - 1) = (n - i)F_i(n - 1) \\ F_{i+1}(n - 1) &= \Delta^{-1}[(n - i)F_i(n - 1)]. \end{aligned} \tag{10}$$

Since $F_0(n) = 1$ for all n

$$\begin{aligned} F_1(n - 1) &= \Delta^{-1}n = \frac{n^{(2)}}{2} \\ F_2(n - 1) &= \Delta^{-1}\left[(n - 1)\frac{n^{(2)}}{2}\right] = \frac{3n^{(4)} + 4n^{(3)}}{24} \\ F_3(n - 1) &= \Delta^{-1}\left[(n - 2)\frac{3n^{(4)} + 4n^{(3)}}{24}\right] \\ &= \frac{n^{(6)} + 4n^{(5)} + 2n^{(4)}}{48}. \end{aligned} \tag{11}$$

From these coefficients we can generate the numbers of Laplace (the numbers L_m below must be divided by $m!$ to yield the numbers of Laplace):

$$\begin{aligned} &L_1 = \tfrac{1}{2} \\ &L_1 + L_2 = \tfrac{1}{3} \\ &L_1 + 3L_2 + L_3 = \tfrac{1}{4} \\ &L_1 + 7L_2 + 6L_3 + L_4 = \tfrac{1}{5} \\ &L_1 + 15L_2 + 25L_3 + 10L_4 + L_5 = \tfrac{1}{6} \\ &\cdots\cdots\cdots\cdots \\ &L_1 + F_{n-1}(n)L_2 + F_{n-2}(n)L_3 + \cdots + L_{n-1} = \frac{1}{n + 1} \end{aligned} \tag{12}$$

giving

$$L_1 = \tfrac{1}{2}, \quad L_2 = -\tfrac{1}{6}, \quad L_3 = \tfrac{1}{4}, \quad L_4 = -\tfrac{19}{30}, \quad L_5 = \tfrac{9}{4}.$$

A determinantal solution is also obvious.

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