from which we can find
\[ \rho_{\text{RC}}^2 = \frac{2n(1 - m)}{m(n + 2)(m + n)} \]
and
\[ \rho_{\text{UC}}^2 = \frac{(n + 3)(n + 4)(m + n - 1)}{(n + 1)(m + n + 1)(m + n + 2)}. \]

If \( n/m = \gamma \) (a fixed constant) and \( n \) is large
\[ \rho^2 \approx \frac{n}{n + m}. \]
\( \rho^2 \) will be near 1 when \( n \) is much larger than \( m \). This corresponds, in computing \( C^2 \), to dividing the smaller sample into subgroups by the larger. In this case \( U \) and \( C^2 \) give essentially the same information. When \( m \) and \( n \) are more nearly equal the two criteria are quite different. For \( n > m \), \( C^2 \) has fewer possible values than for \( n < m \), and is therefore a more sensitive test when \( n < m \).

While it is doubtful that this test is biased for large samples, this question will not be considered in the present note.

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SIGNIFICANCE TEST FOR SPHERICITY OF A NORMAL n-VARIATE DISTRIBUTION
BY JOHN W. MAUCHLY

1. Introduction. This note is concerned with testing the hypothesis that a sample from a normal \( n \)-variate population is in fact from a population for which the variances are all equal and the correlations are all zero. A population having this symmetry will be called “spherical.” Under a linear orthogonal transformation of variates, a spherical population remains spherical, and consequently the features of a sample which furnish information relevant to this hypothesis must be invariant under such transformations.

A situation for which this test is indicated arises when the sample consists of \( N \) \( n \)-dimensional vectors, for which the variates are the \( n \) components along coordinate axes known to be mutually perpendicular, but having an orientation which is, a priori at least, quite arbitrary. A specific application for two dimensions, treated elsewhere [1], may be mentioned. Each of \( N \) days furnishes a sine and a cosine Fourier coefficient for a given periodicity, and these, when plotted as ordinate and abscissa, yield a somewhat elliptical cloud of \( N \) points. The sine and cosine functions are orthogonal, and their variances have
equal expectancies for a random series. The arbitrary nature of the orientation of axes appears here as the arbitrary choice of phase, or origin of time. Of the five ellipses studied, three could easily have come from circular populations (random), and two showed highly significant ellipticity.

2. Likelihood ratio criterion for sphericity. The method of Neyman and Pearson [2] will be used to derive a test criterion which seems entirely suitable. Let \( \Omega \) be the class of all normal \( n \)-variate populations, and let \( \omega \) be the subclass of all normal \( n \)-variate populations satisfying the hypothesis of "sphericity." The likelihood ratio criterion is obtained by taking the ratio of the maximum of the likelihood for variation of all population parameters specifying \( \omega \), to the maximum of the likelihood for variation of all population parameters specifying \( \Omega \). That is,

\[
\lambda_n = \frac{P(\omega \max)}{P(\Omega \max)}.
\]

For the set \( \Omega \), the probability law for a single observation of the \( n \) variates may be written:

\[
P = K \left| a_{ij} \right|^{-1} e^{-\frac{1}{2} \sum_{i,j} \epsilon_{ij}(s_i - a_i)(x_j - a_j)} \quad (i, j = 1, 2 \ldots n),
\]

where \( c_{ij} \) is an element of the matrix \( \| a_{ij} \|^{-1} \), the \( a_{ij} \) being variances and covariances, \( a_i \) is the mean value of the variate \( x_i \) in the population, and \( K \) is a constant the value of which does not concern us here. Then a sample of \( N \) from \( \Omega \) has the probability,

\[
P = K^N \left| a_{ij} \right|^{-1} e^{-\frac{1}{2} \sum_{i,j} \epsilon_{ij}(s_i - a_i)(x_j - a_j)}.\]

Letting

\[
\sum_{a=1}^{N} x_{ia} = N \bar{x}_i \quad \text{and} \quad \sum_{a=1}^{N} (x_{ia} - \bar{x}_i)(x_{ja} - \bar{x}_j) = N s_{ij},
\]

differentiating the logarithm of \( P \) with respect to the parameters \( a_i \) and \( a_{ij} \), and setting these derivatives equal to zero, the maximum likelihood estimates,

\[
\hat{a}_i = \bar{x}_i \quad ; \quad \hat{a}_{ij} = s_{ij},
\]

are obtained. Substituting these values in equation (3) we find that the maximum value of the likelihood is

\[
P(\Omega \max) = K^N \left| s_{ij} \right|^{-1} e^{-\frac{1}{2} s_{ij}} e^{-\frac{1}{2} s_{ij}}.
\]

The derivation of \( P(\omega \max) \) proceeds upon similar lines, but is simpler, for the probability law for the set \( \omega \) is obtained from (3) by setting

\[
c_{ij} = c \hat{a}_{ij},
\]
where \( c \) is any positive constant, and \( \delta_{ij} = 0 \) if \( i \neq j \) and 1 if \( i = j \). The result is found to be

\[
P(\omega \text{ max}) = K^N(s_0)^{-1Nn}e^{-1N}
\]

where \( s_0 \) is defined by

\[
n s_0 = \sum_{i=1}^{n} s_{ii}.
\]

The likelihood ratio criterion is therefore

\[
\lambda_s = \left[ \frac{|s_{ij}|}{(s_0)^{1n}} \right]^N.
\]

It will be convenient to designate the \( N \)th root of this statistic as \( L_{sn} \), where the second subscript indicates the number of variates:

\[
L_{sn} = \frac{|s_{ij}|^{1/2}}{s_0^{1n}}.
\]

3. The moments of the distribution of \( L_{sn} \) when the population is spherical.

The distribution of \( L_{sn} \) cannot be easily obtained in explicit form for a general \( n \), but the moments of \( L_{sn} \) when the hypothesis tested is true are easily found.

Note first that \( L_{sn} \) may be resolved into two factors which are, when the population is spherical, statistically independent:

\[
L_{sn} = \frac{(s_1 s_2 s_3 \cdots s_n)^{1/2}}{s_0^{1n}} \cdot |r_{ij}|^{1/2}.
\]

The first factor is just the one appropriate for testing the equality of the \( n \) variances when the orientation of the coordinate axes is fixed in advance, while the second factor is the square root of the determinant of correlation coefficients. The moments of the distributions of these two statistics are known [3], and since the two are independent (for zero correlation in the population), we may write:

\[
M_h(L_{sn}) = M_h(A)M_h(B),
\]

where \( A \) and \( B \) are used to indicate the two factors, and \( M_h \) indicates the \( h \)th moment. The moments are given by

\[
M_h(L_{sn}) = \prod_{i=1}^{n} \left[ \frac{\Gamma_{1/2}(N - i + h)}{\Gamma_{1/2}(N - i)} \right] n^{1nh} \frac{\Gamma_{1/2}(n(N - 1))}{\Gamma_{1/2}(n(N - 1 + h))}.
\]

4. Significance test for \( n = 2 \). For \( n = 1 \), \( M_h(L_{s1}) = 1 \) for any \( h \), as it should, since \( L_{s1} \) is then identically 1, and the concept of sphericity is meaningless. For \( n = 2 \), the expression (14) reduces to,

\[
M_h(L_{s2}) = \frac{\Gamma(N - 2 + h)\Gamma(N - 1)}{\Gamma(N - 1 + h)\Gamma(N - 2)} = \frac{N - 2}{N - 2 + h}
\]
and the distribution is thus found to be
\[(16) \quad D(L_{n2}) = (N - 2)L_{n2}^{N-3} dL_{n2}.\]

Thus for \( n = 2 \), the significance of the value of \( L_{n2} \) obtained from a given sample of \( N \) points in a plane is simply
\[(17) \quad P(L_{n2} < L_{n2}') = L_{n2}^{N-2}.\]

These results for \( n = 2 \) were obtained by another method in [1].

5. Significance test for \( n = 3 \). For \( n = 3 \) and higher values of \( n \), no simple expression for the distribution seems obtainable. In this case it appears reasonable to fit a Pearson curve of the type,
\[(18) \quad y = Kx^{p-1}(1 - x)^{q-1},\]
by adjusting \( p \) and \( q \) as to obtain agreement with the first two moments of the actual distribution. The calculations were carried out for \( L_{n3}^2 \) rather than \( L_{n3} \) itself, to simplify the moment expressions. The first moment of \( L_{n3}^2 \) is the second moment of \( L_{n3} \), and is given as a function of \( N \) by the equation,
\[(19) \quad \mu_1(N) = \frac{(3N - 6)(3N - 9)}{(3N - 2)(3N - 1)}.\]

Recurrence relations, similar to those noted by Lengyel [4] in carrying out a similar task, hold for the moments of \( L_{n3}^2 \); hence,
\[(20) \quad \mu_3(N) = \mu_1(N)\mu_1(N + 2).\]

Explicit solution of the equations for \( p \) and \( q \) in terms of \( N \) is possible:
\[(21) \quad p = \frac{(9N + 5)(N - 2)(N - 3)}{2(9N^2 - 8N - 15)},\]
\[(22) \quad q = \frac{2(9N - 13)(9N + 5)}{9(9N^2 - 8N - 15)}.\]

For values of \( N > 30 \), acceptable approximations to \( p \) and \( q \) are obtained by carrying out the division indicated in (21) and (22):
\[(23) \quad p = \frac{1}{2}(N - 4) + 2/9 + 70/81(N + 1) \ldots ,\]
\[(24) \quad q = 2 + \frac{140}{9(3N - 2)^2} \ldots .\]

The values of \( p \) and \( q \) are given in Table I so that those desiring other than the standard significance levels may readily enter the Pearson tables.

For \( N \) a multiple of 4 from 8 to 48, and a multiple of 10 from 50 to 100, the significance levels were taken from the Incomplete Beta-Function Tables, using adequate interpolation. The final Table I was then prepared by filling in the skeleton table by interpolation with respect to \( N \).

From the results of Wilks [5] it follows that \(-2N \log \ _2 L_{n2}\) is, for large \( N \),
TABLE I

5%, 1%, and 0.1% levels of significance for the 3-dimensional sphericity criterion, 
$L^2_3 = \chi^2 / n$, and the values of $p$ and $q$ for the Pearson Type I curves used in 
calculating these levels

<table>
<thead>
<tr>
<th>$N$</th>
<th>5%</th>
<th>1%</th>
<th>0.1%</th>
<th>$p$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0.172</td>
<td>0.083</td>
<td>0.030</td>
<td>2.3239</td>
<td>2.0312</td>
</tr>
<tr>
<td>10</td>
<td>0.278</td>
<td>0.165</td>
<td>0.080</td>
<td>3.3044</td>
<td>2.0194</td>
</tr>
<tr>
<td>12</td>
<td>0.366</td>
<td>0.243</td>
<td>0.139</td>
<td>4.2911</td>
<td>2.0131</td>
</tr>
<tr>
<td>14</td>
<td>0.436</td>
<td>0.312</td>
<td>0.197</td>
<td>5.2816</td>
<td>2.0095</td>
</tr>
<tr>
<td>16</td>
<td>0.494</td>
<td>0.372</td>
<td>0.252</td>
<td>6.2744</td>
<td>2.0072</td>
</tr>
<tr>
<td>18</td>
<td>0.541</td>
<td>0.423</td>
<td>0.301</td>
<td>7.2688</td>
<td>2.0057</td>
</tr>
<tr>
<td>20</td>
<td>0.580</td>
<td>0.466</td>
<td>0.346</td>
<td>8.2642</td>
<td>2.0046</td>
</tr>
<tr>
<td>22</td>
<td>0.614</td>
<td>0.504</td>
<td>0.386</td>
<td>9.2605</td>
<td>2.0038</td>
</tr>
<tr>
<td>24</td>
<td>0.642</td>
<td>0.538</td>
<td>0.422</td>
<td>10.2574</td>
<td>2.0032</td>
</tr>
<tr>
<td>26</td>
<td>0.667</td>
<td>0.567</td>
<td>0.454</td>
<td>11.2548</td>
<td>2.0027</td>
</tr>
<tr>
<td>28</td>
<td>0.689</td>
<td>0.593</td>
<td>0.483</td>
<td>12.2526</td>
<td>2.0023</td>
</tr>
<tr>
<td>30</td>
<td>0.708</td>
<td>0.616</td>
<td>0.510</td>
<td>13.2506</td>
<td>2.0020</td>
</tr>
<tr>
<td>32</td>
<td>0.724</td>
<td>0.637</td>
<td>0.534</td>
<td>14.2488</td>
<td>2.0018</td>
</tr>
<tr>
<td>34</td>
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<td>0.555</td>
<td>15.2473</td>
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<tr>
<td>36</td>
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<td>0.575</td>
<td>16.2458</td>
<td>2.0014</td>
</tr>
<tr>
<td>38</td>
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<td>0.687</td>
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<td>17.2447</td>
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</tr>
<tr>
<td>40</td>
<td>0.776</td>
<td>0.701</td>
<td>0.610</td>
<td>18.2435</td>
<td>2.0011</td>
</tr>
<tr>
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<td>0.714</td>
<td>0.626</td>
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</tr>
<tr>
<td>44</td>
<td>0.795</td>
<td>0.726</td>
<td>0.640</td>
<td>20.2416</td>
<td>2.0009</td>
</tr>
<tr>
<td>46</td>
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<td>0.736</td>
<td>0.653</td>
<td>21.2408</td>
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</tr>
<tr>
<td>48</td>
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<td>0.665</td>
<td>22.2400</td>
<td>2.0008</td>
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<td>0.677</td>
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<td>2.0007</td>
</tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
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<td>0.793</td>
<td>0.725</td>
<td>28.2365</td>
<td>2.0005</td>
</tr>
<tr>
<td>65</td>
<td>0.859</td>
<td>0.808</td>
<td>0.744</td>
<td></td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>0.869</td>
<td>0.821</td>
<td>0.760</td>
<td>33.2345</td>
<td>2.0004</td>
</tr>
<tr>
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<td>0.877</td>
<td>0.832</td>
<td>0.775</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.842</td>
<td>0.788</td>
<td>38.2328</td>
<td>2.0003</td>
</tr>
<tr>
<td>85</td>
<td>0.891</td>
<td>0.851</td>
<td>0.799</td>
<td></td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>0.897</td>
<td>0.859</td>
<td>0.809</td>
<td>43.2317</td>
<td>2.0002</td>
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<tr>
<td>95</td>
<td>0.902</td>
<td>0.866</td>
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</tr>
<tr>
<td>100</td>
<td>0.907</td>
<td>0.872</td>
<td>0.827</td>
<td>48.2308</td>
<td>2.0002</td>
</tr>
</tbody>
</table>

* No values for $p$ and $q$ were calculated for these values of $N$; the levels were obtained 
  by interpolation (see text).

distributed approximately like $\chi^2$ with $n(n - 1)/2$ degrees of freedom. However, equation (24) above suggests that for large $N$ one may get a very good
CONFIDENCE INTERVALS

approximation (for \( n = 3 \)) by setting \( q = 2 \); the significance test for \( n = 3 \) then becomes,

\[
P(L_{3} < L'_{3}) = \frac{1}{2} L^{N-4}_{3} ((N - 2) - (N - 4)L^2_{3}).
\]

Probably similar approximations can be found for other values of \( n \). It is a pleasure to acknowledge the helpful comments and advice which I received from Mr. A. M. Mood of Princeton. Recognition is also due Mr. Wallace Brey, a student assistant under the National Youth Administration, who aided in the computations.

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A SIMPLE SAMPLING EXPERIMENT ON CONFIDENCE INTERVALS

BY S. KULLBACK AND A. FRANKEL

1. Introduction. In order to illustrate some of the notions of the theory of confidence or fiducial limits in connection with a course in Statistical Inference at the George Washington University, we had the class carry out certain simple experiments, following a suggestion in one of Neyman’s papers on Statistical Estimation [1]. In the belief that the experimental data may be of interest to others, we present the results herein.

2. The problem. We consider the problem of estimating the range \( \theta \) of a rectangular population defined by \( p(x, \theta) \, dx = dx/\theta \), \( 0 \leq x \leq \theta \) and in particular, for simplicity, we limit ourselves to samples of two and four. We consider three possible approaches to the problem, viz., by using (a) the sample range (b) the sample average or total (c) the larger (largest) sample value. Let us consider each in turn.

(a) Sample range. Wilks [2] has shown that for samples of \( n \) and confidence coefficient \( 1 - \alpha \), the confidence or fiducial limits for the population range \( \theta \) are given by \( r \) and \( r/\psi^\alpha \), where \( r \) is the sample range and \( \psi^\alpha \) is determined by

\[
\psi^\alpha [n - (n - 1)\psi^\alpha] = \alpha.
\]

For \( n = 2, \alpha = 0.19 \) and \( n = 4, \alpha = 0.1792 \), (1) yields \( \psi^\alpha = 0.1 \) and \( \psi^\alpha = 0.4 \) respectively. Accordingly, for samples of two with confidence coefficient