A LEAST SQUARES ACCUMULATION THEOREM

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The following simple least squares theorem does not seem to have been mentioned in the literature, and has at least one practical application.

If $A^*(x)$ and $B^*(x)$ are polynomials of the same degree which are least squares representations of the functions $A(x)$ and $B(x)$ respectively, for the values $x_1, x_2, x_3, \cdots, x_p$, then

$$\sum_{t=1}^{p} A^*(x_t)B(x_t) = \sum_{t=1}^{p} A(x_t)B^*(x_t) = \sum_{t=1}^{p} A^*(x_t)B^*(x_t). \tag{1}$$

To prove the theorem let

$$A^*(x) = \sum_{i=0}^{m} a_i x^i \tag{2}$$

and

$$B^*(x) = \sum_{j=0}^{n} b_j x^j. \tag{3}$$

Then the normal equations for the determination of a_i and b_j are

$$\sum_{i=0}^{m} a_i s_{i+k} = \sum_{t=1}^{p} x_t^k A(x_t), \qquad k = 0, 1, 2, \cdots, m, \tag{4}$$

and

$$\sum_{j=0}^{n} b_j s_{j+h} = \sum_{t=1}^{p} x_t^h B(x_t), \qquad h = 0, 1, 2, \cdots, n, \tag{5}$$

where $s_r = \sum_{t=1}^{p} x_t^r$. Hence, by (2) and (5)

$$\begin{aligned} \sum_{t=1}^{p} A^*(x_t)B(x_t) &= \sum_{t=1}^{p} \left[\sum_{i=0}^{m} a_i x_t^i \right] B(x_t) \\ &= \sum_{i=0}^{m} a_i \sum_{t=1}^{p} x_t^i B(x_t) \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} a_i b_j s_{j+i} \quad \text{if} \quad n \geqq m, \\ &= \sum_{t=1}^{p} A^*(x_t)B^*(x_t) \quad \text{if} \quad n \geqq m. \end{aligned} \tag{6}$$

Similarly it can be shown that

$$\sum_{t=1}^{p} A(x_t)B^*(x_t) = \sum_{t=1}^{p} A^*(x_t)B^*(x_t) \quad \text{if} \quad m \geqq n. \tag{7}$$

Combining (6) and (7) we have

$$(8)\qquad \sum_{t=1}^{p} A^*(x_t)B(x_t) = \sum_{t=1}^{p} A(x_t)B^*(x_t) = \sum_{t=1}^{p} A^*(x_t)B^*(x_t) \quad \text{if} \quad m = n.$$

In the particular case $A(x) = B(x)$, equation (8) gives the interesting result

$$(9)\qquad \sum_{t=1}^{p} A^*(x_t)[A(x_t) - A^*(x_t)] = 0.$$

An obvious extension of equation (6) is

$$(10)\qquad \sum_{t=1}^{p} x_t^q A^*(x_t)B(x_t) = \sum_{t=1}^{p} x_t^q A^*(x_t)B^*(x_t), \quad \text{if} \quad n \geqq m + q,$$

where q is a positive integer.

A practical application of (8) has been made by one large insurance company in the case $m = n = 1$. Suppose that $A(x)$ represents an annual payment made x years ago and is an approximately linear function, and that $B(x)$ represents a compound interest function. Then, even if $B(x)$ is not a linear function, we may write approximately

$$(11)\qquad \begin{aligned} \sum_{x=1}^{p} A(x)B(x) &\cong \sum_{x=1}^{p} A(x)B^*(x) \\ &\cong \sum_{x=1}^{p} A(x)(b_0 + b_1 x) \\ &\cong b_0 \sum_{x=1}^{p} A(x) + b_1 \sum_{x=1}^{p} xA(x) \\ &\cong b_0 \sum_{x=1}^{p} A(x) + b_1 \sum_{x=1}^{p} \sum_{y=x}^{p} A(y). \end{aligned}$$

Thus if a year-by-year record is kept of the annual payments $A(x)$, the sum $\sum_{x=1}^{p} A(x)$, and the double sum $\sum_{x=1}^{p} \sum_{y=x}^{p} A(y)$, and if b_0 and b_1 are tabulated functions of p, equation (11) affords a convenient method of evaluating $\sum_{x=1}^{p} A(x)B(x)$ approximately.

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