A METHOD FOR RECURRENT COMPUTATION OF ALL THE PRINCIPAL MINORS OF A DETERMINANT, AND ITS APPLICATION IN CONFLUENCE ANALYSIS

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1. Recurrent computation of all the principal minors of a determinant. The formulae which I develop in this paper have been worked out for use in statistical confluence analysis. By means of recurrent computation they shorten considerably the amount of work required to compute all principal minors of a square matrix. Originally I elaborated this method as a simplification of one given by Frisch (not published).

Subsequently I found that the method could more easily be deduced from the pivotal method. This method has been described, for example, by Whittaker and Robinson [5] and by Aitken [1].

Let us consider a square n-rowed matrix

$$(1) \qquad \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{Vmatrix}.$$

Let the adjoint of this matrix be $\| p_{ij} \|$ and let us denote its determinant value by $D_{12 \cdots n}$.

Then we have the following identity

$$(2) \qquad \begin{vmatrix} p_{n-1,n-1} & p_{n-1,n} \\ p_{n,n-1} & p_{n,n} \end{vmatrix} = D_{12 \cdots n} D_{12 \cdots n-2}.$$

As Aitken points out, the pivotal method is based upon this identity.

Next consider the following matrix which is formed from the matrix (1) by striking out the nth row and the $(n - 1)$th column:

$$(3) \qquad \begin{Vmatrix} a_{11} & \cdots & a_{1,n-2} & a_{1,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-2,1} & \cdots & a_{n-2,n-2} & a_{n-2,n} \\ a_{n-1,1} & \cdots & a_{n-1,n-2} & a_{n-1,n} \end{Vmatrix}.$$

Let us denote its adjoint by $\| q_{ij} \|$, its determinant value by $A_{12\cdots n} = -p_{n-1,n}$. The determinant

$$\begin{vmatrix} a_{11} & \cdots & a_{1,n-2} & a_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-2,1} & \cdots & a_{n-2,n-2} & a_{n-2,n-1} \\ a_{n,1} & \cdots & a_{n,n-2} & a_{n,n-1} \end{vmatrix} = -p_{n,n-1},$$

we shall denote by $B_{12\cdots n}$.

The identity (2) can now be written

$$D_{12\cdots n} = \frac{D_{12\cdots n-2,n} D_{12\cdots n-2,n-1} - A_{12\cdots n} B_{12\cdots n}}{D_{12\cdots n-2}}. \tag{2'}$$

If we apply the identity (2) to the matrix (3) we get

$$\begin{vmatrix} q_{n-2,n-2} & q_{n-2,n-1} \\ q_{n-1,n-2} & q_{n-1,n-1} \end{vmatrix} = A_{12\cdots n} D_{12\cdots n-3},$$

which may also be written

$$A_{12\cdots n} = \frac{A_{12\cdots n-3,n-1,n} D_{12\cdots n-2} - A_{12\cdots n-3,n-2,n} B_{12\cdots n-1}}{D_{12\cdots n-3}}. \tag{4}$$

To simplify the notation we will not write the affixes present, but write the affixes not present in inverted parentheses. Then our formulae (2′) and (4) can be written

$$D = \frac{D_{)n-1(} D_{)n(} - AB}{D_{)n-1,n(}},$$

$$A = \frac{A_{)n-2(} D_{)n-1,n(} - A_{)n-1(} B_{)n(}}{D_{)n-2,n-1,n(}}.$$

In an analogous way we get

$$B = \frac{B_{)n-2(} D_{)n-1,n(} - B_{)n-1(} A_{)n(}}{D_{)n-2,n-1,n(}}.$$

We may apply these formulae to an arbitrary principal minor $D_{v_1 v_2 \cdots v_k}$. Let us now denote $D_{v_1 v_2 \cdots v_k}$ by D and denote the absence of one or more of the numbers $v_1, v_2, \cdots v_k$ by placing them into inverted parentheses. We then have the formulae:

$$A = \frac{A_{)v_{k-2}(} D_{)v_{k-1},v_k(} - A_{)v_{k-1}(} B_{)v_k(}}{D_{)v_{k-2},v_{k-1},v_k(}}, \tag{5a}$$

$$B = \frac{B_{)v_{k-2}(} D_{)v_{k-1},v_k(} - B_{)v_{k-1}(} A_{)v_k(}}{D_{)v_{k-2},v_{k-1},v_k(}}, \tag{5b}$$

$$D = \frac{D_{)v_{k-1}(} D_{)v_k(} - AB}{D_{)v_{k-1},v_k(}}. \tag{5c}$$

By means of these formulae we can recurrently compute all principal minors. We begin with $D_i = a_{ii}$, $i = 1, 2 \cdots n$, $A_{ij} = a_{ij}$, $B_{ij} = a_{ji}$, where $i < j$. Then we compute the D's with two affixes,

$$D_{ij} = D_i D_j - A_{ij} B_{ij},$$

and then the quantities A, B, D with three affixes,

$$A_{ijk} = A_{jk} D_i - A_{ik} B_{ij}$$

$$B_{ijk} = B_{jk} D_i - B_{ik} A_{ij}$$

$$D_{ijk} = \frac{D_{ik} D_{ij} - A_{ijk} B_{ijk}}{D_i}, \qquad i < j < k.$$

Then we compute the quantities A, B, D with four affixes, and so on.

If we carry through the computations without dropping any figures we have as a control that all divisions will be exact without remainder. If we are dropping figures we can control the result by computing the determinant $D_{12\cdots n}$ in another way. If we wish to control the computation before it is completed, we may use our recurrence formulae on the matrix which we get from the original matrix when the rows and the columns are subjected to the same permutation. For example we can reverse the order of the rows and columns. Then we can control the $(k - 1)$ rowed minors before computing the k-rowed minors.

If all the D's are different from zero, we may reduce the necessary number of multiplications and divisions in the following way. We introduce the following notations:

$$d = \frac{D}{D_{)v_k(}}$$

$$a = \frac{A}{D_{)v_{k-1},v_k(}} \qquad b = \frac{B}{D_{)v_{k-1},v_k(}}$$

$$c = -\frac{b}{d_{)v_k(}}.$$

Substituting in (5), we get the following system of recurrence formulae:

(6a) $$a = a_{)v_{k-2}(} + a_{)v_{k-1}(} c_{)v_k(}$$

(6b) $$b = b_{)v_{k-2}(} + a_{)v_k(} c_{)v_{k-1}(}$$

(6c) $$c = -\frac{b}{d_{)v_k(}}$$

(6d) $$d = d_{)v_{k-1}(} + ac$$

(6e) $$D = D_{)v_k(} d.$$

An affix v_k on a letter indicates the deletion of the last row and column in the determinants making up the definition of that letter, even though those determinants are of lower order than v_k. Similarly, an affix v_{k-1} indicates the deletion of the next to the last row and column.

The a's with two affixes in these formulae are identical with the elements a_{ij} of the matrix (1) where $i < j$. Further, $b_{ij} = a_{ji}$, $i < j$, $d_i = a_{ii}$. Applying the recurrence formulae (6) we start with these values.

If the matrix (1) is symmetric, i.e. if $a_{ij} = a_{ji}$, then we get

$$B_{v_1 v_2 \cdots v_k} = A_{v_1 v_2 \cdots v_k}$$

and

$$b_{v_1 v_2 \cdots v_k} = a_{v_1 v_2 \cdots v_k} .$$

In this case we can therefore replace B by A in the formulae (5) and replace b by a in the formulae (6).

Numerical example. Let us compute all the scatterances in the constructed example given by Frisch, [3, p. 121]. The correlation matrix in this example is:

1.000000	−0.121551	0.656809	0.752502	−0.224549
−0.121551	1.000000	0.657698	−0.732862	0.212165
0.656809	0.657698	1.000000	0.014385	−0.040183
0.752502	−0.732862	0.014385	1.000000	−0.280223
−0.224549	0.212165	−0.040183	−0.280223	1.000000

Using our recurrence formulae (6) we get the following table:

	a	c	d	D
12	−0.121 551	0.121 551	0.985 225	0.985 225
13	0.656 809	−0.656 809	0.568 602	0.568 602
23	0.657 698	−0.657 698	0.567 433	0.567 433
14	0.752 502	−0.752 502	0.433 741	0.433 741
24	−0.732 862	0.732 862	0.462 913	0.462 913
34	0.014 385	−0.014 385	0.999 793	0.999 793
15	−0.224 549	0.224 549	0.949 578	0.949 578
25	0.212 165	−0.212 165	0.954 986	0.954 986
35	−0.040 183	0.040 183	0.998 385	0.998 385
45	−0.280 223	0.280 223	0.921 475	0.921 475
123	0.737 534	−0.748 594	0.016 489	0.016 245
124	−0.641 395	0.651 014	0.016 184	0.015 945
134	−0.479 865	0.843 938	0.028 765	0.016 356
234	0.496 387	−0.874 794	0.028 677	0.016 272
125	0.184 871	−0.187 643	0.914 888	0.901 371
135	0.107 303	−0.188 714	0.929 328	0.528 418
235	−0.179 723	0.316 730	0.898 062	0.509 590
145	−0.111 249	0.256 487	0.921 044	0.399 495
245	−0.124 735	0.269 457	0.921 272	0.426 516
345	−0.279 645	0.279 703	0.920 167	0.919 977

	a	c	d	D
1234	0.000 279	−0.016 6	0.016 179	0.000 262 83
1235	−0.031 090	1.885 5	0.856 268	0.013 910
1245	0.009 105	−0.562 6	0.909 766	0.014 506
1345	−0.020 692	0.719 35	0.914 443	0.014 957
2345	0.032 486	−1.132 8	0.861 262	0.014 014
12345	0.009 621	−0.594 7	0.850 546	0.000 223 55

2. Computation of the coefficients of the characteristic polynomial of a matrix. The characteristic polynomial of the matrix (1) is

$$P(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nn} - \lambda \end{vmatrix}$$

$$= P_n - P_{n-1}\lambda + P_{n-2}\lambda^2 - \cdots + (-1)^n\lambda^n.$$

As is well known, the coefficient P_k can be calculated as the sum of all the k-rowed principal minors of the matrix (1). Our method of computing all the principal minors of a matrix therefore gives us as a by-product a method of computing the coefficients of the characteristic polynomial. Another method for the determination of these coefficients has been given by Paul Horst [4].

We may obtain a comparison between the work of computation entailed by the two methods by calculating the number of multiplications and divisions necessary when using one or the other method. If our recurrence formulae (6) are used, two multiplications and one division are necessary for computing a 2-rowed minor, and 4 multiplications and one division for every minor with 3 or more rows. Consequently the total number of multiplications and divisions will be:

$$S_n = 3\binom{n}{2} + 5\sum_{k=3}^{n}\binom{n}{k}$$

$$= 5 \cdot 2^n - (n^2 + 4n + 5).$$

On using Horst's method, the number of necessary multiplications and divisions will be found to be

$$H_n = (\tfrac{1}{2}n - 1)n^3 + \tfrac{1}{2}n^3 + \tfrac{1}{2}(n-1)(n+2)$$

$$H_n = \tfrac{1}{2}(n-1)(n^3 + n + 2) \qquad n \text{ even},$$

$$H_n = \tfrac{1}{2}(n-1)(n^3 + n^2 + n + 2) \qquad n \text{ odd}.$$

When $n = 2, 3, \cdots 12$, S_n and H_n acquire the following values:

n	S_n	H_n
2	3	6
3	14	41
4	43	105
5	110	314
6	255	560
7	558	1203
8	1179	1827
9	2438	3284
10	4975	4554
11	10070	7325
12	20283	9581

We see that our method of computing the coefficients of the characteristic polynomial involves less calculation when $n < 10$, while Horst's method is superior when $n \geqq 10$.

If our purpose is to find the characteristic roots of the matrix we can do this with less amount of computation without first finding the coefficients of the characteristic polynomial. See Aitken, [2].

3. Applications in confluence analysis. The confluence analysis of Frisch is set forth in his book: "Statistical Confluence Analysis by Means of Complete Regression Systems," [3].

The main method of this book is the "bunch analysis," which includes the computation of the adjoints of the correlation matrices of all sets of variates contained in the total set. In section 1, Frisch has described a preliminary analysis by means of scatterances. The scatterances are the principal minors of the correlation matrix of the total set of variates. If we carry through such an analysis, the recurrence formulae of section 1 of this paper will give a rapid method for the calculation of all the scatterances.

Another application of the computation of all the scatterances arises in the determination of the correct time lags between variates in a structural equation. This problem will be treated in a paper on confluence analysis which will appear in the near future.

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