ENUMERATION AND CONSTRUCTION OF BALANCED INCOMPLETE BLOCK CONFIGURATIONS[1]

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1. **Introduction.** One of the general problems of experimental design is to avoid extraneous effects in making desired comparisons. The method employed is to use experimental materials as nearly homogeneous as possible. Such materials, however, are seldom available in large quantities. On the contrary, field soils vary in fertility from block to block, animals vary with both litter and sex, and leaves on one young plant differ from those on another. Differences between blocks, between litters and sex, and between plants, being irrelevant to the comparisons usually contemplated, must be avoided.

When the number of treatments to be compared is small, well known methods of design, such as the Latin square or randomized complete block, are available and efficient. As the number of treatments increases, however, these designs tend to become less efficient through failure to eliminate heterogeneity. Furthermore, they become cumbersome, the Latin square design requiring replicates equal in number to the treatments and the complete block design providing that each treatment occur in every block. (Blocks are defined as an assemblage of experimental units chosen to be as nearly alike as possible.)

Because of such limitations, several modifications of the complete block design have been devised. These new designs all have the common characteristic that the experimental material is divided into groups or blocks containing fewer units than the number of treatments to be compared. These more homogeneous small blocks are referred to as incomplete blocks.

It is desirable to have all comparisons between pairs of treatments made with equal accuracy. This requires of the design that every pair of treatments occur in the same block an equal number of times. Such a design is referred to as balanced. Balanced incomplete block designs can be arranged (for any given number of treatments) only for certain combinations of block size and number of replications.[2]

The construction of balanced incomplete block designs is mathematically a part of the theory of configurations. A configuration is an assemblage of elements into sets, each element occurring in the same number of sets, and each

[1] A revision of an expository paper presented under a different title at a joint meeting of the Institute of Mathematical Statistics and Biometric Section of the American Statistical Association, December 27, 1939.

[2] Numerous additional designs are available in the partially balanced incomplete blocks [3].

set containing the same number of elements. The configurations to be considered here are the complete configurations, i.e., those in which each element occurs an equal number of times in the same set with every other element. It would be useful to know, (a) what configurations (within the useful range) exist. (b) how these configurations may be constructed.

The typical requirement of the experimenter is this: "I wish to test t treatments and can use blocks of size $k(t > k)$. I should like a design which will involve as little experimental material as feasible." The designer must then determine what configuration of t elements in sets of k will satisfy the incidence relation that each pair of elements occur together in a set an equal number of times, and for which the total number of sets is a minimum. There are still many configurations which the experimenter needs but which have not as yet been constructed.

In order better to explain the construction of these balanced incomplete block designs, it is essential to specify the underlying combinatorial problems. A configuration satisfying the condition of balance can be obtained by writing down all possible combinations, b, of the t elements taken k at a time,

$$b = {}_tC_k = \frac{t!}{k!(t-k)!}.$$

The simplest example is that in which each set contains only two elements and all possible combinations of the t elements, taken in pairs, appear in the different sets. This series of pairs can be written out by the experimenter, and the method of analysis is given by Yates [20].

Let us take another example; given six elements to be taken three at a time,

$$b = {}_6C_3 = \frac{6!}{3!\,3!} = 20.$$

The 20 combinations are,

123	134	*146*	*236*	*345*
124	*135*	*156*	*245*	*346*
125	136	234	246	356
126	145	235	*256*	456.

Such unreduced designs are not necessarily economical or feasible in experimental work. It is often desirable to find some less extensive configuration. In this example half of the combinations, either those in italics or the other half, fulfill the restriction that every element occur with every other element in the same number of sets. Each pair of elements occurs twice in either group of sets. Thus, a balanced incomplete block design can be based on either half of the 20 sets as well as on all 20.

2. Combinatorial methods. Combinatorial considerations of a simple nature enable us to set up necessary conditions which balanced designs must satisfy.

We have t elements arranged in b sets of k elements each; each element occurs in r sets, and each pair of elements occurs together in a set exactly λ times. Then we must have

$$tr = bk, \qquad r(k - 1) = \lambda(t - 1).$$

The first of these equations expresses the fact that the total number of plots must be equal both to the product of elements by replications and to the product of sets by number of elements per set; the second, that the number of pairs into which a given element enters must equal λ times the remaining number of elements.

It is convenient to write

$$r = \frac{\lambda(t - 1)}{k - 1}, \qquad b = \frac{\lambda t(t - 1)}{k(k - 1)}.$$

Since the numbers t, b, r, k, λ must be integers, it is easy to obtain lower limits for any three in terms of the other two.

To give a general classification, the configurations have been divided into classes according to the value of λ. Because of the practical limitations in experimentation, table I has been expanded only to include $\lambda = 6$ and the k values from 1–14. It may be well to call attention to the fact that duplications occur in the different classes of table I. For instance in the class, $\lambda = 1$, for $k = 6$, $t = 15m + 1$, and $m = 1$, then $b = 8$, and $r = 3$. In order to construct a design, the following condition is necessary; $r \geq k$ and therefore $b \geq t$. In this example, the condition is met if b, r and λ are multiplied by 2, the resulting design is $t = 16$, $b = 16$, $r = 6$, $k \doteq 6$ and $\lambda = 2$. This configuration is a duplicate of the design in the class, $\lambda = 2$, for $k = 6$ and $m = 1$. In many of the configurations where λ is 3, 4, 5, or 6, a common factor can be cancelled from b, r and λ giving a design listed in the classes , $\lambda = 1, 2$ or 3.

It should be emphasized that the conditions under which table I was derived are necessary, but not sufficient, for the existence of a complete configuration. For example, consider the following configurations which satisfy the necessary conditions for a design.

Sub class (table I)	m	t	b	r	k	λ
$10m + 5$	1	15	21	7	5	2
$21m + 1$	1	22	22	7	7	2
$15m + 6$	2	36	42	7	6	1
$42m + 1$	1	43	43	7	7	1
$45m + 10$	2	100	110	11	10	1
$110m + 1$	1	111	111	11	11	1.

No configurations of the above specification can actually be constructed.

A selected group of configurations from table I is given in table II. Only those configurations whose k, r and λ lie within practical limits, and whose

TABLE I

Classes of Configurations

	Class											
k	$\lambda = 1$		$\lambda = 2$		$\lambda = 3$		$\lambda = 4$		$\lambda = 5$		$\lambda = 6$	
	t	t	t	t	t	t	t	t	t	t	t	t
3	$6m + 1$	$6m + 3$	$3m + 1$	$3m + 3$	$2m + 1$	$2m + 3$						
4	$12m + 1$	$12m + 4$	$6m + 1$	$6m + 4$	$4m + 1$	$4m + 4$	$3m + 1$	$3m + 4$				
5	$20m + 1$	$20m + 5$	$10m + 1$	$10m + 5$	$20m + 1$	$20m + 5$	$5m + 1$	$5m + 5$	$4m + 1$	$4m + 5$		
6	$15m + 1$	$15m + 6$	$15m + 1$	$15m + 6$	$10m + 1$	$10m + 6$	$15m + 1$	$15m + 6$	$3m + 1$	$3m + 6$	$5m + 1$	$5m + 6$
7	$42m + 1$	$42m + 7$	$21m + 1$	$21m + 7$	$14m + 1$	$14m + 7$	$21m + 1$	$21m + 7$	$42m + 1$	$42m + 7$	$7m + 1$	$7m + 7$
8	$56m + 1$	$56m + 8$	$28m + 1$	$28m + 8$	$56m + 1$	$56m + 8$	$14m + 1$	$14m + 8$	$56m + 1$	$56m + 8$	$28m + 1$	$28m + 8$
9	$72m + 1$	$72m + 9$	$36m + 1$	$36m + 9$	$24m + 1$	$24m + 9$	$18m + 1$	$18m + 9$	$72m + 1$	$72m + 9$	$12m + 1$	$12m + 9$
10	$45m + 1$	$45m + 10$	$45m + 1$	$45m + 10$	$30m + 1$	$30m + 10$	$45m + 1$	$45m + 10$	$18m + 1$	$18m + 10$	$15m + 1$	$15m + 10$
11	$110m + 1$	$110m + 11$	$55m + 1$	$55m + 11$	$110m + 1$	$110m + 11$	$55m + 1$	$55m + 11$	$22m + 1$	$22m + 11$	$55m + 1$	$55m + 11$
12	$132m + 1$	$132m + 12$	$66m + 1$	$66m + 12$	$44m + 1$	$44m + 12$	$33m + 1$	$33m + 12$	$132m + 1$	$132m + 12$	$22m + 1$	$22m + 12$
13	$156m + 1$	$156m + 13$	$78m + 1$	$78m + 13$	$52m + 1$	$52m + 13$	$39m + 1$	$39m + 13$	$156m + 1$	$156m + 13$	$26m + 1$	$26m + 13$
14	$91m + 1$	$91m + 14$	$91m + 1$	$91m + 14$	$182m + 1$	$182m + 14$	$91m + 1$	$91m + 14$	$91m + 1$	$91m + 14$	$91m + 1$	$91m + 14$
.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.

$m = 1, 2, 3, \ldots$.

existence has not been disproved, have been included. The practical limits of k, r and λ, of course, are dependent upon the conditions surrounding the experiment. We have chosen to keep k within the range 3 to 10 except for a few special configurations in which t is greater than 100, in which cases k was allowed to equal 11–14. Also r has been kept within a similar limited range. (Those configurations in table II, with an asterisk preceding t, have not been constructed.)

The above limitations upon k and r give a small, selected group of configurations. However, many others either have been constructed or are known to exist. For balanced incomplete block designs, Yates [20] gives the lower limits of r for t from 4 to 25 and k from 2 to 12 but not greater than $\frac{1}{2}t$. Fisher and Yates [8] have tabulated the configurations which are known to exist having ten or less replications including all arithmetically possible configurations the existence of which has not been disproved.

Even if the existence of a configuration has not been disproved, there still remains the difficult problem of writing out the elements which are to appear in each set. Some discussion of the structure of such configurations is presented by Fisher and Yates [8] by Yates [20, 21] by Goulden [9, 10] and by Bose [4]. Additional descriptions are to follow.

While a search of the literature revealed a number of constructed configurations, yet the general theory of their formation has received relatively little consideration. The question of combinations related to the theory of configurations which is of interest here was first set forth by Kirkman [11] in 1847. He states the problem thus: "If Q_x denote the greatest number of triads that can be formed with x symbols, so that no duad shall be twice employed, then

$$3Q_x = x(x - 1)/2 - V_x$$

if for V_x we put 0, when $x = 6m + 1$ or $6m + 3$." This gives the formula for b which was given earlier in this article. Put $x = t$ and $V_x = 0$

$$b = Q_x = \frac{t(t-1)}{3 \cdot 2} = \frac{t(t-1)}{k(k-1)}.$$

Besides the theory connected with these combinatorial problems, considerable information related to the construction of the configurations has been found in the literature on finite projective geometry, especially the geometry which applies to the theory of groups.

An extensive discussion of the $\lambda = 1$ class of configurations (as listed in table I) can be found in the literature. The theory of the formation of the configurations for the sub-class $t = 6m + 3$ has been summarized by Ball [1]. This is the Kirkman "school-girl problem" for which Eckenstein [7] lists 48 papers and 5 books written during the years 1847–1911 dealing with this subject. The problem was first published in the Lady's and Gentleman's Diary for 1850 [12]. It is usually stated that "a schoolmistress was in the habit of taking her girls for a daily walk. The girls were fifteen in number, and were arranged in five rows of three each, so that each girl might have two companions. The problem is to dispose of them so that for seven consecutive days no girl will walk with any of her school-fellows in any triplet more than once." For this particular subclass ($t = 6m + 3$, $k = 3$), this type of configuration has been shown to exist

TABLE II

Selected Group of Configurations

(Balanced Incomplete Block Designs)

t	b	r	k	λ	t	b	r	k	λ
7	7	3	3	1 Y.S.[1]	*25	50	8	4	1
7	7	4	4	2	25	30	6	5	1
8	14	7	4	3	25	15 + 15	3	5	1 L.S.
9	12	4	3	1	*25	25	9	9	3
9	6 + 6	2	3	1 L.S.[2]	28	63	9	4	1
9	18	8	4	3	28	36	9	7	2
9	18	10	5	5	*29	29	8	8	2
9	12	8	6	5	31	31	6	6	1 Y.S.
10	30	9	3	2	*31	31	10	10	3
10	15	6	4	2	*36	45	10	8	2
10	18	9	5	4	37	37	9	9	2
10	15	9	6	5	*41	82	10	5	1
11	11	5	5	2	*46	69	9	6	1
11	11	6	6	3	*46	46	10	10	2
13	26	6	3	1	49	56	8	7	1
13	13	4	4	1 Y.S.	49	28 + 28	4	7	1 L.S.
13	13	9	9	6	*51	85	10	6	1
15	35	7	3	1	57	57	8	8	1 Y.S.
15	15	7	7	3	64	72	9	8	1
15	15	8	8	4	64	72 + 72	9	8	2 L.S.
16	20	5	4	1	73	73	9	9	1 Y.S.
16	20 + 20	5	4	2 L.S.	81	90	10	9	1
16	16	6	6	2	81	45 + 45	5	9	1 L.S.
16	16	10	10	6	91	91	10	10	1 Y.S.
19	57	9	3	1	121	132	12	11	1
19	19	9	9	4	121	66 + 66	6	11	1 L.S.
19	19	10	10	5	133	133	12	12	1 Y.S.
21	70	10	3	1	169	182	14	13	1
21	21	5	5	1 Y.S.	169	91 + 91	7	13	1 L.S.
*21	28	8	6	2	183	183	14	14	1 Y.S.
*21	30	10	7	3					

* Have not been constructed.

[1] Youden squares.

[2] Lattice squares.

for every possible value of t. Most of the solutions were worked by H. E. Dudeney and O. Eckenstein. They are given by Ball [1] for all t's less than 100, that is, for $t = 9$, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93 and 99. Ball describes several methods of constructing such configurations, as cycles, combinations of cycles, scalene triangles inscribed in the circle, focal and analytical methods. As an illlustration of the school-girl problem, the construction of the configuration for $t = 9$, $b = 12$, $r = 4$, $k = 3$ and $\lambda = 1$ will be shown. Scalene triangles are inscribed in a circle with certain specifications (to be fulfilled) giving the three sets of triplets for the first day as follows,

Set	Group I		
(1)	k	1	5
(2)	3	4	6
(3)	7	8	2.

By rotation or by cyclic substitution the other three groups are secured:

Set	Group II				Group III				Group IV		
(4)	k	2	6	(7)	k	3	7	(10)	k	4	8
(5)	4	5	7	(8)	5	6	8	(11)	6	7	1
(6)	8	1	3,	(9)	1	2	4,	(12)	2	3	5.

Then placing $k = 9$, we have the configuration for $t = 9$, $b = 12$, and $r = 4$. Note that in the school-girl problem the sets are grouped into complete replications of the elements. This problem of 9 girls taken 3 at a time has been subjected to an exhaustive examination. There are 840 arrangements but only one fundamental solution. In the case of 15 girls, the number of fundamental solutions according to Mulden [14] and Cole [6], is seven. Ball mentions the Kirkman problem in quartets which is the sub-class $t = 12m + 4$, for $k = 4$. He states that this has been solved for cases where m does not exceed 49. He also states, "I conjecture that similar methods are applicable to corresponding problems about quintets, sextets, etc."

Before leaving the school-girl problem, an illustration will be given of $t = 28$, $b = 63$, $r = 9$, $k = 4$ and $\lambda = 1$. The following framework was set up by Dr. C. P. Winsor using suggestions from Netto [15].

k	a	b	c
a_1	a_8	b_3	b_6
a_2	a_7	b_1	b_8
a_3	a_6	c_4	c_5
a_4	a_5	c_1	c_8
b_2	b_7	c_3	c_6
b_4	b_5	c_2	c_7.

a, b and c each have every internal difference once and only once; and each pair a-b, a-c and b-c must have every external difference once and only once. The nine groups are given in table III. The cyclic substitution is within three sets, a, b and c. That is,

in group I, $a = 1,\ a_1 = 2,\ a_2 = 3,\ \cdots,\ a_8 = 9;$
in group II, $a = 2,\ a_1 = 3,\ a_2 = 4,\ \cdots,\ a_8 = 1;$
in group III, $a = 3,\ a_1 = 4,\ a_2 = 5,\ \cdots,\ a_8 = 2;$
etc.

Netto [15] discusses t elements in sets of k, every set of 2 elements to occur together in a set exactly λ times. He deals with $\lambda = 1$, and gives a discussion of both sub-classes when $k = 3$, that is, for $t = 6m + 1$ and $t = 6m + 3$. Reiss [16] and Moore [13] have proved that configurations can be constructed for all values of t if $k = 3$. This is the type of information which is valuable in answering the first question in the introduction of this article; "what configurations exist?" Carmichael [5] mentions the quadruple systems $6m + 2$ and $6m + 4$ and states that the general problem of their existence appears not to have been solved. Also for the higher values of k there seems to be very little known of any generality, but it is known that for $k > 3$ there are certain configurations which are not possible.

TABLE III

Configuration for $t = 28,\ b = 63,\ r = 9,\ k = 4,\ \lambda = 1$

				Group I				Group II				Group III				Group IV			
k	a	b	c	28	1	10	19	28	2	11	20	28	3	12	21	28	4	13	22
a_1	a_8	b_3	b_6	2	9	13	16	3	1	14	17	4	2	15	18	5	3	16	10
a_2	a_7	b_1	b_8	3	8	11	18	4	9	12	10	5	1	13	11	6	2	14	12
a_3	a_6	c_4	c_5	4	7	23	24	5	8	24	25	6	9	25	26	7	1	26	27
a_4	a_5	c_1	c_8	5	6	20	27	6	7	21	19	7	8	22	20	8	9	23	21
b_2	b_7	c_3	c_6	12	17	22	25	13	18	23	26	14	10	24	27	15	11	25	19
b_4	b_5	c_2	c_7	14	15	21	26	15	16	22	27	16	17	23	19	17	18	24	20

Group V				Group VI				Group VII				Group VIII				Group IX			
28	5	14	23	28	6	15	24	28	7	16	25	28	8	17	26	28	9	18	27
6	4	17	11	7	5	18	12	8	6	10	13	9	7	11	14	1	8	12	15
7	3	15	13	8	4	16	14	9	5	17	15	1	6	18	16	2	7	10	17
8	2	27	19	9	3	19	20	1	4	20	21	2	5	21	22	3	6	22	23
9	1	24	22	1	2	25	23	2	3	26	24	3	4	27	25	4	5	19	26
16	12	26	20	17	13	27	21	18	14	19	22	10	15	20	23	11	16	21	24
18	10	25	21	10	11	26	22	11	12	27	23	12	13	19	24	13	14	20	25

3. **The method of geometrical configuration.** Another aid in the construction of balanced incomplete block designs is found in some of the finite projective geometries. These are described by Carmichael [5]. A tactical configuration of rank two is defined as a combination of l elements into m sets, each set containing λ distinct elements, and each element occurring in μ distinct sets,

$l \;=\; (t) \;=$ number of points in the geometry,
$m \;=\; (b) \;=$ number of lines,
$\lambda \;=\; (k) \;=$ number of points,
$\mu \;=\; (r) \;=$ number of lines on a point.

The series of finite projective geometries $PG(\kappa,\ p^n)$ for $\kappa > 1$ furnishes a certain infinite class of these tactical configurations. The following list gives those which have been incorporated in the list (table II) of useful balanced incomplete block designs.

Two dimensional space, $PG(2,\ p^n)$

p^n	$l(t)$	$m(b)$	$\lambda(k)$	$\mu(r)$
2	7	7	3	3
3	13	13	4	4
2^2	21	21	5	5
5	31	31	6	6
7	57	57	8	8
2^3	73	73	9	9
3^2	91	91	10	10
11	133	133	12	12
13	183	183	14	14.

Three dimensional space, $PG(3,\ p^n)$

p^n	l	m	λ	μ
2	15	35	7	3.

From the Euclidean geometry $EG(\kappa,\ p^n)$ for $\kappa > 1$ other tactical configurations can be constructed. These are formed from the $PG(\kappa,\ p^n)$ by omitting a given line from the two dimensional space and a plane from the three dimensional space configurations. Some of the resulting designs are:

Two dimensional space, $EG(2,\ p^n)$

p^n	l	m	λ	μ
2	4	6	3	2
3	9	12	4	3
2^2	16	20	5	4
5	25	30	6	5
7	49	56	8	7
2^3	64	72	9	8
3^2	81	90	10	9
11	121	132	12	11
13	169	182	14	13.

Methods are available for constructing the two dimensional space $PG(\kappa, p^n)$ and the corresponding $EG(\kappa, p^n)$ configurations where p is a prime number. This being true, we can also construct the completely orthogonalized squares from the $EG(\kappa, p^n)$ geometry. The reverse situation in which these configurations are constructed by using the completely orthogonalized squares is to be illustrated. These squares consist of superimposed Latin squares, fulfilling the condition that each number from the second Latin square occurs once and only once with each number in the first Latin square. As an example take the two Latin squares:

Latin Square I			Latin Square II		
1	2	3	1	3	2
2	3	1	2	1	3
3	1	2,	3	2	1.

Superimpose square II upon square I to get the completely orthogonalized 3 x 3 square,

11	23	32
22	31	13
33	12	21.

The first number in each cell is a value from square I; the second number in each cell is from square II. Note that the numbers in the second place in each cell occur once and only once with each of the first numbers, that is 1-1, 1-3, and 1-2. The completely orthogonalized squares have been proven to exist for all prime numbers and for powers of prime numbers. The solution of this problem was secured independently by Bose [2] and by Stevens [18]. Those of sides 2, 2^2, 2^3, 2^4, 2^5, 2^6, 3, 3^2, 3^3, 3^4, 5, 5^2, 5^3, 7, 7^2, 11 and 13 have been given.

The completely orthogonalized 3 x 3 square may be used to construct

11 *1*	23 *4*	32 *7*
22 *2*	31 *5*	13 *8*
33 *3*	12 *6*	21 *9*

a balanced incomplete block design. The italic numbers, which follow the cell numbers, designate the 9 elements which are to be arranged in four groups of three sets. Group I is formed by placing the elements from each row into separate sets, in group II the elements from the three columns are placed in three sets; in group III the first set (7) consists of the elements which follow 1 in the first place in the cells, set (8) consists of the elements which follow 2 in the first place in the cells; and group IV is assembled in the same way as group III except the numbers in the second place in the cells are used to select the elements for each set. Thus we have the configuration:

Set	Group I (rows)				Group II (columns)				Group III (first place)				Group IV (second place)		
(1)	1	4	7	(4)	1	2	3	(7)	1	6	8	(10)	1	5	9
(2)	2	5	8	(5)	4	5	6	(8)	2	4	9	(11)	2	6	7
(3)	3	6	9	(6)	7	8	9	(9)	3	5	7	(12)	3	4	8

In the 12 sets of 3 elements, each of the 9 elements occurs with every other element once and only once in a set.

This is an illustration of one series of configurations which can be constructed with the aid of the completely orthogonalized squares. These are the $EG(\kappa, p^n)$ in two dimensional space when $\kappa = 2$ and $p^n = 2, 3, 2^2, 5, 7, 2^3, 3^2, 11, 13, \ldots$ The $PG(\kappa, p^n)$ configurations can be written by adding $(k + 1)$ elements to the previous group of configurations. For example, the elements 10, 11, 12 and 13 may be added to the groups, one to each group. That is, 10 is added to each set in group I, 11 is added to each set in group II, 12 to group III and 13 to group IV. An additional set must be added to include these four new elements. A configuration for $t = 13, b = 13, k = 4, r = 4$ and $\lambda = 1$ results.

Set																			
(1)	1	4	7	10	(4)	1	2	3	11	(7)	1	6	8	12	(10)	1	5	9	13
(2)	2	5	8	10	(5)	4	5	6	11	(8)	2	4	9	12	(11)	2	6	7	13
(3)	3	6	9	10	(6)	7	8	9	11	(9)	3	5	7	12	(12)	3	4	8	13
															(13)	10	11	12	13.

The 13 sets are made up of 4 elements each. These designs are symmetrical for sets and elements, that is, every pair of elements occurs together in the same number of sets, also, every pair of sets has the same number of elements in common. Discussion of the construction of these designs with illustrations are given in references [20, 8, 9] and [19].

In the $PG(\kappa, p^n)$ series of designs, as constructed by means of completely orthogonalized squares, the sets cannot be arranged in replication groups. However, these configurations can be arranged in Youden squares [22] in which all the sets are placed side by side and all the elements in a single row form a complete replication. This method of arrangement has been of considerable value in experimentation with plants. The Youden squares are the $PG(\kappa, p^n)$ when $\kappa = 2$. Singer [17] gives a partial list of the (reduced) perfect difference sets (table IV), only a single set for each p^n. The number of distinct perfect difference sets (or the number of distinct perfect partitions) for a given p^n is equal to $\varphi(q)/3n$. Since each perfect difference set can be paired with its inverse, the number is even.

The construction of one of the Youden squares from its perfect difference set will be illustrated. Consider $p^n = 3$ then $q = p^{2n} + p^n + 1 = 3^2 + 3 + 1 = 13$. There are two perfect difference sets with their inverses for $q = 13$. One perfect difference set is 0, 1, 3, 9 which has the perfect partition 1, 2, 6, 4 which will add in succession to each number from 1 to and including 13, and also 1, 2, 6, 4 add to 13. The elements of the perfect difference set are put in set (1) except that 13 replaces 0. Set (2) is secured by a one-step cyclic substitution, 1 for 13, 2 for 1, 4 for 3 and 10 for 9. This process is continued until there are thirteen sets. If the substitution is applied to set (13), the elements in set (1) are secured.

		Set												
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
Replica-	A	13	1	2	3	4	5	6	7	8	9	10	11	12
tion	B	1	2	3	4	5	6	7	8	9	10	11	12	13
	C	3	4	5	6	7	8	9	10	11	12	13	1	2
	D	9	10	11	12	13	1	2	3	4	5	6	7	8.

This is the Youden square for $t = 13, b = 13, r = 4, k = 4$, and $\lambda = 1$. The elements in each row form a complete replication.

TABLE IV

Singer's list of perfect difference sets

p^n	q	$\frac{\varphi(q)}{3^n}$	Perfect difference set															
2	7	2	0	1	3													
2^2	21	2	0	1	4	14	16											
2^3	73	8	0	1	3	7	15	31	36	54	63							
2^4	273	12	0	1	3	7	15	31	63	90	116	127	136	181	194	204	233	238	255
3	13	4	0	1	3	9												
3^2	91	12	0	1	3	9	27	49	56	61	77	81						
5	31	10	0	1	3	8	12	18										
7	57	12	0	1	3	13	32	36	43	52								
11	133	36	0	1	3	12	20	34	38	81	88	94	104	109				
13	183	40	0	1	3	16	23	28	42	76	82	86	119	137	154	175		

$$t = q = p^{2n} + p^n + 1$$

A third series of configurations, called Lattice squares or quasi-Latin squares [21] can be constructed by using the completely orthogonalized squares. The groups of sets on page 78 are taken in pairs. For each pair a square is constructed having its rows formed by the sets of one group and its columns by the sets of another group. For example, square I below is made so that the sets of group I form the rows and the sets of group II form the columns. Square II is the combination of groups III and IV.

Square I

1	4	7
2	5	8
3	6	9

Square II

1	6	8
9	2	4
5	7	3

In this lattice square each pair of elements occurs together once only in either a row or a column of either one of the squares. Also, every element occurs with every other element once in one column and one row from each square.

A device known as "complements" gives several configurations. From an arrangement having $k \neq \frac{1}{2}t$, a second one can be obtained for the same number of elements, in sets of $t - k$ units. This is done by replacing each set by its complement, that is, by a set containing all the elements missing from the original set. An illustration follows:

$t = 7, \; b = 7$
$r = 3, \; k = 3$
$\lambda = 1$

Set			
(1)	1	2	4
(2)	2	3	5
(3)	3	4	6
(4)	4	5	7
(5)	5	6	1
(6)	6	7	2
(7)	7	1	3,

$t = 7, \; b = 7$
$r = 4, \; k = 4$
$\lambda = 2$

Set				
(1)	3	5	6	7
(2)	1	4	6	7
(3)	1	2	5	7
(4)	1	2	3	6
(5)	2	3	4	7
(6)	1	3	4	5
(7)	2	4	5	6.

While the triple systems, quadruple systems, etc., which have been considered by some mathematicians, do furnish designs meeting the balance requirements, they are usually not suitable for experimental purposes. A quadruple system requires that every possible triple of elements occur once and only once together in a block. Since we need only every pair together once ($\lambda = 1$) or more, only the triple systems are generally useful.

4. **Summary.** The mathematical theory of configuration has been helpful in the construction of the balanced incomplete block designs. It would be useful to know (a) what configurations (within the useful range) exist, (b) how these configurations may be constructed. In table I the configurations have been classified according to the value of λ, while in table II configurations within a useful range have been listed. Of the designs in this table which have not been constructed, some are known to exist. Those aids which have been used in the construction of the balanced incomplete block designs have been briefly discussed.

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