more general problem by the expedient of imagining $n - m$ blank cards to be added at the end of the call deck and regarding these as an additional kind. It is thus apparent that formulae (13) and (14) apply without modification to this altered situation.

7. Application to contingency table. Stevens[5] has considered the distribution of entries in a contingency table with fixed marginal totals, and has pointed out that the problem of matching two decks of cards may be dealt with from that standpoint. A contingency table classifies data into n columns and m rows, and we may consider the row as indicating the kind of card which occupies a given position in the call deck, the columns having the same function with respect to the target deck. Stevens defines a quantity c as the sum of entries in a prescribed set of cells, subject to the condition that no two cells of the set are in the same row or column, and mentions as unsolved the problem of the exact sampling distribution of c.

We now have at our disposal the machinery for solving this problem. Following Stevens's notation, let $a_1, a_2, \cdots, a_m$ denote the fixed row totals and $b_1, b_2, \cdots, b_n$ the fixed column totals, while x_{rs} denotes the frequency of the cell in the rth row and the sth column. Then, let $c = \sum_{h=1}^{l} x_{r_h s_h}$, where l does not exceed either m or n. Imagine two decks of N cards $\left(N = \sum_{h=1}^{m} a_h = \sum_{h=1}^{n} b_h\right)$, the first containing a_1 cards of one kind, a_2 of another, etc., and the second containing b_1 cards of one kind, b_2 of another, etc. Moreover, let the r_hth kind in the first deck and the s_hth kind in the second deck be the same kind ($h = 1, 2, \cdots, l$), the other kinds being all different. Evidently c is the number of matchings between the two decks. Hence, the methods of this paper can be used to obtain the distribution of c. The formulae we have obtained agree with those for the expected value and variance of c given by Stevens.

ON METHODS OF SOLVING NORMAL EQUATIONS

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There seems to be considerable disagreement concerning what is the most satisfactory method of solving a set of normal equations. Since such information as errors of estimate and significance of results is usually desired in addition to the solution, in its broader aspects the problem is one of deciding what is the most satisfactory method of calculating the inverse of a symmetric matrix.

For equations with several unknowns some compact systematic method of

[5] W. L. Stevens, *Annals of Eugenics*, loc. cit.

calculation is necessary to eliminate much of the labor involved in the ordinary method of calculating the inverse from its definition. Among the more common of such systematic methods are those associated with the names of Chio,[1] Gauss,[1] Doolittle,[2] and Aitken.[3] In addition, A. A. Albert[4] recently called attention to a method implicit in elementary matrix theory. There are also various iterative schemes, and schemes which are but slight variations of the above methods. In this note only the methods associated with the above names will be considered, and for convenience they will be labeled with those names, regardless of who should be given credit for them.

The purpose of this note is to show that when the calculation of the inverse is systematized, all of the above methods are fundamentally equivalent and merely involve a different arrangement of work. Consequently, any advantage in calculating time for any particular method will arise through such features as a simpler technique or less copying, rather than through fewer multiplications and divisions.

By the method of Chio is meant the evaluation of determinants by the pivotal method of reduction. Since all of the methods mentioned above use pivotal reduction, the method of Chio will not be treated as a distinct method. Furthermore, since Gauss' method is incorporated in that of Aitken, it will be necessary to consider only the methods of Aitken, Doolittle, and Albert as distinct.

First consider the method of Albert, which is based on the following matrix properties. Let the matrix $\mathbf{A}$ be subjected to a sequence of row transformations leading to the matrix $\mathbf{A}'$. Then, writing $\mathbf{A} = \mathbf{IA}$, it follows from a theorem in matrix theory that $\mathbf{A}' = \mathbf{I}'\mathbf{A}$, and consequently that $\mathbf{A}'\mathbf{A}^{-1} = \mathbf{I}'$. If row transformations are chosen which make $\mathbf{A}' = \mathbf{I}$, then $\mathbf{A}^{-1} = \mathbf{I}'$. This states that if the same row transformations are applied to the identity matrix as were used to reduce $\mathbf{A}$ to the identity matrix, then the resulting matrix will be the desired inverse. The customary manner of reducing $\mathbf{A}$ to $\mathbf{I}$ is to work for zeros in columns as follows:

$$\left\|\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}\right\|, \qquad a_{11}\left\|\begin{matrix} 1 & \dfrac{a_{12}}{a_{11}} & \cdots & \dfrac{a_{1n}}{a_{11}} \\ 0 & \left(a_{22} - a_{12}\dfrac{a_{21}}{a_{11}}\right) & \cdots & \left(a_{2n} - a_{1n}\dfrac{a_{21}}{a_{11}}\right) \\ \vdots & \vdots & & \vdots \\ 0 & \left(a_{n2} - a_{12}\dfrac{a_{n1}}{a_{11}}\right) & \cdots & \left(a_{nn} - a_{1n}\dfrac{a_{n1}}{a_{11}}\right) \end{matrix}\right\|,$$

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[1] See, for example, Whittaker and Robinson, *The Calculus of Observations*, p. 71 and p. 234.

[2] See, for example, Croxton and Cowden, *Applied General Statistics*, 1939, p. 716.

[3] *Roy. Soc. Edin. Proc.*, Vol. 57 (1936-37), p. 172.

[4] Am. Math. Monthly, Vol. 48, No. 3 (1941), p. 198.

$$a_{11}b_{22}\begin{Vmatrix} 1 & \dfrac{a_{12}}{a_{11}} & \dfrac{a_{13}}{a_{11}} & \cdots & \dfrac{a_{1n}}{a_{11}} \\ 0 & 1 & \dfrac{b_{23}}{b_{22}} & \cdots & \dfrac{b_{2n}}{b_{22}} \\ 0 & 0 & \left(b_{33} - b_{23}\dfrac{b_{32}}{b_{22}}\right) & \cdots & \left(b_{3n} - b_{2n}\dfrac{b_{32}}{b_{22}}\right) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \left(b_{n3} - b_{23}\dfrac{b_{n2}}{b_{22}}\right) & \cdots & \left(b_{nn} - b_{2n}\dfrac{b_{n2}}{b_{22}}\right) \end{Vmatrix}, \cdots,$$

where new letters are introduced for new elements after each reduction. After zeros are obtained below the main diagonal, zeros are obtained above the diagonal by starting with the last column. If now these operations are performed in the same order on **I**, the result will be $\mathbf{A}^{-1}$.

Next consider the method of Aitken, which is based on the evaluation of a bordered determinant, namely,

$$\begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} & 0 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} & 1 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} & 0 \\ 0 & \cdots & -1 & \cdots & 0 & 0 \end{vmatrix} = \text{cofactor of } a_{ij}.$$

To obtain $\mathbf{A}^{-1}$ it is merely necessary to evaluate determinants of this type and divide them by $|A|$. Aitken's method evaluates all such determinants simultaneously, using Chio's reduction technique in much the same manner as illustrated above with Albert's method. Thus,

$$\left\Vert\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \\ \hline -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \end{array}\right\Vert,$$

$$a_{11}\left\|\begin{array}{cccc|cccc} 1 & \dfrac{a_{12}}{a_{11}} & \cdots & \dfrac{a_{1n}}{a_{11}} & \dfrac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \left(a_{22}-a_{12}\dfrac{a_{21}}{a_{11}}\right) & \cdots & \left(a_{2n}-a_{1n}\dfrac{a_{21}}{a_{11}}\right) & -\dfrac{a_{21}}{a_{11}} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \left(a_{n2}-a_{12}\dfrac{a_{n1}}{a_{11}}\right) & \cdots & \left(a_{nn}-a_{1n}\dfrac{a_{n1}}{a_{11}}\right) & -\dfrac{a_{n1}}{a_{11}} & 0 & \cdots & 1 \\ \hline 0 & \dfrac{a_{12}}{a_{11}} & \cdots & \dfrac{a_{1n}}{a_{11}} & \dfrac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \end{array}\right\|,$$

$$a_{11}b_{22}\left\|\begin{array}{ccccc|cccc} 1 & \dfrac{a_{12}}{a_{11}} & \dfrac{a_{13}}{a_{11}} & \cdots & \dfrac{a_{1n}}{a_{11}} & \dfrac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & 1 & \dfrac{b_{23}}{b_{22}} & \cdots & \dfrac{b_{2n}}{b_{22}} & -\dfrac{a_{21}}{a_{11}b_{22}} & \dfrac{1}{b_{22}} & \cdots & 0 \\ 0 & 0 & \left(b_{33}-b_{23}\dfrac{b_{32}}{b_{22}}\right) & \cdots & \left(b_{3n}-b_{2n}\dfrac{b_{32}}{b_{22}}\right) & \left(\dfrac{a_{21}b_{32}}{a_{11}b_{22}}-\dfrac{a_{31}}{a_{11}}\right) & -\dfrac{b_{32}}{b_{22}} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \left(b_{n3}-b_{23}\dfrac{b_{n2}}{b_{22}}\right) & \cdots & \left(b_{nn}-b_{2n}\dfrac{b_{n2}}{b_{22}}\right) & \left(\dfrac{a_{21}b_{n2}}{a_{11}b_{22}}-\dfrac{a_{n1}}{a_{11}}\right) & -\dfrac{b_{2n}}{b_{22}} & \cdots & 1 \\ \hline 0 & 0 & \left(\dfrac{a_{13}}{a_{11}}-\dfrac{a_{12}b_{23}}{a_{11}b_{22}}\right) & \cdots & \left(\dfrac{a_{1n}}{a_{11}}-\dfrac{a_{12}b_{2n}}{a_{11}b_{22}}\right) & \left(\dfrac{a_{21}^2}{a_{11}^2b_{22}}+\dfrac{1}{a_{11}}\right) & -\dfrac{a_{12}}{a_{11}b_{22}} & \cdots & 0 \\ 0 & 0 & \dfrac{b_{23}}{b_{22}} & \cdots & \dfrac{b_{2n}}{b_{22}} & -\dfrac{a_{12}}{a_{11}b_{22}} & \dfrac{1}{b_{22}} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \end{array}\right\|, \cdots .$$

When zeros are obtained below the main diagonal to the left of the vertical dividing line, the matrix in the lower right section will be $\mathbf{A}^{-1}$. This follows from the fact that the elements of this matrix will be the evaluations of bordered determinants, like those of the previous paragraph, divided by $a_{11}b_{22}\cdots = |A|$.

It will be observed that the operations on **A** in Albert's method which produce zeros below the main diagonal are the same as those which occur above the horizontal dividing line in Aitken's method. This set of operations is performed simultaneously on **I**, since the upper right section of Aitken's scheme is **I**. Furthermore, obtaining a zero for an element below the horizontal line and to the left of the vertical line, is equivalent to obtaining a zero for the element corresponding to the same row and column in the section above the horizontal, provided the preceding columns contain zeros above the diagonal. But obtaining zeros above the main diagonal of **A** constitutes the second set of operations in Albert's method to obtain $\mathbf{A}' = \mathbf{I}$. Thus, the operations in Aitken's method which produce zeros in a given column for elements above the horizontal line are merely the first set of operations in Albert's method, while those which produce zeros below the horizontal line are the second set of operations in reverse order. Since, in Aitken's scheme, the first set of operations is performed on **I** in the upper right section and the results are transferred a row at a time to the lower right section, where they are in turn operated upon by the second set of operations, this lower right section is merely **I** operated upon by the entire set of operations of Albert's method. Consequently, Aitken's and Albert's methods are the same except for the order in which operations are performed and differences arising therefrom. Since Aitken's method performs these operations more compactly, it is to be preferred to that of Albert.

Next consider the method of Doolittle, which is described by following the instructions given in the first column in the table shown on page 348. The forward solution is completed after n such sectional operations. For a given k column, the backward solution is obtained as usual by substitution in the last row of each section taken in reverse order.

If all summations in each section are performed in pairs and the sums recorded each time, rather than being performed in one operation, the forward solution of the Doolittle method will be found to be a rearrangement of the work occurring above the horizontal line in Aitken's method. Thus the first lines of each section give the matrix above the horizontal line in Aitken's scheme. Then, except for signs, I' and the sums of the first two lines of the remaining sections give the result of Aitken's first sequence of operations above the horizontal. Then, except for signs, II' and the sums of the first three lines of the remaining sections give the result of Aitken's second sequence of operations above the horizontal, etc.

The back solution involves precisely the same operations as those making up the second set of Albert's sequence of operations to obtain zeros above the main diagonal. Since these were shown to be a rearrangement of operations in Aitken's method, it follows that the methods of Aitken and Doolittle are the same except for the order of operations and differences arising therefrom. Hence all three methods are basically the same when systematized for a calculating machine.

Because of this equivalence, the number of necessary multiplications and divisions will be the same for all three methods, and will be found to be $\frac{1}{2}n^2(n + 1)$. Since Aitken's method is to be preferred to that of Albert, it will suffice to compare the methods of Aitken and Doolittle for calculating convenience.

The Doolittle method possesses several distinct advantages. First, its multiplications occur a row at a time with one of the factors constant for that row; consequently the keyboard remains unchanged for a given row of operations.

	1	2	3	$\cdots$	n	k_1	k_2	$\cdots$	k_n
I and ΣI	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}	-1	0		0
I′	-1	$-\frac{a_{12}}{a_{11}}$	$-\frac{a_{13}}{a_{11}}$	$\cdots$	$-\frac{a_{1n}}{a_{11}}$	$\frac{1}{a_{11}}$	0	$\cdots$	0
II	a_{21}	a_{22}	a_{23}	$\cdots$	a_{2n}	0	-1		0
$\Sigma I \cdot I'_2$	$-a_{12}$	$-a_{12}\frac{a_{12}}{a_{11}}$	$-a_{13}\frac{a_{12}}{a_{11}}$	$\cdots$	$-a_{1n}\frac{a_{12}}{a_{11}}$	$\frac{a_{12}}{a_{11}}$	0		0
ΣII	0	$\left(a_{22} - a_{12}\frac{a_{12}}{a_{11}}\right)$	$\left(a_{23} - a_{13}\frac{a_{12}}{a_{11}}\right)$	$\cdots$	$\left(a_{2n} - a_{1n}\frac{a_{12}}{a_{11}}\right)$	$\frac{a_{12}}{a_{11}}$	-1	$\cdots$	0
II′	0	-1	$-\frac{b_{23}}{b_{22}}$	$\cdots$	$-\frac{b_{2n}}{b_{22}}$	$-\frac{a_{12}}{a_{11}b_{22}}$	$\frac{1}{b_{22}}$		0
III	a_{31}	a_{32}	a_{33}	$\cdots$	a_{3n}	0	0		0
$\Sigma I \cdot I'_3$	$-a_{13}$	$-a_{12}\frac{a_{13}}{a_{11}}$	$-a_{13}\frac{a_{13}}{a_{11}}$	$\cdots$	$-a_{1n}\frac{a_{13}}{a_{11}}$	$\frac{a_{13}}{a_{11}}$	0		0
$\Sigma II \cdot II'_3$	0	$-b_{23}$	$-b_{23}\frac{b_{23}}{b_{22}}$	$\cdots$	$-b_{2n}\frac{b_{23}}{b_{22}}$	$-\frac{a_{12}b_{23}}{a_{11}b_{22}}$	$\frac{b_{23}}{b_{22}}$	$\cdots$	0
ΣIII	0	0	c_{33}	$\cdots$	c_{3n}	c_{3n+1}	$\frac{b_{23}}{b_{22}}$		0
III′	0	0	-1	$\cdots$	$-\frac{c_{3n}}{c_{33}}$	$-\frac{c_{3n+1}}{c_{33}}$	$-\frac{b_{23}}{b_{22}c_{33}}$		0
$\vdots$			$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$

Aitken's method, however, consists of calculating successive cross products' which requires clearing of the keyboard after each such operation. Secondly, there are fewer additions in the Doolittle method. It sums i quantities at a time in section i, while Aitken's cross products always involve the sum of two quantities. Because of the necessity of calculating the complements of negative sums, this difference becomes important when the number of variables is large. A third feature in favor of the Doolittle method is the ease of performing the calculations without previous experience. It may be easier to understand how to calculate cross products, but actually the calculations of the Doolittle method are easier to perform. Aitken's method requires some experience with it, if one is to avoid repeating certain calculations which would result from calculating all cross products mechanically. The comparative amount of copying in the two methods depends upon the number of variables involved.

From the above considerations, it may be concluded that the Doolittle method is to be preferred among those considered in this paper for solving a set of normal equations or calculating the inverse of a symmetric matrix. However, if a single calculating technique is desired which can be used for nonsymmetrical equations as well, then the method of Aitken is to be preferred.