THE LENGTH OF THE CYCLES WHICH RESULT FROM THE GRADUATION OF CHANCE ELEMENTS

By Edward L. Dodd

1. **Introduction.** Eugen Slutzky[1] found that under certain conditions repeated summations of chance elements lead to a sinusoidal configuration. Generalizations were made by V. Romanovsky.[2] A more recent paper by Slutzky[3] has appeared, summarizing his original Russian memoir, and making extensions. Contributions to this subject have also been made by H. E. Jones,[4] E. J. Moulton,[5] and A. Wald.[6]

Readers who wish to get into touch with recent literature on periodicity are referred to two excellent books, that of Karl Stumpff[7] with bibliography of 319 references, and that of Herman Wold,[8] with bibliography of nearly 70 references.

In this paper, I deal with the wavy configuration resulting from a *single* application of a specified graduation formula. For this purpose, only linear operators are considered. For actual graduation it is customary to require that the sum of the coefficients or "weights" be equal to unity. But for the present purpose, this requirement is irrelevant. For example, summing and averaging are here essentially identical. The graduation formula considered may or may not be the combination of simple summations or averages. Indeed, formulas preferred by actuaries and statisticians include terms with *negative* coefficients; and thus involve an operation other than addition. F. R. Mac-

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[1] Eugen Slutzky, "Sur un théorème limite relatif aux series des quantités éventuelles." *Comptes Rendus*, Vol. 185 (1927) pp. 169–171.

[2] V. Romanovsky, "Généralisations d'un théorème de M. E. Slutzky." *Comptes Rendus*, Vol. 192(1931) pp. 718–721. "Sur la loi sinusoidale limite." *Rendiconto Circolo Mathematico di Palermo*, Vol. 56 (1932) pp. 82–111. "Sur une généralisation de la loi sinusoidale limite." Ibid., Vol. 57 (1933) pp. 130–136.

[3] E. Slutzky, "The summation of random causes as a source of cyclic processes." *Econometrica*, Vol. 5 (1937) pp. 105–146.

[4] H. E. Jones, "The theory of runs applied to time series," *Report of Third Annual Conference of Cowles Commission for Research in Economics* (1937) pp. 33–36. This abstract itself does not include reference to repetitions, mentioned by Moulton and Wald.

[5] E. J. Moulton, "The periodic function obtained by repeated accumulation of a statistical series. *American Mathematical Monthly*, Vol. 45 (1938), pp. 583–586.

[6] A. Wald, "Long cycles as a result of repeated integration." *American Mathematical Monthly*, Vol. 46 (1939), pp. 136–141.

[7] Karl Stumpff, *Grundlagen und Methoden der Periodenforschung*, Berlin, 1937, Julius Springer.

[8] Herman Wold, *A Study in the Analysis of Stationary Time Series.* Uppsala, 1938, Almqvist and Wiksells.

aulay[9] gives a chart of 24 weight diagrams. Of these only the first four are without negative coefficients.

Of course, the "waves" produced are irregular, and the difficulty of defining a cycle-length confronts us. The apparently naïve definition of a cycle-length as the average distance between successive maxima (or minima) is, I believe, worth consideration as a rough first approximation of the cycle length for graduated values delivered by formulas with negative coefficients or by those involving at least triple summations. But the cycle length thus determined is somewhat too short; for, slight undulations will occur—Slutzky calls them "ripples"—which should be eliminated if we want a cycle-length *intuitionally reasonable.* On the other hand, the cycle-length defined as the average distance between alternate intersections of the graduated curve with the base line is likely to be decidedly too long,—as illustrated by Slutzky's Figure 2 (loc. cit., p. 109) which exhibits 1,000 graduated items, with 41 marked maxima and 41 marked minima—after *elimination* of what he considers ripples—but with only 23 up-crossings and 23 down-crossings of the base line. I indicate in what follows an analytic method for removing ripples. And I describe *several methods* for obtaining a number which might be called a cycle-length. Often these seem to *cluster about a central value,* which appears to me to be a reasonable estimate of the *"length of the cycle"* created by the *specified graduation formula.*

The *theory* to be presented here assumes that the chance elements are *normally distributed* about zero with constant variance. But the data used by Slutzky came from lottery drawings, with a "rectangular" distribution; and for checking I have likewise used *rectangular distributions*; mainly, three sets of 600 numbers each, taken from the tenth figures of logarithms in the Vega Tables. It is known, however, that the average of a few elements distributed rectangularly is nearly normal. From many tests that I have made, it would seem that rectangular distributions react as if normal. To illustrate: When normal data are given a twelve-fold summation or averaging by twos, the probabilities that at a specified point there will be an upcrossing of the base line, a maximum, or an inflection from concave to convex are respectively, 0.0628, 0.106, and 0.134. These numbers multiplied by 100 give 6.28, 10.6, and 13.4, as the expected number of occurrences per hundred graduated values. Slutzky exhibits in Figure 4 (loc. cit., p. 111) 100 ordinates as the result of applying to lottery drawings 12-fold summation by twos. The figure shows 6 or 7 up-crossings, ten maxima, and 13 or 14 such inflections—in close agreement with *expectations based upon normal distributions.*

2. Derivation of Probabilities and Comparison of Actual with Expected Occurrences. A "cycle length" is first conceived of as the *reciprocal of a relative frequency or probability.* Thus, if the probability that a graduated value will

[9] F. R. Macaulay, *The Smoothing of Time Series.* Publications of the National Bureau of Economic Research, incorporated, No. 19 (1931). See pp. 77–79.

be a maximum is 0.05, we expect 5 maxima per hundred graduated values, making the "cycle length" for maxima equal to 20. It will be recalled that if p is the probability of an occurrence of an event in a single trial, then in s trials the expected number of occurrences is sp, whether the trials are *independent or not.*

It is assumed that the *data,* $x_1, x_2, \cdots$ are *independent and normally distributed about zero with constant variance* V. Then any linear function

$$y_r = a_{-m}x_{r-m} + \cdots + a_0x_r + a_1x_{r+1} + \cdots + a_mx_{r+m} \tag{1}$$

is likewise normally distributed about zero; and the variance of y_r is $V = \Sigma a_i^2$.

(a) *Probabilities When Two Conditions Are Imposed.* Consider first the "planes" $y_{r-1} = 0$ and $y_r = 0$, each in $2m + 1$ dimensions; and jointly in $2m + 2$ dimensions. They form four "dihedral" angles. Let

$$\theta = \text{angle between } y_{r-1} = 0 \quad \text{and} \quad y_r = 0, \tag{2}$$

the inside points $(x_{r-m-1}, \cdots, x_{r+m})$ being such that $y_{r-1} < 0$, and $y_r > 0$. Now, an orthogonal transformation or "rotation "leaves invariant this angle θ and also the normal probability function:

$$\text{Probability} = \text{Const.}\cdot\exp[-\Sigma x_i^2/2V]. \tag{3}$$

The angle θ may be found[10] from

$$\cos\theta = \sum_{i=-m}^{m-1} a_i a_{i+1} \Big/ \sum_{-m}^{m} a_i^2. \tag{4}$$

Let us think of the rotation which carries the intersection of the planes into the "vertical" position. To find the probability that $y_{r-1} < 0$ and $y_r > 0$, we integrate over all $2m + 2$ dimensional space which lies between the two planes in the dihedral angle thus characterized. For $2m$ of such variables, the integration extends from $-\infty$ to $+\infty$ yielding unity as a factor. If u and v are the two variables that remain, then we are to find the volume of that portion of the solid

$$z = (1/2\pi V)\exp[-(u^2 + v^2)/2V] \tag{5}$$

which lies between two vertical planes through the origin making the angle θ with each other. Then,

$$\text{Probability of up-crossing} = \theta/360°. \tag{6}$$

$$\text{Cycle length for up-crossing} = 360°/\theta. \tag{7}$$

Let

$$\Delta y_r = y_{r+1} - y_r.$$

[10] D. M. Y. Sommerville. *An introduction to the Geometry of N Dimensions.* Methuen and Co., Ltd., London, 1929. See p. 76.

Then y_r is a maximum if $\Delta y_{r-1} > 0$ and $\Delta y_r < 0$. Suppose

(8) $$\theta_1 = \text{angle between } \Delta y_{r-1} = 0 \quad \text{and} \quad \Delta y_r = 0,$$

inside points making $\Delta y_{r-1} > 0$ and $\Delta y_r < 0$. Then

(9) $$\text{Probability for maximum at } y_r = \theta_1/360°$$

(10) $$\text{Cycle length for maxima} = 360°/\theta.$$

The same equations apply to minima; since for minima we simply reverse the two foregoing inequalities, and pass to the equal "vertical" dihedral angle.

Likewise, from $\Delta^2 y_{r-1} < 0$ and $\Delta^2 y_r > 0$ we obtain an angle θ_2 such that $\theta_2/360°$ is the probability for change of inflection from concave downward to convex downward. This is also equal to the probability for change of inflection from convex to concave. Such changes of inflection have some interest on their own account and in checking; but do not seem to have any direct bearing upon the main problem under discussion here.

(b) *Probabilities When Three Conditions Are Imposed.* We consider now the *elimination of ripples.* To make y_r a maximum, two linear conditions are required. A third linear condition such as $y_r > \frac{1}{2}(y_{r-k} + y_{r+k})$, or simply $y_r > y_{r+k}$, with $k > 1$, will remove some ripples. Suppose we have given three planes through the origin,

(11) $$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= 0, \\ b_1x_1 + b_2x_2 + \cdots + b_nx_n &= 0, \\ c_1x_1 + c_2x_2 + \cdots + c_nx_n &= 0. \end{aligned}$$

The angles between these planes in pairs are

(12) $$\cos\alpha = \frac{\Sigma b_i c_i}{(\Sigma b_i^2 \cdot \Sigma c_i^2)^{\frac{1}{2}}}; \qquad \cos\beta = \frac{\Sigma a_i c_i}{(\Sigma a_i^2 \cdot \Sigma c_i^2)^{\frac{1}{2}}}; \qquad \cos\gamma = \frac{\Sigma a_i b_i}{(\Sigma a_i^2 \cdot \Sigma b_i^2)^{\frac{1}{2}}}$$

In general, eight-trihedral angles are thus formed at the origin; since we may take acute angles for α, β, and γ or their supplements. By an orthogonal transformation or "rotation about the origin" we are led to the three dimensional problem of finding the portion of a sphere lying in a specified spherical pyramid with base a spherical triangle, ABC, having spherical excess $E = A + B + C - 180°$. Now the spherical surface is 4 great circles or 720°. Hence, for a maximum, subject to an additional linear homogeneous inequality,

(13) $$\text{Probability of conditioned maximum} = E/720°$$

care having been taken to enter the proper trihedral angle.

(c) *Probabilities When Four Conditions Are Imposed.* To avoid complexities involved in the use of four intersecting planes, the following expedient was employed. Consider a set of values of y_r such that this y_r is a maximum. Among these there is theoretically a certain fraction or proportion p at which also

$y_r > y_{r+k}$, with $k > i$, and the same proportion p satisfying $y_r > y_{r-k}$. Let p' be the proportion satisfying both inequalities. Then $1 - p' \leqq 1 - p + 1 - p$ leads to

$$(14) \qquad p' \geqq 2p - 1 = p^2 - (1 - p)^2.$$

If p is fairly close to unity; a good approximation for p' would seem to be

$$(15) \qquad p' = p^2.$$

This p^2 would have been exact for p', had the graduated values been independent. That p' is here only slightly above $2p - 1$ seems likely, from the graduations that I have examined; for, the failure of one of the inequalities $y_r > y_{r+k}$ or $y_r > y_{r-k}$ was seldom accompanied by the failure of the other.

For graduations with the Spencer 21-term formula, when $k = 5$, $p = 0.936$, and $(1 - p)^2 = 0.0041$, which is fairly small. In practice, we would find in this case directly $P = 0.07125$ = probability of a maximum; $Pp = 0.0668$ = probability of a maximum at y_r with $y_r > y_{r+5}$. Then the probability Pp' of a maximum at y_r with $y_r > y_{r+5}$ and $y_r > y_{r-5}$ would have as lower bound $2Pp - P = 2(0.0668) - 0.07125 = 0.06235$.

But a closer approximation to the actual value would seem to be $Pp^2 = (Pp)^2/P = (0.0668)^2/0.07125 = 0.0626$.

This would give a cycle length of $1/0.0626 = 15.97$.

(d) *Indications from Correlation Theory.* We may also attempt to estimate a cycle length with the aid of correlation theory. If for graduation, we use a linear operator with coefficients proportional to successive ordinates of a cosine curve with a specified period, it is, I presume, fairly well known that the graduated values tend to exhibit the period of that cosine curve. Moreover, this quasi period may be induced very strongly with the use of formulas which represent "damped vibration" as shown by H. Labrouste[11] and Mrs. Labrouste. Now many standard graduation formulas have plots resembling somewhat damped vibration. Here, the central symmetrical arch leading down to the lowest negative terms on each side is usually large in comparison with the flanking waves. Now for a *strict cosine* curve of period $2k$, the coefficient of correlation of y_r and y_{r+k} is -1, at least theoretically. For *chance* material y_r, with mean zero and constant variance, the coefficient of correlation between y_r and y_{r+j} is defined in terms of expected values, thus:

$$(16) \qquad \rho_j = E(y_r y_{r+j})/E(y_r^2).$$

For graduated values, y_r, we might then *seek the value j which will make ρ_j as close to -1 as possible.* But for most common graduation formulas, ρ_j does not approximate -1. This difficulty, however, disappears if the graduation

[11] H. and Mrs. Labrouste, "Harmonic analysis by means of linear combinations of ordinates," *Terrestrial Magnetism and Atmospheric Electricity*, Vol. 41 (1936) pp. 15–28. See pp. 17, 18.

formula is properly centered. In a Fourier series, there is a constant term, which is of no significance in indicating oscillations, and is sometimes eliminated. The analogous modification for a linear graduation formula with n coefficients—of which the sum is unity—would seem to be the subtraction of $1/n$ from each coefficient, forming what I regard as a *residual*. *For this residual, negative correlations of substantial size appear.* And that j with which the numerically largest negative correlation ρ_j is associated may be considered as indicating a half-cycle length.

In the case of the Spencer 21-term formula, $j = 8$, making cycle-length $= 16$, just about identical with the cycle length for maxima at y_r with $y_r > y_{r-5}$ and $y_r > y_{r+5}$.

(e) *The period of a Closely Fitting Cosine Curve.* By another route, also, we may approach the problem of associating with a specified linear graduation a number as the length of induced cycles. We shall consider here only those formulas in which the coefficients are symmetrical with respect to the center. In equation (1), this means that $a_{-j} = a_j; j = 1, 2, \cdots, m$. Suppose now that the x's are no longer chance elements, but are the successive terms of a cosine curve with period k. That is:

$$x_r = \cos(r\theta + \alpha); \qquad \theta = 2\pi/k = 360°/k. \tag{17}$$

Then, if $a_{-j} = a_j$, it follows that

$$a_{-j}x_{r-j} + a_jx_{r+j} = 2a_j \cos j\theta \cdot \cos(r\theta + \alpha). \tag{18}$$

Then, from (1),

$$y_r = C \cos(r\theta + \alpha), \tag{19}$$

where C is independent of r. For a given graduation formula, with a's specified, this C depends upon θ, or we may say, upon $k = 360°/\theta$. *We may regard the graduation formula y_0 as "fitting best" the curve* cos $[r(360°)/k]$ *when k is so chosen as to give to C a largest value.* The presumption is that the graduation formula will curl chance data up into cycles in about the same fashion as a cosine curve to which it is closely akin. The actual period of this closely fitting cosine curve may then be taken as the quasi-period or "cycle-length" of the graduation formula.

If, relying upon intuition, we were to select a cosine curve to fit a given graduation formula, we might easily decide to disregard the small waves that usually flank the central arch, and to take a cosine curve with a span—distance between minima—equal to the span of this central arch. *In fact, this span gives, I believe, a good first estimate of the cycle length of the induced waves. This first estimate seems, however, a trifle too small.*

3. Size of Ripples, Simple Summation, Variability, and Height of Waves.

(a) *Size of Ripples.* In the use of $y_r > y_{r+k}$ to remove ripples, what integer should we take for k? The dividing line between ripples and waves is of course arbitrary. As Figure 2, p. 109, Slutzky exhibits 1,000 graduated values from two-fold summations by 10, with ripples removed. He states (p. 119): "maxima and minima with amplitudes of ten units or less being discarded as ripples." For this double summation, I find that the probability that y_r will be a maximum with $y_r > y_{r+10}$ and $y_r > y_{r-10}$ is approximately 0.0437. Among 1,000 graduated values, 43.7 such maxima would then be expected. Slutzky marks with arrows the 41 maxima which remain after the elimination of what he regards as ripples. The reciprocal of 0.0437 gives 22.9 as cycle length. Then $k = 10$ is less than half this cycle-length. For standard graduation formulas, it would seem likely that a *value of k about one-third the span of its central arch would eliminate fairly well the inconsequential fluctuations*; and likewise for graduations, with coefficients forming an arch with nearly horizontal ends, like twelve-fold summation by twos, with arch span 12. For this twelve-fold summation, I find that 0.0831 is the probability that a maximum will occur at y_r, with $y_r > y_{r+4}$ and $y_r > y_{r-4}$, giving 8.31 such maxima per hundred graduated values. Slutzky's Figure IVa shows eight such maxima, and two ripples.

(b) *Simple Summation.* I shall not discuss in detail the cycles produced by simple summation or averaging. Formulas for probability here are relatively simple. Thus, for the sum or average of n normal chance data, the probability of a maximum is 1/4, irrespective of the value of n. This appears to be about valid for rectangular data if we count the weak maxima. *A simple average of chance data, however, seems to inherit largely the chaotic character of the present data. But some sinuosity is, after all, implanted.*

(c) *Variability.* A general discussion of the variability of induced waves is beyond the scope of this paper. However, I record a numerical result. For the Spencer 21-term graduation formula, the probability of a maximum is 0.07125. Among 580 graduated values, then, 41.3 maxima would be expected. Actually, 42 maxima were found. Now, if $n - 1$ points are placed "at random" on a line of unit length—here dx is the probability that a point will fall in an interval of length dx—then the expected value[12] of the sum of the squares of the resulting n segments is $2/(n + 1)$. Thus, if 42 points are placed at random on an interval of 580 units, the expected sum of the squares of the segments is $(2/44)\ (580)^2 = 15{,}290.9$. But, if the points are placed at equal intervals, this sum of squares takes its least value, $(580)^2/43 = 7{,}823.3$. Then, $15{,}290.9 - 7{,}823.3 = 7{,}467.6$. On the other hand, the 42 maxima among *Spencer graduated values* gave segments for which the sum of the squares was 8,656.5; that is, *only* 833.2 *in excess of the above* 7,823.3, *which represents perfect periodicity for maxima.* Of course, this excess of 833.2 indicates considerable departure from perfect periodicity; but it is nowhere near the 7,467.6 to be expected from a random distribution of points. In spite of irregularities, *the sinusoidal character of graduated values is conspicuous.*

(d) *The Height or Amplitude of Induced Waves.* While our chief interest

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[12] W. Burnside, *Theory of Probability*, Cambridge University Press, 1928. See p. 71.

here lies in what is called the *length* of a cycle, a brief reference may well be made to the amplitude or *height* of the induced waves. The operation of the linear function y_r in (1) upon data with variance V yields graduated values with variance $V\Sigma a_i^2$. This particular statement does not require the assumption of normality. Thus the Spencer 21-term formula is expected to produce graduated values with a standard deviation 37.8% of that of the data. This represents some reduction, of course; but, nevertheless, the *"waves" stand out in bold relief. They are not diminutive.*

4. Data and Graduations Examined. Slutzky's graduations, exhibited in *Econometrica*, Vol. 5, have already been mentioned. Three sets of chance data were graduated by students at the University of Texas, Mr. Victor W. Pfeiffer in 1936, Mr. C. M. Tolar and Miss Anna Velma Martin, in 1938, to make tests with regard to smoothing coefficients,[13] the results appearing in M.A. theses. The data were figures in the tenth place of the Vega logarithm tables, 600 numbers in each set, as follows: Logarithms from 200 to 799; logarithms of cosines of angles from 0° to 59° 54′, by intervals of 6′; logarithms of sines of angles from 6′ to 60° by intervals of 6′. The graduation formulas used were all symmetric, with $a_{-j} = a_j$. Mr. Pfeiffer used the Spencer 21-term formula, with coefficients 1/350 of:

$$-1, -3, -5, -5, -2, 6, 18, 33, 47, 57, 60, 57, \text{etc.}$$

The other two formulas used were 11-term formulas which I devised, correct to third differences, and with fourth differences rather small, described by: $-1.13\,D^4$ and $-0.97\,D^4$, where $D = \log_e E$ (see Henderson, loc. cit., pp. 26–37); as compared with $-5.4\,D^4$ for Woolhouse 15-term, and $-12.6\,D^4$ for Spencer 21-term. These two 11-term formulas are:

(i) Averaging by twos, threes, and fours, applied to (1/12) (−4, 3, 14, 3, −4) yielding (1/288) (−4, −9, 3, 36, 73, 90, 73, 36, 3, −9, −4);

(ii) Triple averaging by threes, applied to (1/10) (−3, 2, 12, 2, −3) yielding (1/270) (−3, −7, 0, 29, 71, 90, 71, 29, 0, −7, −3). From part of the foregoing data, also, I made other graduations to check certain probabilities.

5. Cycle Lengths for the Spencer 21-Term Graduation Formula. All the various determinations of cycle length mentioned in the foregoing were applied to the Spencer formula, and to some other formulas. The results obtained for the Spencer formula seem representative, and will be given here in detail. Our main conception of a cycle-length is that it is the reciprocal of a probability or relative frequency. The probability of a minimum is the same as that of a maximum; of a down-crossing of the base line, the same as that of an up-crossing. Probabilities are listed that the representative ordinate y_r will be a maximum—

[13] Robert Henderson, *Graduation of Mortality and Other Tables*, Actuarial Society of America, New York, 1919, p. 34.

with or without further restrictions. The probability is given for an up-cross at the representative abscissa x_r. In the table which follows, a middle entry for a cycle length of 16 is obtained from the "residual" described in (d) of Section 2.

The Expected Length of Cycles Produced When Normal Chance Data Are Graduated by the Spencer 21-term Formula in Accordance with Various Specifications for the Cycle

Specification	Probability	Cycle-Length
Maximum at y_r	0.07125	14.0
Maximum at y_r with $y_r > y_{r+5}$	0.0668	15.0
Maximum at y_r with $y_r > \frac{1}{2}(y_{r-7} + y_{r+7})$	0.0657	15.2
Maximum at y_r with $y_r > y_{r+5}$, and $y_r > y_{r-5}$	0.0626	16.0
By use of "residual". (See 2(d))		16.0
Maximum at y_r with $y_r > y_{r+7}$	0.0623	16.1
Period of "best fitting" cosine curve. (See 2(e))		16.7
Maximum at y_r, $y_r > 0$. (Or: $y_r >$ Mean y_r)	0.0591	16.9
Maximum at y_r with $y_r > y_{r+7}$ and $y_r > y_{r-7}$	0.0545	18.3
Up cross at x_r	0.0469	21.3

The foregoing exhibit seems to suggest a cycle length of something like 16 *for the cycles created by the operation of the Spencer* 21-*term formula upon chance data.* This is just about the reciprocal of the probability that a maximum will occur at y_r with $y_r > y_{r+5}$ and $y_r > y_{r-5}$. If 16 is thus set up as the standard wave length, a wave of 10 units extending from x_{r-5} to x_{r+5} would not be regarded as insignificant.

Now 16 is also the interval between the outermost low coefficients, -5, in the Spencer formula. The plot of a curve through ordinates equal to the Spencer coefficients would probably make the central arch have a span of about 15. This 15 seems a little too small as a representative of cycle lengths obtained by the foregoing different methods.

From the theory set forth, 0.0626 is the probability that a maximum will occur at y_r with $y_r > y_{r+5}$ and $y_r > y_{r-5}$. Then among 580 graduated values, 36.3 such maxima would be expected. Among the Pfeiffer graduated values 38 were actually found.

6. Comparative Results of Seven Graduation Formulas. An exhibit will now be made of results obtained from seven graduation formulas. Of these, the simplest is double averaging or summation by tens, with coefficients forming a triangular arch, with a "span" which will be set down here as 18. Next in order of simplicity—avoiding negative coefficients—is 12-fold averaging by twos. Probabilities are given that a maximum will occur at a point y_r, with $y_r > y_{r-k}$ and $y_r > y_{r+k}$ for what seems to be appropriate values of k. In the five cases where graduations were made, the number of the maxima of specified character actually found are set down in line with their expected values. Also the span of the central arch is compared with cycle lengths.

Macaulay (loc. cit., pp. 73, 74) mentions favorably a 43-term formula obtained as follows: Summation by 8, by 12, doubly by 5, applied to weights: +7, −10, 0, 0, 0, 0, 0, 0, +10, 0, 0, 0, 0, 0, 0, −10, +7. This is the longest formula to be considered here.

As noted before, my theory is based upon the assumption of a normal distribution for the data. The data actually tested had a rectangular distribution. Nevertheless, close agreement was found between the expected number of maxima and the number actually found.

Results of Applying Seven Graduation Formulas to Chance Data. Comparison of the Expected Number of Conditioned Maxima with the Actual Number Found Among Graduated Values, and Comparison of Cycle Length with Span of Central Arch

(1) Graduation Formula	(2) k	(3) Probability Max. at y_r $y_r > y_{r-k}$ $y_r > y_{r+k}$	(4) Number of Graduated Items, y_r	(5) Expected Number of Such Maxima	(6) Actual Number of Such Maxima	(7) Cycle as 1/(3)	(8) Span of Central Arch
11-term by Tolar	3	0.110	590	64.9	67	9.09	8
11-term by Martin	3	0.114	590	67.3	65	8.77	8
13-term $(2)^{12}$ by Slutzky	4	0.0831	100	8.31	8	12.0	12
19-term $(10)^2$ by Slutzky	10	0.0437	1,000	43.7	41	22.9	18
21-term Spencer by Pfeiffer	5	0.0668	580	36.3	38	16.0	15
29-term Kenchington	8	0.0428				23.4	20
43-term Macaulay	9	0.0389				25.7	22

7. **Summary.** E. Slutzky found that the summing of chance data resulted in series of numbers with something like a cyclic appearance,—this being intensified by repetition of the summing. Slutzky and others have proven limit theorems. In this paper, I study the effects of a single application of a graduation process upon chance data. The most acceptable graduation formulas contain negative coefficients, and thus involve something more than repeated summations. Several methods are discussed for assigning to a given graduation formula a number as the length of the cycles it tends to produce. One of the most satisfactory of these is in line with the suggestion of Slutzky that before counting maxima, any insignificant "ripples" should be eliminated. The proba-

bility is found that a graduated value y_r should be a maximum—greater than the two adjacent values y_{r-1} and y_{r+1}—with the further condition that for some appropriate k, y_r shall be greater than y_{r-k} or y_{r+k} or both. The reciprocal of this latter probability is suggested as the length of the cycle which the given graduation tends to implant in the graduated values.

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