3. Two practical formulas for computations. It seems worthwhile to note here two simple formulas in connection with Mills' ratio (2) which will be useful for computations. The first is the obvious relationship

$$R_{-u} = \sqrt{2\pi}\, e^{u^2/2} - R_u = 1/z - R_u \tag{9}$$

in the notation of Pearson's tables. The second applies to large values of x, and may be written

$$\frac{x}{x^2 + \sigma_x^2} < \gamma(x) = \frac{1}{\sigma_x} R_{x/\sigma_x} < \frac{1}{x} \tag{10}$$

(10) is true for $x > 0$, and can be proved by means of the differential equation which $\gamma(x)$ satisfies.

4. Remarks. The estimate $\gamma(x)$ has the following inadequacy: If only a single observation x is known, then it is unknown whether a is of like or unlike sign compared to x. It turns out then that the mathematical expectation for the value of $\gamma(x)$ vanishes identically. This difficulty of course disappears if more than one observation is available. Methods of avoiding this difficulty for time series, e.g. by noting relative frequencies for observations separated by 1, 2, 3 etc. intervals to agree in sign, will be discussed elsewhere in connection with practical applications.

It may be worthwhile to note that Geary[5] developed certain characteristics of the distribution of a quotient, which however are not adapted to our purposes.

NOTE ON CONFIDENCE LIMITS FOR CONTINUOUS DISTRIBUTION FUNCTIONS

By A. Wald* and J. Wolfowitz

In a recent paper [1] we discussed the following problem: Let X be a stochastic variable with the cumulative distribution function $f(x)$, about which nothing is known except that it is continuous. Let $x_1, \cdots, x_n$ be n independent, random observations on X. The question is to give confidence limits for $f(x)$. We gave a theoretical solution when the confidence set is a particularly simple and important one, a "belt."

A particularly simple and expedient way from the practical point of view is to construct these belts of uniform thickness ([1], p. 115, equation 50). If the appropriate tables, as mentioned in our paper, were available, the construction of confidence limits, no matter how large the size of the sample, would be immediate.

Our formulas (11), (16), (19), (27) and (29) are not very practical for computation, particularly when the samples are large. We have recently learned that

[5] Geary, R. C., "The Frequency Distribution of a Quotient," *Jour. Roy. Stat. Soc.*, Vol. 93 (1930), pp. 442–446.

* Columbia University, New York City.

there exists a result by Kolmogoroff [2], generalized by Smirnoff [3],[1] which for large samples gives an easy method for constructing tables, i.e. of finding α when c and n are given (all notations as in [1]). The result of Kolmogoroff-Smirnoff is:

Let $c = \lambda/\sqrt{n}$. Then for any fixed $\lambda > 0$,

$$\lim_{n=\infty} \overline{P} = \lim_{n=\infty} \underline{P} = 1 - e^{-2\lambda^2}$$

$$\lim_{n=\infty} P = 1 - 2\sum_{m=1}^{\infty} (-1)^{(m-1)} e^{-2m^2\lambda^2}.$$

This series converges very rapidly.

REFERENCES

[1] Wald and Wolfowitz, "Confidence limits for continuous distribution functions," *Annals of Math. Stat.*, Vol. 10(1939), pp. 105–118.

[2] A. Kolmogoroff, "Sulla determinazione empirica di una leggi di distribuzione," *Giornale dell'Instituto Italiana degli Attuari*, Vol. 11(1933).

[3] N. Smirnoff, "Sur les ecarts de la courbe de distribution empirique," *Recueil Mathematique* (*Mathematicheski Sbornik*), New series, Vol. 6(48)(1939), pp. 3–26.

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[1] In the French résumé of Smirnoff's article, on page 26, due to a typographical error this formula is given with a factor $(-1)^m$ instead of the correct factor $(-1)^{m-1}$. The correct result follows from equation (112), page 23, of the Russian text when t is set equal to zero.