ON SOME PROPERTIES OF MULTIDIMENSIONAL DISTRIBUTIONS

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If, in a system of random variables $x_1, x_2, \cdots, x_n$, some variables are connected by a functional (exact) dependence, the n-dimensional distribution law has a degenerated character. In other words, in this case the probability is not distributed over the whole n-dimensional space, but is concentrated on a manifold of a smaller number of dimensions which may be called the *skeleton of the distribution.*

The character and the dimensionality of this manifold are determined by the character and the number of functional connections between the variables $x_1, x_2, \cdots, x_n$. If all these connections are linear, the skeleton will be a linear manifold (hyperplane). The investigation of the skeleton of distribution represents obviously an interest from the theoretical as well as from the practical point of view.

In the present paper we establish some criteria which enable us to determine, for any distribution possessing finite moments of the first and second order, the linear skeleton and to find the variations of the dimensionality of this manifold when the variables are subjected to a linear transformation.[1]

We also apply the obtained results to the case of a multidimensional normal distribution (generalized by H. Cramér to the case of linear dependence between variables).

§1

Let

(1) $$x_1, x_2, \cdots, x_n$$

be a system of random variables defined in the n-dimensional euclidean space R_n by the multidimensional distribution function $F(x_1, x_2, \cdots, x_n)$. The function F is defined on all Borel sets in R_n. We assume the existence of the following moments:

$$E(x_i) = \int\int \cdots \int_{R_n} x_i \, dd \cdots dF(x_1, x_2, \cdots x_n) = 0$$

$$E(x_i x_j) = \int\int \cdots \int_{R_n} x_i x_j \, dd \cdots dF(x_1, x_2, \cdots x_n) = \mu_{ij}$$

where the integrals are to be understood in the sense of Lebesgue-Radon.

[1] The questions of degeneracy of a statistical distribution were for the first time considered—from a somewhat different point of view—by R. Frisch [1].

If the variables $x_1, x_2, \cdots, x_n$ are connected by a relation of the form $C_1x_1 + C_2x_2 + \cdots + C_nx_n = 0$ ($\Sigma C^2 \neq 0$) (are linearly dependent), we call this relation a *linear bond of the distribution F*.

We shall call a system of linear bond of the distribution F *complete*, if all bonds of the system are linearly independent and every linear bond of the distribution depends linearly on the bonds of the system.

By the (linear) *decrement* of the distribution F (we denote it by $k(F)$ or simply k) we understand the number of bonds in a complete system. We may, correspondingly, call the difference between the number of variables and the decrement of the distribution the (linear) *rank* of the distribution, or the dimensionality of the linear skeleton.

The decrement (rank) is given by the following

THEOREM 1.[2] *The decrement (rank) of the distribution F is equal to the decrement*[3] *(rank) of the matrix*

$$\| \mu_{ij} \| \qquad i, j = 1, 2, \cdots n$$

of the moments of the second order of this distribution; that is

$$k(F) = k(\| \mu_{ij} \|), \qquad i, j = 1, 2, \cdots n \tag{2}$$

PROOF. Consider the form

$$v = t_1x_1 + t_2x_2 + \cdots + t_nx_n \tag{3}$$

where $t_1, t_2, \cdots, t_n$ are arbitrary real numbers, not all equal to zero. Let

$$\begin{aligned} Q^2 = E(v^2) &= \int\int \cdots \int_{R_n} (t_1x_1 + t_2x_2 + \cdots + t_nx_n)^2 dd \cdots \\ &\qquad \cdots dF(x_1, x_2, \cdots x_n) \\ &= \sum_{i,j=1}^{n} t_it_j \int\int \cdots \int_{R_n} x_ix_j dd \cdots dF(x_1, x_2, \cdots x_n) = \sum_{i,j=1}^{n} t_it_j\mu_{ij}. \end{aligned} \tag{4}$$

Q^2 is a non-negative quadratic form in the variables $t_1, t_2, \cdots, t_n$. The system of values $t_1, t_2, \cdots, t_n$, for which the expression (3) becomes zero is a double point of the form Q^2.

The coordinates of the double point can be found from the system of homogeneous equations:

$$\begin{aligned} \mu_{11}t_1 + \mu_{12}t_2 + \cdots + \mu_{1n}t_n &= 0 \\ \mu_{21}t_1 + \mu_{22}t_2 + \cdots + \mu_{2n}t_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mu_{n1}t_1 + \mu_{n2}t_2 + \cdots + \mu_{nn}t_n &= 0. \end{aligned} \tag{5}$$

[2] This theorem was proved by a different method by R. Frisch [1].

[3] By the decrement of a (rectangular) matrix we call, after B. Kagan, the difference between the number of its rows and its rank.

It is, however, known that the number of the independent double points of the form, Q^2, i.e. the number of linearly independent untrivial solutions of the system (5) is equal to the decrement of the matrix $\| \mu_{ij} \|$, $i, j = 1, 2, \cdots n$.

Consequently, there exist only $k(\| \mu_{ij} \|)$ independent linear connections between the variables $x_1, x_2, \cdots, x_n$, which proves the theorem.

Hence it follows that the variables $x_1, x_2, \cdots, x_n$ are linearly independent ($k(F) = 0$) if and only if the form Q^2 is positive definite and, consequently, the discriminant $| \mu_{ij} |$ of the form is positive.

The following two theorems may be used for determination of a complete system of linear bonds. The first of them is a special case of the second, but is stated separately in order to simplify the proof.

THEOREM 2. *If $k(F) = 1$, we obtain the linear bond of the distribution by replacing in the determinant on the left hand side of the equation*

$$\begin{vmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nn} \end{vmatrix} = 0 \tag{6}$$

the elements of one (arbitrary) row by $x_1, x_2, \cdots, x_n$ respectively.

For instance, replacing the first row, we have

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nn} \end{vmatrix} = 0. \tag{7}$$

PROOF. Since the decrement of the matrix $\| \mu_{ij} \|$, $i, j = 1, 2, \cdots n$ is equal to 1, for the unique nontrivial independent solution of the system (5) ($t_1, t_2, \cdots, t_n$) may be taken, as we know, the system of algebraical supplements of the elements of any row of the determinant $| \mu_{ij} |$, $i, j = 1, 2, \cdots n$. (Among the algebraical supplements of elements of each row there is at least one different from zero, since the algebraical supplements of corresponding elements of any pair of rows are proportional to each other.)

Hence, since $t_1x_1 + t_2x_2 + \cdots + t_nx_n = 0$, the theorem follows.

THEOREM 3. *If $k(F) > 0$, we obtain a complete system of linear bonds of the distribution F replacing in each of the k equations*

$$\begin{vmatrix} \mu_{ki} & \mu_{k,k+1} & \cdots & \mu_{kn} \\ \mu_{k+1,i} & \mu_{k+1,k+1} & \cdots & \mu_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{ni} & \mu_{n,k+1} & \cdots & \mu_{nn} \end{vmatrix} = 0, \qquad i = 1, 2, \cdots k \tag{8}$$

one (arbitrary) row of the determinant respectively by $x_i, x_{k+1}, \cdots, x_n$, where $x_{k+1}, \cdots, x_n$ are chosen in such a way that

$$\begin{vmatrix} \mu_{k+1,k+1} & \cdots & \mu_{k+1,n} \\ \cdots & \cdots & \cdots \\ \mu_{n,k+1} & \cdots & \mu_{nn} \end{vmatrix} > 0.$$

Replacing, for example, the first rows, we obtain:

(9)
$$\begin{vmatrix} x_1 & x_{k+1} & \cdots & x_n \\ \mu_{k+1,1} & \mu_{k+1,k+1} & \cdots & \mu_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n1} & \mu_{n,k+1} & \cdots & \mu_{nn} \end{vmatrix} = 0$$
$$\begin{vmatrix} x_2 & x_{k+1} & \cdots & x_n \\ \mu_{k+1,2} & \mu_{k+1,k+1} & \cdots & \mu_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n2} & \mu_{n,k+1} & \cdots & \mu_{nn} \end{vmatrix} = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\begin{vmatrix} x_k & x_{k+1} & \cdots & x_n \\ \mu_{k+1,k} & \mu_{k+1,k+1} & \cdots & \mu_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{nk} & \mu_{n,k+1} & \cdots & \mu_{nn} \end{vmatrix} = 0.$$

PROOF. The theorem is already proved for $k(F) = 1$ (Theorem 2). We have to prove it for $k(F) > 1$.

Let us in the first place show that the matrix $\| \mu_{ij} \|$, $i, j = 1, 2, \cdots n$ possesses at least one positive chief algebraical supplement of the order $n - k$.

In fact, in the system of n variables $x_1, x_2, \cdots, x_n$, connected by k independent linear relations there must exist a subsystem of $n - k$ linearly independent variables. Let these variables be $x_{k+1}, x_{k+2}, \cdots, x_n$. The determinant of the moments of the second order of this subsystem: $| \mu_{ij} |$, $i, j = k + 1, \cdots, n$ is different from zero and, by the property of Gramm's determinants, is positive. Further, each of the subsystems $x_i, x_{k+1}, \cdots, x_n$, is subjected to the distribution law $F_i(x_i, x_{k+1}, \cdots, x_n)$ with the decrement $k_i = 1$ and, consequently, by Theorem 2, the relations (9) are satisfied. (Arguing as before we find that any (not necessarily the first) row in each of the determinants in (8) may be replaced by $x_i, x_{k+1}, \cdots x_n$).

In order to show the independence of the relations (9), write the system (9) in the form:

(9′)
$$\sum_{j=1}^{n} C_{ij} x_j = 0, \qquad i = 1, 2, \cdots, k$$

and consider the matrix of its coefficients:

(10)
$$\begin{Vmatrix} C_{11} & 0 & \cdots & 0 & C_{1,k+1} & C_{1,k+2} & \cdots & C_{1n} \\ 0 & C_{22} & \cdots & 0 & C_{2,k+1} & C_{2,k+2} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & C_{kk} & C_{k,k+1} & C_{k,k+2} & \cdots & C_{kn} \end{Vmatrix}.$$

The matrices (10) have the rank k, since the determinant of order k

$$\begin{vmatrix} C_{11} & 0 & \cdots & 0 \\ 0 & C_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & C_{kk} \end{vmatrix} = C_{11} \cdot C_{22} \cdots C_{kk}$$

belonging to the matrix, is positive; this follows from

$$C_{11} = C_{22} = \cdots = C_{kk} = |\mu_{ij}| > 0, \qquad i, j = k + 1, \cdots, n.$$

Thus the independence of the relations (9) is proved and the theorem is established.

§2

In this section we consider the question of the variation of decrement of the distribution in the case when the variables are subjected to a linear transformation.

Let $x_1, x_2, \cdots, x_n$ be a system (1) of random variables and

$$\begin{aligned} u_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ u_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned} \tag{11}$$

a system of linear forms in the variables (1).

The distribution function of the variables $u_1, u_2, \cdots, u_m$ we denote by F_1, the decrement of the distribution by $k(F_1)$, or, shorter, by k_1.

The two systems of equations (11) and (9) form together the system:

$$\begin{aligned} u_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \\ 0 &= a_{m+1,1}x_1 + a_{m+1,2}x_2 + \cdots + a_{m+1,n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 &= a_{m+k,1}x_1 + a_{m+k,2}x_2 + \cdots + a_{m+k,n}x_n \end{aligned} \tag{12}$$

where the last k equations represent, in new notation, the equations (9).

We call the matrix of the coefficients of the variables in the system (12): $\| a_{ij} \|$ $i = 1, 2, \cdots, m + k; j = 1, 2, \cdots, n$, *the elongated matrix of the transformation.*

We prove the following

THEOREM 4. *The decrement of the distribution $F_1(u_1, u_2, \cdots u_m)$ is equal to the decrement of the elongated matrix of the transformation.*

$$k(F_1) = k(\| a_{ij} \|). \qquad \begin{matrix} i = 1, 2, \cdots, m + k \\ j = 1, 2, \cdots, n \end{matrix} \tag{13}$$

PROOF. Consider a system of forms in arbitrary linearly independent parameters $\xi_1, \xi_2, \cdots, \xi_n$:

$$
\begin{aligned}
v_1 &= a_{11}\xi_1 + a_{12}\xi_2 + \cdots + a_{1n}\xi_n \\
v_2 &= a_{21}\xi_1 + a_{22}\xi_2 + \cdots + a_{2n}\xi_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
v_m &= a_{m1}\xi_1 + a_{m2}\xi_2 + \cdots + a_{mn}\xi_n \\
v_{m+1} &= a_{m+1,1}\xi_1 + a_{m+1,2}\xi_2 + \cdots + a_{m+1,n}\xi_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
v_{m+k} &= a_{m+k,1}\xi_1 + a_{m+k,2}\xi_2 + \cdots + a_{m+k,n}\xi_n
\end{aligned}
\tag{14}
$$

such that the matrix of the system (14) coincides with the elongated matrix of the transformation.

For

$$v_{m+1} = 0, \qquad v_{m+2} = 0, \cdots, \qquad v_{m+k} = 0 \tag{15}$$

the system (14) reduces to the system (12).

If the decrement of the matrix of the system is equal to s, there exist only $m + k - s$ linearly independent forms v_i, and each of the remaining s forms is a linear combination of the first.

By Steinitz's theorem we can always include in a subsystem of independent forms the forms $v_{m+1}, \cdots, v_{m+k}$ (since these forms are independent).

Denoting all forms of the subsystem by $v_{s+1}, \cdots, v_m, v_{m+1}, \cdots, v_{m+k}$, let us write the s relations connecting each of the remaining forms with the forms of our subsystem in the form:

$$
\begin{aligned}
g_{11}v_1 + g_{1,s+1}v_{s+1} + \cdots + g_{1m}v_m + g_{1,m+1}v_{m+1} + \cdots + g_{1,m+k}v_{m+k} &= 0 \\
g_{22}v_2 + g_{2,s+1}v_{s+1} + \cdots + g_{2m}v_m + g_{2,m+1}v_{m+1} + \cdots + g_{2,m+k}v_{m+k} &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
g_{ss}v_s + g_{s,s+1}v_{s+1} + \cdots + g_{sm}v_m + g_{s,m+1}v_{m+1} + \cdots + g_{s,m+k}v_{m+k} &= 0
\end{aligned}
\tag{16}
$$

where $g_{11}, g_{22}, \cdots, g_{ss} \neq 0$.

Assigning to the variables in these equations the values (15) we clearly obtain s linear relations between the variables $u_1, u_2, \cdots, u_m$

$$
\begin{aligned}
g_{11}u_1 + g_{1,s+1}u_{s+1} + \cdots + g_{1m}u_m &= 0 \\
g_{22}u_2 + g_{2,s+1}u_{s+1} + \cdots + g_{2m}u_m &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
g_{ss}u_s + g_{s,s+1}u_{s+1} + \cdots + g_{sm}u_m &= 0.
\end{aligned}
\tag{17}
$$

The equations (17) are linearly independent, since the matrix of the system (17)

$$\begin{Vmatrix} g_{11} & 0 & \cdots & 0 & g_{1,s+1} & \cdots & g_{1m} \\ 0 & g_{22} & \cdots & 0 & g_{2,s+1} & \cdots & g_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & g_{ss} & g_{s,s+1} & \cdots & g_{sm} \end{Vmatrix}$$

has the rank s; this follows from the fact that it contains the determinant

$$\begin{vmatrix} g_{11} & 0 & \cdots & 0 \\ 0 & g_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & g_{ss} \end{vmatrix} = g_{11} \cdot g_{22} \cdots g_{ss}$$

of the order s, which is different from zero.

We proceed now to prove that there exists no other linear relation between the variables u_i linearly independent of the relations (17).

From the equations (17) the variables $u_1, u_2, \cdots, u_s$ may be determined as linear combinations of the variables $u_{s+1}, \cdots, u_m$ (we suppose that $m > s$, since for $m = s$ the proposition under consideration is trivial).

It is thus to be proved that the variables $u_{s+1}, \cdots, u_m$ are linearly independent (since every new linear relation between the variables $u_1, u_2, \cdots, u_m$, independent of (17) must, after corresponding substitutions, lead to a linear relation between $u_{s+1}, \cdots, u_m$).

In the equations (12) the linearly independent variables $u_{s+1}, \cdots, u_{m+k}$ are linear forms in n linearly independent parameters $\xi_1, \xi_2, \cdots, \xi_n$.

We may instead of the $\xi_1, \xi_2, \cdots, \xi_n$ take for the system of linearly independent parameters $v_{m+1}, \cdots, v_{m+k}, \xi_{k+1}, \cdots, \xi_n$ (changing the indices of the ξ in an appropriate manner), defining $\xi_1, \xi_2, \cdots, \xi_k$[4] by the system of equations

$$\begin{aligned} v_{m+1} &= a_{m+1,1}\xi_1 + \cdots + a_{m+1,n}\xi_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ v_{m+k} &= a_{m+k,1}\xi_1 + \cdots + a_{m+k,n}\xi_n \end{aligned}$$

which is always possible, since the forms $v_{m+1}, \cdots, v_{m+k}$ are independent.

Substituting the expressions obtained for the $\xi_1, \xi_2, \cdots, \xi_n$ into the forms $v_{s+1}, \cdots, v_m$, we find

$$\begin{aligned} v_{s+1} &= \varphi_{s+1}(v_{m+1}, \cdots v_{m+k}) + \psi_{s+1}(\xi_{k+1}, \cdots \xi_n) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ v_m &= \varphi_m(v_{m+1}, \cdots v_{m+k}) + \psi_m(\xi_{k+1}, \cdots \xi_n) \end{aligned} \tag{18}$$

where φ and ψ are linear forms in the corresponding arguments.

The variables $v_{s+1}, \cdots, v_m$ remain, of course, independent.

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[4] The indices of the ξ adequately chosen.

Performing in the equations (18) the substitution (15), we obtain:

$$(19)\qquad \begin{aligned} u_{s+1} &= \psi_{s+1}(\xi_{k+1}, \cdots \xi_n) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_m &= \psi_m(\xi_{k+1}, \cdots \xi_n). \end{aligned}$$

If there exists a linear dependence between the $u_{s+1}, \cdots, u_m$, we can find $\alpha_{s+1}, \cdots, \alpha_m$, not all equal to zero, such that

$$(20)\qquad \alpha_{s+1}u_{s+1} + \cdots + \alpha_m u_m = 0.$$

Multiplying the equations (18) by the coefficients $\alpha_{s+1}, \cdots, \alpha_m$ respectively, and adding, we obtain, by virtue of (19) and (20)

$$\alpha_{s+1}v_{s+1} + \cdots + \alpha_m v_m = \alpha_{s+1}\varphi_{s+1}(v_{m+1}, \cdots v_{m+k}) + \cdots + \alpha_m\varphi_m(v_{m+1}, \cdots v_{m+k})$$

i.e. the variables $v_{s+1}, \cdots, v_{m+k}$ are linearly dependent, which contradicts the assumption.

The required proposition is thus proved.

It follows that the s equations (17) form a complete system of bonds of the distribution F_1, which proves our theorem.

The moments of the second order of the distribution F_1 are connected with the moments of the distribution F by the following formulae

$$(21)\qquad \begin{aligned} v_{ij} = E(u_i u_j) &= E\left[\left(\sum_{r=1}^{n} a_{ir}x_r\right)\left(\sum_{s=1}^{n} a_{js}x_s\right)\right] \\ &= \sum_{r,s=1}^{n} a_{ir}a_{js}E(x_r x_s) = \sum_{r,s=1}^{n} a_{ir}a_{js}\mu_{rs} \qquad (i, j = 1, 2, \cdots m). \end{aligned}$$

§3

Let the normal law of distribution G (generalized by H. Cramér) be given by its multidimensional characteristic function [2], [3]:

$$(22)\qquad \begin{aligned} f(t_1, t_2, \cdots t_n) &= \int\int\cdots\int_{R_n} e^{i(t_1x_1+t_2x_2+\cdots+t_nx_n)}\, d\, d \cdots dG(x_1, x_2, \cdots x_n) \\ &= e^{-\frac{1}{2}Q^2} \end{aligned}$$

where $Q^2 = \sum_{r,s=1}^{n} c_{rs}t_rt_s$ $(c_{rs} = c_{sr})$ is a non-negative quadratic form in the real variables $t_1, t_2, \cdots, t_n$. (The integrals, as above, to be understood in the sense of Lebesgue-Radon.)

As is easily seen, the coefficients c_{rs} are the moments of the second order of the distribution G for which

$$\mu_{rs} \equiv E(x_r x_s) \equiv i^2\left[\frac{\partial^2 f}{\partial t_r\, \partial t_s}\right]_{\Sigma t^2=0} = c_{rs}.$$

If Q^2 is positive definite, we have a proper normal distribution.

If Q^2 is non-negative, the distribution G possesses a positive decrement.

The decrement and the linear bonds of the distribution may be determined from the matrix of the coefficients $\| c_{rs} \|$ $r, s = 1, 2, \cdots, n$ on ground of the general theorems of §1.

Let, as before,

$$(11) \qquad \begin{aligned} u_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ u_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

be a system of linear forms in the variables $x_1, x_2, \cdots, x_n$. We shall prove the following

THEOREM 5. *The variables $u_1, u_2, \cdots, u_m$ are subject to the generalized normal distribution law the decrement of which is equal to the decrement of the elongated matrix of the transformation*

$$\left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ a_{m+1,1} & a_{m+1,2} & \cdots & a_{m+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m+k,1} & a_{m+k,2} & \cdots & a_{m+k,n} \end{matrix} \right\|.$$

PROOF. Consider the characteristic function of the distribution $G_1(u_1, u_2, \cdots u_m)$,

$$(23) \quad f_1(t_1, t_2, \cdots t_m) = \int\int \cdots \int_{R_m} e^{i(t_1u_1+t_2u_2+\cdots+t_mu_m)}\, d\, d \cdots dG_1(u_1, u_2 \cdots, u_m).$$

Performing in this expression the substitution (11), we obtain

$$(24) \quad \begin{aligned} f_1(t_1, t_2, \cdots t_m) \\ &= \int\int \cdots \int_{R_n} e^{i\left(t_1 \sum_{j=1}^{n} a_{1j}x_j + t_2 \sum_{j=1}^{n} a_{2j}x_j + \cdots + t_m \sum_{j=1}^{n} a_{mj}x_j\right)}\, d\, d \cdots \\ &\qquad \cdots dG_1\left\{\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{ij}x_j, \cdots \sum_{j=1}^{n} a_{mj}x_j\right\} \\ &= \int\int \cdots \int_{R_n} e^{i\left(x_1 \sum_{p=1}^{m} a_{p1}t_p + x_2 \sum_{p=1}^{m} a_{p2}t_p + \cdots + x_n \sum_{p=1}^{m} a_{pn}t_p\right)}\, d\, d \\ &\qquad \cdots dG(x_1, x_2, \cdots x_n). \end{aligned}$$

($d\, d \cdots dG(x_1, x_2, \cdots x_n)$ in the expression (24) does not, in general, coincide with $d\, d \cdots dG(x_1, x_2, \cdots x_n)$ in the expression (22)).

Taking into account (22), we obtain

$$f_1 = e^{-\frac{1}{2}Q_1^2}$$

where

$$Q_1^2 = \sum_{r,s=1}^{n} \left\{ c_{rs} \left(\sum_{p=1}^{m} a_{pr} t_p \right) \left(\sum_{q=1}^{m} a_{qs} t_q \right) \right\}$$

(26)
$$= \sum_{r,s=1}^{n} \left\{ c_{rs} \sum_{p,q=1}^{m} a_{pr} q_{qs} t_p t_q \right\}$$

$$= \sum_{p,q=1}^{m} \left\{ t_p t_q \sum_{r,s=1}^{n} a_{pr} a_{qs} c_{rs} \right\} = \sum_{p,q=1}^{m} t_p t_q v_{pq} .$$

Q_1^2 is a non-negative quadratic form in $t_1, t_2, \cdots, t_m$, the coefficients of which coincide with the moments of the second order of the distribution $G_1(u_1, u_2, \cdots u_m)$.

Consequently, the distribution G_1 is a generalized normal distribution.

By Theorem 4 the decrement of the distribution G_1 is equal to the decrement of the matrix $\| a_{pr} \|$ $p = 1, 2, \cdots, m + k$; $r = 1, 2, \cdots, n$, the last k rows of which consist of the coefficients of the complete system of linear bonds of the distribution G.

Let now $x_1, x_2, \cdots, x_n$ be a system of random variables subjected to a proper Gaussian law. The density function of the distribution of the system is

$$y = Ce^{-\chi^2} = \frac{1}{(2\pi)^{\frac{n}{2}} \prod_{i=1}^{n} \sqrt{\mu_{ii}} \sqrt{R}} \exp\left[-\tfrac{1}{2}R \left\{ \sum_{i=1}^{n} R_{ii} \frac{x_i^2}{\mu_{ii}} + 2 \sum_{i} \sum_{j>i} R_{ij} \frac{x_i x_j}{\sqrt{\mu_{ii}\mu_{jj}}} \right\} \right]$$

where

$$R = \begin{vmatrix} 1 & r_{12} & \cdots & r_{1n} \\ r_{21} & 1 & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{2n} & \cdots & 1 \end{vmatrix} .$$

R_{ij} are the algebraical supplements in R, $r_{ij} = \dfrac{\mu_{ij}}{\sqrt{\mu_{ii}\mu_{jj}}}$, and χ^2 is a positive definite quadratic form in the variables $x_1, x_2, \cdots, x_n$.

Again let $u_1, u_2, \cdots, u_m$ be a system of linear forms in the variables $x_1, x_2, \cdots, x_n$

(11)
$$\begin{aligned} u_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ u_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n. \end{aligned}$$

Then from Theorem 5 follows the

COROLLARY. *The random variables* $u_1, u_2, \cdots, u_m$ *are subject to the m-dimensional properly normal distribution law of Gauss if and only if the matrix*

$$\left\| \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right\|$$

of the system of forms (11) *has the rank m.*

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