with the associated indicial equation

\[ f(x) = x^4 - 0.398x^3 + 0.220x^2 - 0.013x - 0.027 = 0. \]

Its roots have been computed and are known to be less than unity in absolute value. This may be verified by computing

\[
\begin{align*}
\pi_0 &= 0.782 > 0 \\
\pi_1 &= 3.338 > 0 \\
\pi_2 &= 5.398 > 0 \\
\pi_3 &= 4.878 > 0 \\
\pi_4 &= 1.604 > 0 \\
T_2 &= 14.204 > 0 \\
T_3 &= 43.177 > 0
\end{align*}
\]

To compute the same results by cross-multiplication the work is arranged as follows:

\[
\begin{align*}
\pi_0 & \quad \pi_2 & \quad \pi_4 \\
& 0.782 & 5.398 & 1.604 \\
\pi_1 & \quad \pi_3 \\
& 3.338 & 4.878 \\
\pi_1\pi_3 - \pi_0\pi_2 & \quad \pi_2\pi_4 - 0 \\
& 14.204 & 7.824 \\
\pi_3(\pi_1\pi_3 - \pi_0\pi_2) - \pi_1\pi_3\pi_4 & \quad 43.177
\end{align*}
\]

It may be remarked that the presence of a negative coefficient anywhere in the table is an immediate indication of instability, and that there is no necessity to continue the computation until a negative sign appears in a leading coefficient. This fact often saves much labor.

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VALUES OF MILLS’ RATIO OF AREA TO BOUNDING ORDINATE AND OF THE NORMAL PROBABILITY INTEGRAL FOR LARGE VALUES OF THE ARGUMENT

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A pair of simple inequalities is proved which constitute upper and lower bounds for the ratio \( R_x \), valid for \( x > 0 \). The writer has failed to encounter these inequalities in the literature, hence it seems worthwhile to present them for whatever value they may have.

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1 J. P. Mills, "Table of ratio: area to bounding ordinate, for any portion of the normal curve." _Biometrika_ Vol. 18 (1928) pp. 395–400. Also Pearson’s tables, Part II, Table III.
The function $R_z$ is defined by

$$R_z = e^{z^2} \int_z^{\infty} e^{-t^2} dt.$$  

The following relations between $R = R_z$ and its derivatives are easily established by direct differentiations and substitutions:

$$\frac{dR}{dx} = xR - 1,$$

$$\frac{d^2R}{dx^2} = x \frac{dR}{dx} + R = x^2 + 1 \frac{dR}{dx} + \frac{1}{x},$$

$$\frac{d^3R}{dx^3} = \left(1 + \frac{2}{x^2 + 1}\right)x \frac{d^2R}{dx^2} - \frac{2}{x^2 + 1}.$$ 

Also by ordinary rules

$$R_z > 0,$$

$$\lim_{x \to \infty} xR_z = 1.$$ 

1°. Suppose that at any point $x_1 > 0$, $x_1R > 1$. Then by (2) $dR/dx > 0$, and $R_z$ would continue to increase with increasing $x$; still more, $xR_z$ would continue to increase, hence we should have $xR_z > 1$ for $x \geq x_1$, which contradicts (6). Therefore we find $xR_z \leq 1$ for $x > 0$, and

$$R_z \leq \frac{1}{x},$$

which establishes the required upper inequality.

2°. Suppose that at any point $x_2 > 0$, $d^2R/dx^2 < 0$. Then by (4) $d^3R/dx^3 = (d/dx)(d^2R/dx^2) < 0$ at this point. Since these derivatives are continuous this implies that for all $x > x_2$, $d^2R/dx^2 < [d^2R/dx^2]_{x = x_2} < 0$. Then we get the inequalities, for $x > x_2$

$$\frac{dR}{dx} < \left[\frac{dR}{dx}\right]_2 + (x - x_2) \left[\frac{d^2R}{dx^2}\right]_2 < \left[\frac{dR}{dx}\right]_2$$

$$R < R_{x_2} + (x - x_2) \left[\frac{dR}{dx}\right]_2 + \frac{1}{2}(x - x_2)^2 \left[\frac{d^2R}{dx^2}\right]_2$$

where $[\ ]_2$ indicates evaluation at $x = x_2$. Since $[d^2R/dx^2]_2 < 0$, this implies that for sufficiently large $x$, $R_z < 0$, which contradicts (5). It follows then that (3) is positive, and substitution of (2) gives

$$R_z \geq \frac{x}{x^2 + 1}.$$
We combine (7) and (8) in the double inequality:

\[
\frac{x}{x^2 + 1} \leq R_x \leq \frac{1}{x}, \quad \text{if } x \geq 0.
\]

This gives for the probability integral the corresponding inequality

\[
\frac{x}{x^2 + 1} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} \, dt \leq \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}
\]

It can easily be shown (for \( x > 0 \)) that equalities in (9) and (10) are impossible.