ON MECHANICAL TABULATION OF POLYNOMIALS

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1. Introduction. The purpose of this paper is to show how automatic accounting machines, which have been used previously in evaluating such quantities as Σx^n and $\Sigma x^{n-1}y$, may be used in the preparation of mathematical tables of integral powers, of polynomials, and of functions which can be approximated by polynomials. These tables may be prepared for any desired intervals of the argument such as 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{2}$, $\frac{1}{3}$, etc.

The method is an adaptation of the general theory of "cumulative" or "progressive" totals which has proved useful in computing moments and product moments both with and without accounting machines. The reader unfamiliar with the mathematical method and its machine applications might refer to such presentations as those of Hardy [1], Mendenhall and Warren [2, 3], Razram and Wagner [4], Brandt [5], and Dwyer [6, 7]. The main feature of the method is the computation of summed products or of summed powers by means of successive cumulated additions. It is shown in this paper how it is possible to use this same process in constructing tables of powers and tables of polynomials.

2. The Cumulative Formulas. If the numbers F_x are defined and finite for $x = 1, 2, 3, \cdots, (a - 1), a$, and if these values of F_x are cumulated for $x = a, x = a - 1$, etc., then the value in the row headed by $x = 1$ can be written as ${}^1T_1$. If these cumulations are cumulated successively with the superscript indicating the order of the cumulation and the subscript indicating the value of x which heads the row, then

$$ {}^2T_1 = \Sigma x F_x, \qquad {}^3T_1 = \Sigma \frac{(x+1)x}{2!} F_x, \qquad {}^3T_2 = \Sigma \frac{x(x-1)}{2!} F_x, $$

$$ {}^4T_1 = \Sigma \frac{(x+2)(x+1)x}{3!} F_x $$

and in general for $i < j$,

$$ {}^jT_i = \Sigma \frac{[x + j - (i+1)]^{(j-1)}}{(j-1)!} F_x. \tag{1} $$

Formula (1) is basic to much of the previous work involving cumulative totals. Various authors have studied such important special cases as (A) where F_x equals the frequency function f_x, (B) where $F_x = xf_x$, and (C) where F_x equals the sum of all the values of y having the same x value. These special cases have been found very useful in computing moments and product moments.

The moments may be expressed in terms of the cumulations in a variety of ways. The diagonal formulas have the differences of zero as coefficients and are expressed in terms of ${}^1T_1$, ${}^2T_1$, ${}^3T_2$, ${}^4T_3$, ${}^5T_4$, etc. The columnar formulas, whose coefficients have been recently studied [6, 7], are expressed in terms of cumulations of the same order, ${}^jT_x$, with j fixed. Razram and Wagner [4] have given formulas which utilize the entries of different rows and different columns but which demand fewer entries for the formulas. Razram and Wagner worked out the formulas through $\Sigma x^4 f_x$ but the argument holds for $\Sigma x^i F_x$. For purposes of comparison the values of $\Sigma x^i F_x$, $i = 0, 1, 2, 3, 4$, as they appear in the diagonal, columnar, and Razram-Wagner systems are presented in Table I.

TABLE I

Values of $\Sigma x^i F_x$ for $i = 0, 1, 2, 3, 4$.

F_x	Diagonal	Columnar	Razram-Wagner
ΣF_x	${}^1T_1$	${}^1T_1$	${}^1T_1$
ΣxF_x	${}^2T_1$	${}^2T_1$	${}^2T_1$
Σx^2F_x	${}^2T_1 + 2{}^3T_2$	${}^3T_1 + {}^3T_2$	${}^3T_1 + {}^3T_2 = {}^3T_{1+2}$
Σx^3F_x	${}^2T_1 + 6{}^3T_2 + 6{}^4T_3$	${}^4T_1 + 4{}^4T_2 + {}^4T_3$	${}^2T_1 + 6{}^4T_2$
Σx^4F_x	${}^2T_1 + 14{}^3T_2 + 36{}^4T_3 + 24{}^5T_4$	${}^5T_1 + 11{}^5T_2 + 11{}^5T_3 + {}^5T_4$	${}^3T_{1+2} + 12{}^5T_{2+3}$

In developing the theory of the later sections of this paper I have developed further formulas of the type shown by Razram and Wagner since these formulas have fewer terms than do those of the other systems and the coefficients are factorable by $(j-1)!/2$. These formulas for $\Sigma x^s F_x$, with s even, feature such terms as ${}^3T_1 + {}^3T_2 = {}^3T_{1+2}$, ${}^5T_{2+3}$, etc., so that there are two entries from the same column. For the purposes of this paper it is preferable to have a single entry from each column and this situation results from continued application of the formula

$$(2) \qquad {}^jT_{i+(i+1)} = {}^jT_i + {}^jT_{i+1} = {}^{j-1}T_i + 2\,{}^jT_{i+1}.$$

The formulas for $\Sigma x^s F_x$ with $s \leqq 12$ are given. The alternative forms are given for the formulas involving even values of s.

$$\Sigma F_x = {}^1T_1, \quad \Sigma xF_x = {}^2T_1, \quad \Sigma x^2F_x = {}^3T_1 + {}^3T_2 = {}^3T_{1+2} = {}^2T_1 + 2\,{}^3T_2,$$

$$\Sigma x^3F_x = {}^2T_1 + 6\,{}^4T_2, \qquad \Sigma x^4F_x = {}^3T_{1+2} + 12\,{}^5T_{2+3}$$
$$= {}^2T_1 + 2\,{}^3T_2 + 12\,{}^4T_2 + 24\,{}^5T_3,$$

$$\Sigma x^5F_x = {}^2T_1 + 30\,{}^4T_2 + 120\,{}^6T_3,$$

$$\Sigma x^6F_x = {}^3T_{1+2} + 60\,{}^5T_{2+3} + 360\,{}^7T_{3+4}$$
$$= {}^2T_1 + 2\,{}^3T_2 + 60\,{}^4T_2 + 120\,{}^5T_3 + 360\,{}^6T_3 + 720\,{}^7T_4,$$

$$\Sigma x^7F_x = {}^2T_1 + 126\,{}^4T_2 + 1680\,{}^6T_3 + 5040\,{}^8T_4,$$

$$\Sigma x^8F_x = {}^3T_{1+2} + 252\,{}^5T_{2+3} + 5040\,{}^7T_{3+4} + 20160\,{}^9T_{4+5}$$
$$= {}^2T_1 + 2\,{}^3T_2 + 252\,{}^4T_2 + 504\,{}^5T_3 + 5040\,{}^6T_3 + 10080\,{}^7T_4$$
$$+ 20160\,{}^8T_4 + 40320\,{}^9T_5,$$

$$(3)\quad \begin{aligned} \Sigma x\, {}^{9}F_x &= {}^{2}T_1 + 510\, {}^{4}T_2 + 17640\, {}^{6}T_3 + 151200\, {}^{8}T_4 + 362880\, {}^{10}T_5, \\ \Sigma x\, {}^{10}F_x &= {}^{3}T_{1+2} + 1020\, {}^{5}T_{2+3} + 52920\, {}^{7}T_{3+4} + 604800\, {}^{9}T_{4+5} + 1814400\, {}^{11}T_{5+6} \\ &= {}^{2}T_1 + 2\, {}^{3}T_2 + 1020\, {}^{4}T_2 + 2040\, {}^{5}T_3 + 52920\, {}^{6}T_3 + 105840\, {}^{7}T_4 \\ &\quad + 604800\, {}^{8}T_4 + 1209600\, {}^{9}T_5 + 1814400\, {}^{10}T_5 + 3628800\, {}^{11}T_6, \\ \Sigma x\, {}^{11}F_x &= {}^{2}T_1 + 2046\, {}^{4}T_2 + 168960\, {}^{6}T_3 + 3160080\, {}^{8}T_4 + 19958400\, {}^{10}T_5 \\ &\quad + 39916800\, {}^{12}T_6, \\ \Sigma x\, {}^{12}F_x &= {}^{3}T_{1+2} + 4092\, {}^{5}T_{2+3} + 506880\, {}^{7}T_{3+4} + 12640320\, {}^{9}T_{4+5} \\ &\quad + 99792000\, {}^{11}T_{5+6} + 239500800\, {}^{13}T_{6+7} \\ &= {}^{2}T_1 + 2\, {}^{3}T_2 + 4092\, {}^{4}T_2 + 8184\, {}^{5}T_3 + 506880\, {}^{6}T_3 + 1013760\, {}^{7}T_4 \\ &\quad + 12640320\, {}^{8}T_4 + 25280640\, {}^{9}T_5 + 99792000\, {}^{10}T_5 \\ &\quad + 199584000\, {}^{11}T_6 + 239500800\, {}^{12}T_6 + 479001600\, {}^{13}T_7. \end{aligned}$$

The derivation of these formulas is obtained with the use of (1), with the use of

$$(4)\qquad {}^{j}T_i = {}^{j}T_{i+1} + {}^{j-1}T_i,$$

and with the use of formulas of lower order. For example we have from (1)

$$\Sigma\, \frac{(x+4)(x+3)(x+2)(x+1)x}{120} F_x = {}^{6}T_1$$

so that

$$\Sigma x\, {}^{5}F_x = 120\, {}^{6}T_1 - 10\, \Sigma x\, {}^{4}F_x - 35\, \Sigma x\, {}^{3}F_x - 50\, \Sigma x\, {}^{2}F_x - 24\, \Sigma x F_x$$

which after substitution of $\Sigma x\, {}^{4}F_x$, $\Sigma x\, {}^{3}F_x$, etc. and simplification results in the value ${}^{2}T_1 + 30\, {}^{4}T_2 + 120\, {}^{6}T_3$.

3. Tables of powers. If $F_x = 1$ when $x = a$, but is zero otherwise then $\Sigma x\, {}^{s}F_x$ is equal to a^s. It follows that the value of a^s can be obtained from the successive cumulations of this F_x with the use of (3). For example in Table II

TABLE II

Cumulations of $F_x = 1$, *when* $x = 6$,
0, *when* $x \neq 6$.

a	x	F_x	${}^{1}T$	${}^{2}T$	${}^{3}T$	${}^{4}T$	${}^{5}T$
1	6	1	1	1	1	1	1
2	5	0	1	2	3	4	5
3	4	0	1	3	6	10	15
4	3	0	1	4	10	20	*35*
5	2	0	1	5	*15*	*35*	70
6	1	0	1	*6*	21	56	126
7		0	1	7	28	84	210
8		0	1	8	36	120	330

$$6^2 = {}^2T_1 + 2\,{}^3T_2 = 6 + 2(15) = 36,$$

$$6^3 = {}^2T_1 + 6\,{}^4T_2 = 6 + 6(35) = 216,$$

$$6^4 = {}^2T_1 + 2\,{}^3T_2 + 12\,{}^4T_2 + 24\,{}^5T_3 = 6 + 2(15) + 12(35) + 24(35) = 1296.$$

The values of ${}^2T_1$, ${}^3T_2$, ${}^4T_2$ and ${}^5T_3$ for $a = 6$ are italicized in Table II.

To get the values of 5^2, 5^3, 5^4, etc. it would be necessary to start to cumulate from $x = 5$. Now since the values of ${}^1T_i$ are unity, it follows that the values for $a = 5$ can be found by taking the entries above those for $a = 6$. Thus ${}^2T_1 = 5$, ${}^3T_2 = 10$, ${}^4T_2 = 20$, ${}^5T_3 = 15$ with $5^2 = 5 + 2(10)$, $5^3 = 5 + 6(20)$, $5^4 = 5 + 2(10) + 12(20) + 24(15)$. It is evident in general that the values for any a^2, a^3, a^4 can be obtained by taking the row headed by a as the bottom row. Thus using $a = 8$, we have $8^2 = 8 + 2(28)$, $8^3 = 8 + 6(84)$, etc. It then appears that we may omit the x column of Table II and consider the cumulations to be ascending cumulations for a rather than descending cumulations for x.

A more satisfactory course is to cumulate the coefficients so as to eliminate the multiplications. Thus the value of $6\,{}^jT_i$ could be obtained without multiplication by cumulating 6, 0, 0, 0, 0 $\cdots$ rather than 1, 0, 0, 0, $\cdots$. Several cumulations may be carried on at the same time so that the additions are not necessary and the tabulation results in a table of the desired powers.

In preparation of a power table, the formulas (3) become a series of instructions on the way in which we are to do the cumulating. For instance the formula:

$$x^7 = 5040\,{}^8T_4 + 1680\,{}^6T_3 + 126\,{}^4T_2 + {}^2T_1,$$

tells us that to form a table of the seventh power we must cumulate[1] the coefficient 5040 eight times; add in the coefficient 1680 when there are six operations; the coefficient 126 when there are four; and the coefficient 1 when there are two remaining. A change in subscript tells us that the coefficient when first included forms a separate total ahead of the ones already partly figured. When the subscript does not change, the coefficient is to be included in the first summary card total. The final cumulating operation prints the actual table.

To prepare a power table by machine we secure a set of cards punched all alike with the numbers from 1 to 9 punched diagonally in successive columns across the card. The machine is wired to add the coefficient of the highest term by selecting the proper digits from the diagonals, cumulate after each card and summary punch each total. This way of starting saves one cumulation. The summary cards are cumulated repeatedly in the same manner until the number of operations indicated by the highest term is completed. When the number of operations remaining equals j of another term ${}^jT_i$, a card for the coefficient of that term is included in the tabulation ahead of the summary cards. This automatically adds the new coefficient to each term of the series. When the subscript i in ${}^jT_i$ changes, the new coefficient card must form a separate total;

[1] This operation is generally known as *progressive totalling* in machine operation.

when it does not change, the coefficient card must tabulate in the first summary card total.

To illustrate the tabulation of power tables, the formula for the cube table is— $x^3 = 6\,{}^4T_2 + {}^2T_1$.

The successive operations yield the following table:

TABLE III

x	Operation number 1	2	3	$4{:}x^3$
1	0	0	1	1
2	6	6	7	8
3	6	12	19	27
4	6	18	37	64
5	6	24	61	125
6	6	30	91	216
7	6	36	127	343
8	6	42	169	512
9	6	48	217	729
10	6	54	271	1000

In actual machine work, operation 1 can be omitted and work begun with operation 2. The machine is set to add the coefficient 6 of the highest term from each card and an accumulated total is printed and punched for each card tabulated, giving the results shown under operation 2. An additional card is punched for the coefficient of the second term, 1, and placed ahead of the cards produced in operation 2. The cumulation and punching is repeated, giving the results shown under operation 3. The summary cards from this operation are cumulatively tabulated, giving the results shown under operation 4, which is the table of cubes desired.

Similarly, for a table of the fourth power, the formula $x^4 = 24\,{}^5T_3 + 12\,{}^4T_2 + 2\,{}^3T_2 + {}^2T_1$ indicates the following operations—

TABLE IV

x	Operation number 1	2	3	4	$5{:}x^4$
1	0	0	0	1	1
2	0	12	14	15	16
3	24	36	50	65	81
4	24	60	110	175	256
5	24	84	194	369	625
6	24	108	302	671	1296
7	24	132	434	1105	2401
8	24	156	590	1695	4096
9	24	180	770	2465	6561
10	24	204	974	3439	10000
11	24	228	1202	4641	14641
12	24	252	1454	6095	20736

Note in operation 3 where the subscript does not change, the coefficient 2 is added to the first card punched by the machine, while in operation 4 where it changes, the coefficient 1 appears as a separate total.

4. Tables of polynomials. To tabulate values of $f(x) = a + bx + cx^2 \cdots$ (where $a, b, c, \cdots$, are positive or negative coefficients) the method is similar to that of preparing power tables except that the coefficients to be added are determined by multiplying the coefficients of the formulas for the different powers by the values a, b, c etc., adding the coefficients of like terms in the various formulas, and using these resultant coefficients in place of the simple coefficients used in the power tables. Thus if we wish to tabulate values of $f(x) = 4 + 3x + 2x^2 + x^5$ the coefficients are found as follows:

$$\begin{array}{rl} 4x^0 = & 4\,{}^1T_0 \\ +\,3x = & \quad +\,3\,{}^2T_1 \\ +\,2x^2 = & \quad +\,2\,{}^2T_1 + 2\cdot 2\,{}^3T_2 \\ +\,x^5 = & \quad +\,{}^2T_1 \qquad\qquad + 30\,{}^4T_2 + 120\,{}^6T_3 \\ \hline f(x) = & 4\,{}^1T_0 + 6\,{}^2T_1 + 4\,{}^3T_2 + 30\,{}^4T_2 + 120\,{}^6T_3 \end{array}$$

This equation gives instructions to perform six operations with 120 as coefficient; adding the coefficient 30 as a separate total when there are 4 operations remaining; adding 4 to the first summary card total when there are 3 operations; adding 6 as a separate total when there are 2 operations remaining; and adding 4 on the last operation.

The first few totals appear thus—

TABLE V

x	*Operation number* 1	2	3	4	5	6:$f(x)$
0						4
1	0	0	0	0	6	10
2	0	0	30	34	40	50
3	120	120	150	184	224	274
4	120	240	390	574	798	1072
5	120	360	750	1324	2122	3194
6	120	480	1230	2554	4676	7870
7	120	600	1830	4384	9060	*16930*
8	120	720	2550	6934	*15994*	32924
9	120	840	*3390*	*10324*	26318	59242
10	*120*	*960*	4350	14674	40992	100234

It is not necessary to confine these tables to values for whole numbers, as we can tabulate equally well values of $f(x)$ for intervals of x of .1, .01 or .001 or $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ etc. In this case, before combining formulas for different powers we multiply both sides by the desired interval raised to the power to which x is raised in that particular formula, then add like terms as before.

To tabulate the previous example in $.1x$ intervals we proceed as follows:

$$4x^0 = 4.000\ {}^1T_0$$

$$3x/10 = \quad + .3\ {}^2T_1$$

$$2(x/10)^2 = \quad + .02\ {}^2T_1 \quad + .04\ {}^3T_2$$

$$(x/10)^5 = \quad + .00001\ {}^2T_1 \quad + .00030\ {}^4T_2 + .00120\ {}^6T_3$$

$$f(x) = 4\ {}^1T_0 \quad + .32001\ {}^2T_1 + .04\ {}^3T_2 + .00030\ {}^4T_2 + .00120\ {}^6T_3$$

TABLE VI

x	Operation number					
	1	2	3	4	5	6: $f(x)$
1	0	0	0	0	.32001	4.32001
2	0	0	.0003	.0403	.36031	4.68032
3	.0012	.0012	.0015	.0418	.40211	5.08243
4	.0012	.0024	.0039	.0457	.44781	5.53024
5	.0012	0036	.0075	.0532	.50101	6.03125
6	.0012	.0048	.0123	.0655	.56651	6.59776
7	.0012	.0060	.0183	.0738	.64031	7.23807
8	.0012	.0072	.0255	.0993	.73961	8.07768
9	.0012	.0084	.0339	.1332	.87281	8.95049
10	.0012	.0096	.0435	.1767	1.04951	10.00000

Where any coefficients are negative in the equations expressed in ${}^jT_i$ terms, they are simply added in as minus figures.

To round off the preceding function to 3 decimal places, we add 5 to the constant term ${}^1T_0$ in the position to the right of the last decimal retained, i.e. in this case the 4th decimal place. The constant term is then 4.0005.

Exact	*Counter reads*	*Prints*
4.32001	4.32051	4.320
4.68032	4.68082	4.680
5.08243	5.08293	5.082
5.53024	5.53074	5.530
6.03125	6.03175	6.031
6.59776	6.59826	6.598
7.23807	7.23857	7.238
8.07768	8.07818	8.078
8.95049	8.95099	8.950
10.00000	10.00050	10.000

5. Automatic calculation of polynomial coefficients. Frequently when polynomials are being evaluated, the process of forming the coefficients can be performed automatically from a punched-card table. Such a table consists of a set of cards for each power x^s containing the multiples of all the coefficients of each of the terms ${}^jT_i$ in the formula (3) for that power. These multiples are 1, 2, 3, 4, $\cdots$, 9; 10, 20, 30, 40, $\cdots$, 90; 100, 200, $\cdots$, 900; 1000, 2000 etc., and may be produced automatically by making a linear table of each coefficient in the manner described in this paper. Each card is punched with the information called for by the heading of the following card form:

s	j	i	multiple	coeff. × multiple
07	06	03	00005	008400

The particular figures indicated are those which would be punched for the term $5(1680)^6T_3$ in the representation of $5x^7$ according to formula (3).

The table is used by withdrawing the cards for the coefficients a, b, c, d, etc. of the desired polynomial. For instance, if one of the polynomial coefficients is 14485 x^7, we select from the x^7 section of the table all cards containing the multiples 10000, 4000, 400, 80, and 5. In the x^7 table there are 4 cards for each multiple, one each for terms ${}^8T_4$, ${}^6T_3$, ${}^4T_2$, and ${}^2T_1$. These cards are combined with the cards selected for the other coefficients of the polynomial and sorted to bring all cards for each ${}^jT_i$ together. The cards for each term ${}^jT_i$ are then automatically added on the electric accounting machine.

6. Subdividing tables. In preparing tables it may be desired to prepare the table in more detail at certain points, giving values of the function at 1/10, 1/20, 1/50, or 1/100, etc., of the interval of the rest of the table. This may readily be done by recalculating the coefficients of the cumulative terms, and using these values in the same manner as the original ones.

There are many formulas for the determination of the subdivided differences given in various texts on interpolation, such as those given by Comrie [8] and Bower [9]. One effective method is to use formulas (3) to calculate the subdivided differences. The values called for in the formula for the highest power are taken from the table of the function at the regular interval, giving effect to the rule involving subscripts. These coefficients are reduced by an amount sufficient to cancel the coefficient of the highest cumulative term, and the coefficients of the remaining cumulative terms are reduced in proportion according to formula (3) for the highest power. Usually the coefficient of the highest term of the formula will divide evenly into the coefficient taken from the table, and the other reductions are calculated by multiplying this result by the other coefficients of the formula. The highest remaining coefficient is then reduced by an amount sufficient to cancel itself, and, by use of the formula (3) for the power whose highest cumulative term matches the highest remaining coefficient, the reduction to the remaining cumulative terms is calculated and subtracted.

The highest remaining coefficient is reduced in a like manner, and this process is continued until all the cumulative coefficients have been analyzed.

The partial cumulative coefficients thus computed are multiplied by the desired subdivision $1/m$ raised to the power of the corresponding formula (3), and recombined to form the new coefficients, as shown in the example below. In taking values from the table, when the subscript does not change, the tabular value must be reduced by the amount of the higher coefficient with the same subscript, to give effect to the rule that the coefficients in such cases are increments (see last example in section 3).

To subdivide the polynomial of section 4 at $x = 7.0$, we take the italicized values from Table V starting at $f(7)$ as ${}^1T_0$, and proceed as follows:

	${}^6T_3$	${}^5T_3$	${}^4T_2$	${}^3T_2$	${}^2T_1$	${}^1T_0$
		960		10324		
From Table V	120	–120	3390	–3390	15994	16930
$F(x)$..............	120	840	3390	6934	15994	16930
ax^5	120		30		1	
		840	3360	6934	15993	
bx^4		840	420	70	35	
			2940	6864	15958	
cx^3			2940		490	
				6864	15468	
dx^2				6864	3432	
					12036	
ex................					12036	
						16930

If the interval is 1/10 we have:

		${}^6T_3$	${}^5T_3$	${}^4T_2$	${}^3T_2$	${}^2T_1$	${}^1T_0$
$x^5/10^5$	=	.00120		.00030		.00001	
$35x^4/10^4$	=		.0840	.04200	.0070	.00350	
$490x^3/10^3$	=			2.94000		.49000	
$3432x^2/10^2$	=				68.6400	34.32000	
$12036x/10$	=					1203.60000	+16930

$f(x) = .00120\,{}^6T_3 + .0840\,{}^5T_3 + 2.9823\,{}^4T_2 + 68.6470\,{}^3T_2 + 1238.41351\,{}^2T_1 + 16930\,{}^1T_0$ provides the coefficients for subtabulating the function at the desired interval, beginning at the argument $x = 7.0$.

7. Accuracy of Tables. When the values of the coefficients are not exact, owing to the original values for a, b, c etc. or the dropping of decimals in the computation of the coefficients, the errors accumulate fairly rapidly. Each coefficient will introduce its own error into the summation.

To maintain accuracy throughout a long table it is advisable to transform $f(x)$ by Horner's method of decreasing the roots [10, pp. 100–101], compute new coefficients for the transformed equation at intervals, and prepare the table in sections. Decreasing the roots by r gives us a new starting point at $x = r$.

Since two or more functions may be computed at one time, a function for which the coefficients are not exact may be computed by adding in the usual way from the starting values and subtracting from the ending values simultaneously. As many digits as agree in both tabulations of the function may be considered correct.

The tabulations can be made to practically any degree of accuracy on the equipment available, as the newer machines can be formed into counters of any capacity up to 80 digits. In practice, counters of 16, 20 or 24 digits will ordinarily suffice for the accuracy desired and two or more functions can be evaluated simultaneously. Cards are read and added at the rate of 150 per minute, or read, added and listed on the tape at the rate of 80 per minute and new summary cards produced at the rate of 40 per minute (on alphabetic equipment with gang summary punches). Computation may be carried out with additional decimal places and the final tabulation of the function rounded off to the nearest number retained.

8. Summary. The cumulative or progressive-total method is shown to be applicable to the preparation of tables of functions expressed in the form of a power series.

The cumulative formulas for the powers through the twelfth power have been presented, and simple methods are given for transforming a power series into its corresponding cumulative formula, for changing the interval of the table, rounding off the values of the function, and subdividing the table at desired points.

It is hoped that this discussion will make tables in printed or punched-card form more generally available as a tool for the computer. Since tables may be so readily prepared by this process, the usefulness of the tabular method of solving problems is greatly increased.

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