approximation (for $n = 3$) by setting $q = 2$; the significance test for $n = 3$ then becomes,

$$P(L_{s3} < L'_{s3}) = \tfrac{1}{2}L'^{N-4}_{s3}[(N - 2) - (N - 4)L'^{2}_{s3}]. \tag{25}$$

Probably similar approximations can be found for other values of n. It is a pleasure to acknowledge the helpful comments and advice which I received from Mr. A. M. Mood of Princeton. Recognition is also due Mr. Wallace Brey, a student assistant under the National Youth Administration, who aided in the computations.

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A SIMPLE SAMPLING EXPERIMENT ON CONFIDENCE INTERVALS

By S. Kullback and A. Frankel

1. Introduction. In order to illustrate some of the notions of the theory of confidence or fiducial limits in connection with a course in Statistical Inference at the George Washington University, we had the class carry out certain simple experiments, following a suggestion in one of Neyman's papers on Statistical Estimation [1]. In the belief that the experimental data may be of interest to others, we present the results herein.

2. The problem. We consider the problem of estimating the range θ of a rectangular population defined by $p(x, \theta)\,dx = dx/\theta$, $0 \leqq x \leqq \theta$ and in particular, for simplicity, we limit ourselves to samples of two and four. We consider three possible approaches to the problem, viz., by using (a) the sample range (b) the sample average or total (c) the larger (largest) sample value. Let us consider each in turn.

(*a*) *Sample range.* Wilks [2] has shown that for samples of n and confidence coefficient $1 - \alpha$, the confidence or fiducial limits for the population range θ are given by r and r/ψ_α, where r is the sample range and ψ_α is determined by

$$\psi_\alpha^{n-1}[n - (n - 1)\psi_\alpha] = \alpha. \tag{1}$$

For $n = 2$, $\alpha = 0.19$ and $n = 4$, $\alpha = 0.1792$, (1) yields $\psi_\alpha = 0.1$ and $\psi_\alpha = 0.4$ respectively. Accordingly, for samples of two with confidence coefficient

$1 - \alpha = 0.81$, and for samples of four with confidence coefficient $1 - \alpha = 0.8208$, the confidence interval is respectively given by

$$(2) \qquad (r, 10r) \quad \text{and} \quad (r, 2.5r).$$

The length, λ_r, of the confidence interval is respectively $9r$ and $1.5r$. Using the distribution of r, $n(n - 1)(\theta - r)r^{n-2}/\theta^n$, we have for samples of two: $E(\lambda_r) = 3\theta$, $\sigma_{\lambda_r} = 2.12130$, and for samples of four: $E(\lambda_r) = 0.9\theta$, $\sigma_{\lambda_r} = 0.3\theta$.

(*b*) *Sample total.* Following Neyman [1, p. 357] let us denote by $A(\theta)$ the region defined by

$$(3) \qquad \theta - \Delta \leq x_1 + x_2 \leq \theta + \Delta$$

where θ is the population range, x_1 and x_2 the sample values of the sample E_2 and Δ is selected so as to have $P\{E_2 \epsilon A(\theta) \mid \theta\} = 1 - \alpha$. It is readily found that $P\{E_2 \epsilon A(\theta) \mid \theta\} = [\theta^2 - (\theta - \Delta)^2]/\theta^2 = 1 - \alpha$ from which we find that $\Delta = \theta(1 - \alpha^{1/2})$. Accordingly (3) becomes $\theta\alpha^{1/2} \leq x_1 + x_2 \leq \theta(2 - \alpha^{1/2})$, yielding the confidence limits $(x_1 + x_2)/(2 - \alpha^{1/2})$, $(x_1 + x_2)/\alpha^{1/2}$ For the confidence coefficient $1 - \alpha = 0.81$ the confidence interval is given by

$$(4) \qquad [0.6394(x_1 + x_2), 2.2941(x_1 + x_2)].$$

The length of the confidence interval is given by $\lambda_T = 1.6547(x_1 + x_2)$ so that $E(\lambda_T) = 1.6547\theta$, $\sigma_{\lambda_T} = 0.6755\theta$.

Let us denote by $A'(\theta)$ the region defined by

$$(5) \qquad 2\theta - \Delta \leq x_1 + x_2 + x_3 + x_4 \leq 2\theta + \Delta,$$

where θ is the population range, x_1, x_2, x_3, x_4 the sample values of the sample E_4 and Δ is selected so as to have $P\{E_4 \epsilon A'(\theta) \mid \theta\} = 1 - \alpha$. Using the known distribution of the sample average [3] and $1 - \alpha = 0.8208$, it is readily found that

$$P\{E_4 \epsilon A'(\theta) \mid \theta\} = \frac{16}{3}\left\{\frac{\Delta}{4\theta} - 8\left(\frac{\Delta}{4\theta}\right)^3 + 12\left(\frac{\Delta}{4\theta}\right)^4\right\} = 0.8208$$

from which we find that $\Delta = 0.788\theta$. Accordingly, (5) becomes $1.212\theta \leq x_1 + x_2 + x_3 + x_4 \leq 2.788\theta$, yielding the confidence interval

$$(6) \qquad [0.3587(x_1 + x_2 + x_3 + x_4), 0.8251(x_1 + x_2 + x_3 + x_4)]$$

The length of the confidence interval is given by $\lambda_T = 0.4664(x_1 + x_2 + x_3 + x_4)$ so that $E(\lambda_T) = 0.93280$ and $\sigma_{\lambda_T} = 0.2679\theta$.

(*c*) *Larger* (*largest*) *sample value.* Again following Neyman [1, p. 359] let us denote by $A_1(\theta)$ the region defined by

$$(7) \qquad q\theta \leq L \leq \theta$$

where θ is the population range, L the larger of the two sample values x_1 and x_2 and q, a number between zero and unity, to be determined by $P\{E_2 \epsilon A_1(\theta) \mid \theta\} = 1 - \alpha$. It is readily found that $P\{E_2 \epsilon A_1(\theta) \mid \theta\} = (\theta^2 - q^2\theta^2)/\theta^2 = 1 - \alpha$, from which we find that $q = \alpha^{1/2}$. Accordingly, (7) becomes $\theta\alpha^{1/2} \leq L \leq \theta$ yielding the confidence limits $L, L/\alpha^{1/2}$. For the confidence coefficient $1 - \alpha = 0.81$ the confidence interval is given by

$$(L,\ 2.2941L). \tag{8}$$

TABLE I

No. of cases of coverage per set of 100 samples	Frequency					
	Range		Sum		Larger (Largest)	
x	Samples of two	Samples of four	Samples of two	Samples of four	Samples of two	Samples of four
69					1	
70						
71					1	
72						
73						1
74		1			1	
75						
76	4		3		4	1
77	2		6	1	2	
78	3		6		3	1
79	9	2	4	2	3	
80	3	1	6		4	
81	2	2	1		3	
82	2	1	6	1	2	5
83	3	3	3	1	5	3
84	3	2		1	4	1
85	3			3	2	
86	2	2		2	2	1
87	1	1	2	1		1
88			1	2	1	1
89	1		1	1		
90						
91	1					
	39	15	39	15	39	15
Average....	81.1	82.1	80.2	84.2	80.2	82.1

The length of the confidence interval is given by $\lambda_L = 1.2941L$ so that using the distribution of L, $nL^{n-1}\,dL$, we have $E(\lambda_L) = 0.8627\theta$ and $\sigma_{\lambda_L} = 0.3050\theta$.

Incidentally, since $L \leqq x_1 + x_2$ we have $1.2941L < 1.6547(x_1 + x_2)$ so that in every case, for samples of two, the confidence interval of procedure (*c*) is shorter than the confidence interval of procedure (*b*).

For samples of four, we consider the region (7) where L is the largest of the sample values x_1, x_2, x_3 and x_4 of the sample E_4. It is readily found that $P\{E_4 \epsilon A_1(\theta) \mid \theta\} = (\theta^4 - q^4\theta^4)/\theta^4 = 1 - \alpha$, from which we find that $q^4 = \alpha$. For $\alpha = 0.1792$, $q = 0.6506$ so that (7) becomes $0.6506\theta \leq L \leq \theta$ yielding the confidence interval

$$(L, 1.5370L). \tag{9}$$

The length of the confidence interval is given by $\lambda_L = 0.5370L$ so that $E(\lambda_L) = 0.4296\theta$ and $\sigma_{\lambda_L} = 0.0877\theta$.

TABLE II

	Sample size	Range		Sum		Larger (Largest)	
		Theoretical	Observed	Theoretical	Observed	Theoretical	Observed
Confidence Coefficient	2	.8100	.8110	.8100	.802	.8100	.8020
	4	.8208	.8210	.8208	.842	.8208	.8210
Average length of confidence interval per set of 100 samples	2	3.0000	2.9660	1.6547	1.6441	.8627	.8556
	4	.9000	.8976	.9328	.9296	.4296	.4272
Standard deviation of average length of confidence interval	2	.2121	.2133	.0676	.0581	.0305	.0293
	4	.0300	.0335	.0268	.0140	.0088	.0093

3. The Experimental Data. We considered the rectangular population with $\theta = 1$ and obtained the sample values by using pairs of digits obtained from Tippett's random sample tables [4]. Using these observed values the confidence intervals given by (2), (4), (6), (8) and (9) were computed and the number of cases in which the value $\theta = 1$ was covered, noted. In all, 3900 samples of two were observed, subdivided into 39 sets of 100 each. The samples of four were obtained by combining pairs of samples of two and there were studied 1500 samples of four, subdivided into 15 sets of 100 each. Table I gives the observed distribution of the number of cases of coverage per set of 100 samples of two and of four. The length of the confidence interval obtained by each of the three procedures was obtained and the observed mean and standard deviation of the distribution of the average length of the confidence interval per set of 100 samples computed. (Since they are averages of 100 values, these observations are practically normally distributed.) Table II summarizes these results.

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THE NUMERICAL COMPUTATION OF THE PRODUCT OF CONJUGATE IMAGINARY GAMMA FUNCTIONS

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The difference equation

$$\frac{f_{x+1}}{f_x} = \frac{x^2 + c_1x + c_2}{x^2 + c_3x + c_4} \tag{1}$$

was used by Professor Harry C. Carver [1] as the basis for graduating frequency distributions in a manner analogous to the use of the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a - x}{b_0 + b_1x + b_2x^2}$$

in the Pearson system of frequency curves. In order to determine a particular f_x by Professor Carver's method it was necessary to perform the complete graduation from the lower limit of the range up to and including the required f_x. When x is large and only isolated values of f_x are required it seems desirable to have a method for computing f_x directly, and the present note seeks to accomplish this purpose.

It is well known [2] that the difference equation

$$\frac{f_{x+1}}{f_x} = a\frac{(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)}{(x - \beta_1)(x - \beta_2) \cdots (x - \beta_m)} \tag{2}$$

has the solution

$$f_x = w_x a^x \frac{\Gamma(x - \alpha_1) \cdots \Gamma(x - \alpha_n)}{\Gamma(x - \beta_1) \cdots \Gamma(x - \beta^m)}, \tag{3}$$

where w_x is a periodic function of x ($w_x = w_{x+n} = \cdots = k$) and $\Gamma(x + 1)$ for x, a positive real number may be defined in the usual manner by the second Euler integral

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t}\,dt \tag{4}$$