TABULATION OF THE PROBABILITIES FOR THE RATIO OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE TO THE VARIANCE

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with a note

By John von Neumann

In recent publications von Neumann has determined the distribution of δ^2/s^2, the ratio of the mean square successive difference to the variance, for odd values of the sample size n[1] and for even values of n.[2] In this paper the probability function, i.e., the integral of the distribution, is evaluated for specific values of n.

Let x be a stochastic variable normally distributed with mean ζ and the standard deviation σ. The following customary definitions for the sample are:

the mean, $$\bar{x} = \frac{1}{n}\sum_{\mu=1}^{n} x_\mu,$$

the variance, $$s^2 = \frac{1}{n}\sum_{\mu=1}^{n} (x_\mu - \bar{x})^2,$$

and the mean square successive difference, $\delta^2 = \frac{1}{n-1}\sum_{\mu=1}^{n-1} (x_{\mu+1} - x_\mu)^2$. Letting $\frac{\delta^2}{s^2} = \frac{2n}{n-1}(1 - \epsilon)$, von Neumann shows that the distribution of ϵ, $\omega(\epsilon)$, is symmetrical with zero mean and intercepts equal to $\pm \cos\frac{\pi}{n}$ (loc. cit.[1], p. 372), and that $\omega(\epsilon)$ is determined for odd values of n by

$$\frac{d^{\frac{1}{2}(n-1)-1}}{d\epsilon^{\frac{1}{2}(n-1)-1}}\omega(\epsilon) = \pm\frac{(\frac{1}{2}[n-1]-1)!}{\pi}\frac{1}{\sqrt{\prod_{\mu=1}^{n-1}\left(\epsilon - \cos\frac{\mu\pi}{n}\right)}},$$

in the odd intervals

$$\cos\frac{\pi}{n} \geqq \epsilon \geqq \cos\frac{2\pi}{n},$$

$$\frac{\cos 3\pi}{n} \geqq \epsilon \geqq \cos\frac{4\pi}{n}, \cdots, \frac{\cos(n-2)\pi}{n} \geqq \epsilon \geqq \frac{\cos(n-1)\pi}{n},$$

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[1] John von Neumann, "Distribution of the ratio of the mean square successive difference to the variance," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 367–395.

[2] John von Neumann, "A further remark on the distribution of the ratio of the mean square successive difference to the variance," *Annals of Math. Stat.*, Vol. 13 (1942), pp. 86–88.

and by $\dfrac{d^{\frac{1}{2}(n-1)-1}}{d\epsilon^{\frac{1}{2}(n-1)-1}}\,\omega(\epsilon) = 0$ in the even intervals

$$\cos\frac{2\pi}{n} \geqq \epsilon \geqq \cos\frac{3\pi}{n},$$

$$\cos\frac{4\pi}{n} \geqq \epsilon \geqq \cos\frac{5\pi}{n}, \cdots, \cos\frac{(n-3)\pi}{n} \geqq \epsilon \geqq \cos\frac{(n-2)\pi}{n}$$

(loc. cit.[1] pp. 389–390).

For $n = 3$,

$$\omega(\epsilon) = \frac{1}{\pi}\,\frac{1}{\sqrt{\frac{1}{4} - \epsilon^2}} \tag{1}$$

for $\cos\dfrac{\pi}{3} \geqq \epsilon \geqq \cos\dfrac{2\pi}{3}$.

For $n = 5$,

$$\omega'(\epsilon) = \frac{1}{\pi}\,\frac{1}{\sqrt{-\epsilon^4 + \frac{3}{4}\epsilon^2 - \frac{1}{16}}}$$

$$\omega(\epsilon) = \frac{1}{\pi}\,\frac{2}{\cos\frac{\pi}{5} + \cos\frac{2\pi}{5}}\,\mathrm{sn}^{-1}\left(\left[\frac{\cos\frac{\pi}{5} + \cos\frac{2\pi}{5}}{\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}}\cdot\frac{\epsilon + \cos\frac{\pi}{5}}{\cos\frac{\pi}{5} - \epsilon}\right]^{\frac{1}{2}}, \frac{\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}}{\cos\frac{\pi}{5} + \cos\frac{2\pi}{5}}\,{}^{3}\right) \tag{2}$$

for $\cos\dfrac{\pi}{5} \geqq \epsilon \geqq \dfrac{2\pi}{5}$ and $\cos\dfrac{3\pi}{5} \geqq \epsilon \geqq \cos\dfrac{4\pi}{5}$.

But for $\cos\dfrac{2\pi}{5} \geqq \epsilon \geqq \cos\dfrac{3\pi}{5}$, $\omega'(\epsilon) = 0$, thus

$$\omega(\epsilon) = \text{const.} \tag{3}$$

For $n = 7$,

$$\omega''(\epsilon) = \pm\frac{2}{\pi}\,\frac{1}{\sqrt{-\epsilon^6 + \frac{5}{4}\epsilon^4 - \frac{3}{8}\epsilon^2 + \frac{1}{64}}} \tag{4}$$

for $\cos\dfrac{\pi}{7} \geqq \epsilon \geqq \cos\dfrac{2\pi}{7}$ and $\cos\dfrac{5\pi}{7} \geqq \epsilon \geqq \cos\dfrac{6\pi}{7}$ with the $+$ sign, and for $\cos\dfrac{3\pi}{7} \geqq \epsilon \geqq \cos\dfrac{4\pi}{7}$ with the $-$ sign.

But for $\cos\dfrac{2\pi}{7} \geqq \epsilon \geqq \cos\dfrac{3\pi}{7}$ and $\cos\dfrac{4\pi}{7} \geqq \epsilon \geqq \cos\dfrac{5\pi}{7}$, $\omega''(\epsilon) = 0$, thus

$$\omega'(\epsilon) = \text{const.} \tag{5}$$

[3] The square of the modulus. The numerical evaluation of the inverse sine amplitude function used for $n = 4, 5, 6$, is taken from unpublished tables of the Legendrian elliptic integrals by F. V. Reno of the Ballistic Research Laboratory, Aberdeen Proving Ground. The square of the modulus is the argument for this tabulation.

For even values of n von Neumann shows that the distribution of ϵ,

$$\omega_{A+(0)}(\epsilon) = \frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma[\frac{1}{2}(n-2)]\Gamma(\frac{1}{2})} \int_{\frac{|\epsilon|}{\cos\frac{\pi}{n}}}^{1} \omega_A\left(\frac{\epsilon}{\rho}\right) \rho^{\frac{1}{2}n-3}(1-\rho)^{-\frac{1}{2}}\, d\rho \qquad \text{(loc. cit.[2])}.$$

For $n = 4$,

$$\omega_{A+(0)}(\epsilon) = \tfrac{1}{2} \int_{\sqrt{2}\,\epsilon}^{1} \frac{\omega_A\left(\frac{\epsilon}{\rho}\right) d\rho}{\rho\sqrt{1-\rho}},$$

$$\text{where } \omega_A\left(\frac{\epsilon}{\rho}\right) = \frac{1}{\pi}\left[-\left(\frac{\epsilon}{\rho} - \cos\frac{\pi}{4}\right)\left(\frac{\epsilon}{\rho} - \cos\frac{3\pi}{4}\right)\right]^{-\frac{1}{2}}.$$

(6)
$$\omega_{A+(0)}(\epsilon) = \frac{1}{\sqrt{2}\,\pi} \int_{\sqrt{2}\,\epsilon}^{1} \frac{d\rho}{\sqrt{(\rho - \sqrt{2}\,\epsilon)(\rho + \sqrt{2}\,\epsilon)(1-\rho)}}$$

$$= \frac{\sqrt{2}}{\pi\sqrt{1+\sqrt{2}\,\epsilon}}\,\mathrm{sn}^{-1}\left(1, \frac{1-\sqrt{2}\,\epsilon}{1+\sqrt{2}\,\epsilon}^{\frac{1}{2}}\right)$$

for $\cos\frac{\pi}{4} \geqq \epsilon \geqq \frac{3\pi}{4}$.

For $n = 6$, $\omega_{A+(0)}(\epsilon) = \frac{3}{4} \int_{2\epsilon/\sqrt{3}}^{1} \frac{\omega_A\left(\frac{\epsilon}{\rho}\right)}{\sqrt{1-\rho}}\, d\rho$, where

(7) $$\omega_A\left(\frac{\epsilon}{\rho}\right) = \frac{4}{\pi(\sqrt{3}+1)}\,\mathrm{sn}^{-1}\left((\sqrt{3}+1)\left[\frac{1}{2}\cdot\frac{\sqrt{3} - 2\,(\epsilon/\rho)}{\sqrt{3} + 2\,(\epsilon/\rho)}\right]^{\frac{1}{2}}, 7 - 4\sqrt{3}^{\frac{1}{2}}\right)$$

for $\cos\frac{\pi}{6} \geqq \epsilon \geqq \cos\frac{\pi}{3}$ and $\cos\frac{2\pi}{3} \geqq \epsilon \geqq \cos\frac{5\pi}{6}$, and where

(8) $$\omega(\epsilon) = \text{const.}$$

for $\cos\frac{\pi}{3} \geqq \epsilon \geqq \cos\frac{2\pi}{3}$.

The integrals needed to obtain $\omega(\epsilon)$ for $n = 6$ and $\omega'(\epsilon)$ for $n = 7$ have been evaluated by numerical quadrature. Graphs of the distribution of δ^2/s^2, $\omega(\delta^2/s^2)$, for $n = 3, 4, 5, 6, 7$, are shown in Fig. 1.

The probability function, $P(\delta^2/s^2 < k) = \int_0^k \omega(\delta^2/s^2).\ d(\delta^2/s^2)$ has been obtained from $\omega(\delta^2/s^2)$ by numerical quadrature for $n = 4, 5, 6, 7$. The results are given in Table III.

As is mentioned by von Neumann, R. H. Kent has suggested a series approximation of the form

$$\omega(\epsilon) \approx \sum_{h=0}^{\infty} a_h \left(\cos^2\frac{\pi}{n} - \epsilon^2\right)^{\frac{1}{2}n-2+h},$$

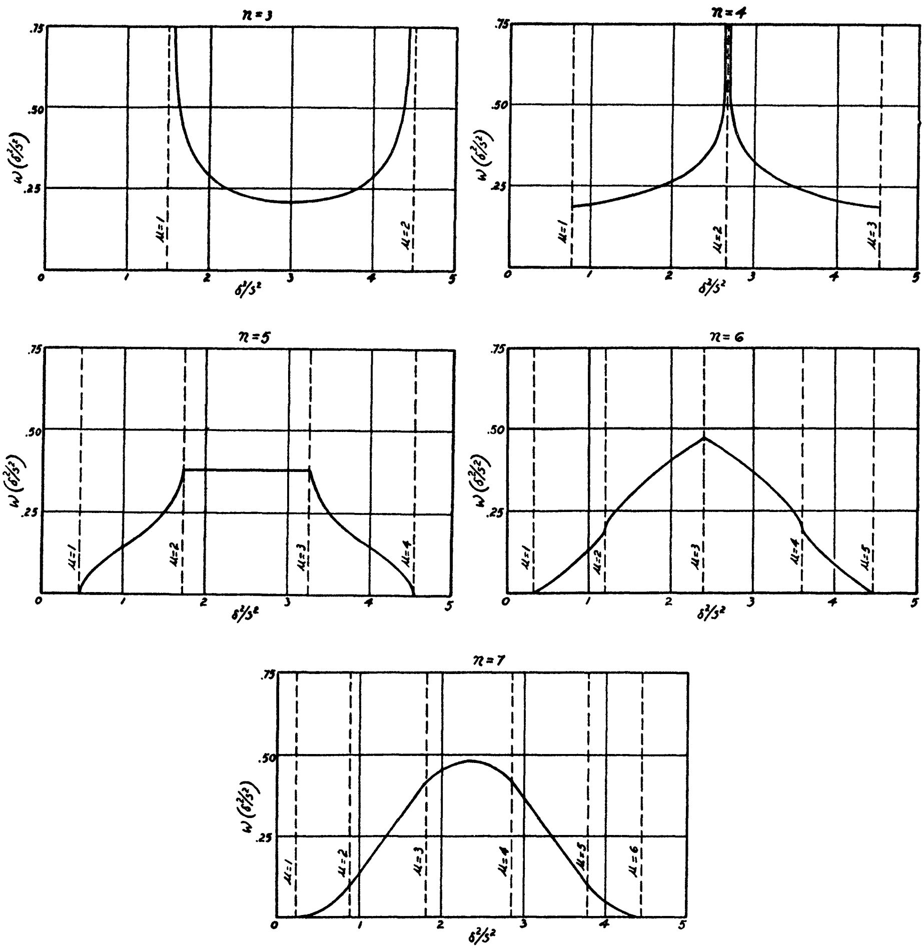

FIG. 1

since the order of vanishing of $\omega(\epsilon)$ is $\frac{1}{2}n - 2$, and since $\omega(\epsilon)$ is an even function of ϵ (loc. cit.[1] p. 391). Determining the a_h by the condition of normalization and by the first three even moments of the actual distribution, M_2, M_4 and M_6 (given on pp. 377–378, loc. cit.[1]), and integrating the result, we obtain

$$P(\epsilon < k') \approx \int_{-\cos\frac{\pi}{n}}^{\epsilon} \sum_{h=0}^{3} a_h \left(\cos^2 \frac{\pi}{n} - \epsilon^2\right)^{\frac{1}{2}n-2+h} d\epsilon$$

$$= \frac{(n-1)(n+1)(n+3)}{2^4} I_x(\tfrac{1}{2}[n-2], \tfrac{1}{2}[n-2])$$

$$\cdot \left[-\frac{1}{3} + \frac{M_2(n+5)}{\cos^2 \frac{\pi}{n}} - \frac{M_4(n+5)(n+7)}{3 \cos^4 \frac{\pi}{n}} + \frac{M_6(n+5)(n+7)(n+9)}{45 \cos^6 \frac{\pi}{n}} \right]$$

$$
\begin{aligned}
&+\frac{(n+1)(n+3)(n+5)}{2^4} I_x(\tfrac{1}{2}n, \tfrac{1}{2}n)\left[1-\frac{M_2(3n+13)}{\cos^2\frac{\pi}{n}}\right. \\
&\qquad\left.+\frac{M_4(3n+11)(n+7)}{3\cos^4\frac{\pi}{n}}-\frac{M_6(n+3)(n+7)(n+9)}{15\cos^6\frac{\pi}{n}}\right] \\
(9)\qquad &+\frac{(n+3)(n+5)(n+7)}{2^4} I_x(\tfrac{1}{2}[n+2], \tfrac{1}{2}[n+2])\left[-1+\frac{M_2(3n+11)}{\cos^2\frac{\pi}{n}}\right. \\
&\qquad\left.-\frac{M_4(3n+19)(n+3)}{3\cos^4\frac{\pi}{n}}+\frac{M_6(n+3)(n+5)(n+9)}{15\cos^6\frac{\pi}{n}}\right] \\
&+\frac{(n+5)(n+7)(n+9)}{2^4} I_x(\tfrac{1}{2}[n+4], \tfrac{1}{2}[n+4])\left[\frac{1}{3}-\frac{M_2(n+3)}{\cos^2\frac{\pi}{n}}\right. \\
&\qquad\left.+\frac{M_4(n+3)(n+5)}{3\cos^4\frac{\pi}{n}}-\frac{M_6(n+3)(n+5)(n+7)}{45\cos^6\frac{\pi}{n}}\right]
\end{aligned}
$$

The Tables of the Incomplete Beta-Function[4] can be used to evaluate (9), with $x = \frac{1}{2}\left(\frac{\epsilon}{\cos(\pi/n)} + 1\right)$. Table I shows the results obtained for the eighth and tenth moments for the distribution (9) and for the true distribution for certain values of n.

Table II gives a tabulation of $P\left(\frac{\delta^2}{s^2} < k\right)$ for $n = 7$ by the use of (9) and by the method of (4) and (5). The approximation (9) has been used for the computation of the probabilities of Table III for $n \geqq 8$.[5]

It has been shown (loc. cit.[1] pp. 378–379) that for $n \to \infty$ the distribution of ϵ becomes asymptotically normal. For $n = 60$ values of δ^2/s^2 are given below for different levels of significance. These values have been computed from Table III and from a table of the integral of the normal function with standard deviation equal to $\frac{2n}{n-1}\sqrt{\frac{n-2}{(n-1)(n+1)}}$, the square root of the second moment of the distribution of δ^2/s^2.

[4] Karl Pearson (Editor), *Tables of the Incomplete Beta-Function*, London: Biometrika Office, 1934.

[5] The results obtained by L. C. Young using the Pearson Type II distribution are sufficiently precise for the significance levels and sample sizes tabulated. Cf. L. C. Young, "On randomness in ordered sequences," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 293–300.

TABLE I

n	M_8 (9)	M_8 True	M_{10} (9)	M_{10} True
7	.00412	.00413	.00201	.00202
8	.00318	.00318	.00150	.00151
9	.00246	.00246	.00111	.00112

TABLE II

$P\left(\frac{\delta^2}{s^2} < k\right)$ for $n = 7$

k	By (9)	By (4) and (5)
.25	.00001	.00001
.30	.00007	.00007
.35	.00027	.00027
.40	.00065	.00065
.45	.00124	.00126
.50	.00209	.00214
.55	.00326	.00333
.60	.00478	.00486
.65	.00671	.00678
.70	.00911	.00913
.75	.01203	.01197
.80	.01552	.01534
.85	.01964	.01932
.90	.02443	.02403
.95	.02995	.02957
1.00	.03624	.03598
1.05	.04333	.04325
1.10	.05126	.05137
1.15	.06006	.06036
1.20	.06976	.07020

Values of δ^2/s^2 for Different Levels of Significance

$n = 60$

	$P = .001$	$P = .005$	$P = .01$	$P = .05$
Table III............	1.2558	1.3779	1.4384	1.6082
Normal.............	1.2358	1.3688	1.4333	1.6092

This work was undertaken at the suggestion of Mr. R. H. Kent. I am much indebted to him and to Professor John von Neumann for many important suggestions and criticisms.

Note to Fig. 1, by John von Neumann. Inspection of the graphs of $\omega(\delta^2/s^2)$ for $n = 3, 4, 5, 6, 7$ (see Fig. 1) discloses certain singularities of the function $\omega(\delta^2/s^2)$, which seem to deserve attention.

TABLE III

$$P\left(\frac{\delta^2}{s^2} < k\right) = \int_0^k \omega\left(\frac{\delta^2}{s^2}\right) d\left(\frac{\delta^2}{s^2}\right)$$

k \ n	4	5	6	7	8	9	10	11	12
.25				.00001	.00001	.00001	.00001		
.30				.00007	.00007	.00005	.00004	.00002	.00001
.35			.00006	.00027	.00021	.00014	.00009	.00005	.00003
.40			.00047	.00065	.00047	.00031	.00019	.00012	.00007
.45			.00126	.00126	.00088	.00059	.00038	.00025	.00016
.50		.00038	.00246	.00214	.00150	.00103	.00069	.00046	.00031
.55		.00223	.00409	.00333	.00237	.00168	.00116	.00080	.00055
.60		.00493	.00615	.00486	.00355	.00259	.00185	.00132	.00094
.65		.00830	.00865	.00678	.00511	.00382	.00282	.00208	.00152
.70		.01225	.01161	.00913	.00710	.00544	.00414	.00313	.00235
.75		.01673	.01505	.01197	.00958	.00753	.00587	.00455	.00351
.80	.00356	.02171	.01900	.01534	.01263	.01015	.00809	.00642	.00508
.85	.01302	.02717	.02348	.01932	.01631	.01338	.01089	.00883	.00714
.90	.02257	.03310	.02851	.02403	.02068	.01729	.01436	.01188	.00980
.95	.03223	.03949	.03412	.02957	.02579	.02196	.01858	.01565	.01316
1.00	.04199	.04634	.04035	.03598	.03171	.02745	.02363	.02025	.01733
1.05	.05186	.05364	.04728	.04325	.03849	.03384	.02959	.02578	.02241
1.10	.06184	.06140	.05500	.05137	.04618	.04120	.03655	.03232	.02852
1.15	.07194	.06963	.06361	.06036	.05482	.04957	.04458	.03997	.03577
1.20			.07323	.07020	.06445	.05901	.05375	.04882	.04425
1.25						.06956	.06412	.05894	.05407
1.30								.07040	.06531

k \ n	15	20	25	30	40	50	60
.35	.00001						
.40	.00002						
.45	.00004						
.50	.00009	.00001					
.55	.00018	.00002					
.60	.00033	.00005	.00001				
.65	.00059	.00012	.00002				
.70	.00100	.00024	.00005	.00001			
.75	.00161	.00044	.00011	.00003			
.80	.00250	.00076	.00023	.00007	.00001		
.85	.00375	.00127	.00044	.00015	.00002		
.90	.00547	.00206	.00079	.00030	.00004	.00001	
.95	.00778	.00323	.00135	.00057	.00010	.00002	
1.00	.01079	.00489	.00222	.00102	.00022	.00005	.00001
1.05	.01465	.00720	.00355	.00176	.00044	.00012	.00003
1.10	.01950	.01033	.00550	.00294	.00085	.00026	.00008
1.15	.02550	.01448	.00826	.00474	.00158	.00054	.00019
1.20	.03280	.01986	.01208	.00738	.00280	.00108	.00043
1.25	.04155	.02670	.01723	.01117	.00476	.00206	.00092
1.30	.05189	.03524	.02402	.01644	.00780	.00376	.00185
1.35	.06396	.04571	.03276	.02357	.01235	.00656	.00355
1.40	.07787	.05834	.04379	.03298	.01892	.01098	.00649
1.45		.07333	.05743	.04511	.02810	.01769	.01133
1.50			.07398	.06038	.04055	.02750	.01893
1.55				.07920	.05696	.04131	.03034
1.60					.07797	.06006	.04675
1.65						.08465	.06942
1.70							.09949

Values of k for which $P\left(\frac{\delta^2}{s^2} < k\right) = 0$

n	k	n	k
4	.7811	15	.0468
5	.4775	20	.0259
6	.3215	25	.0164
7	.2311	30	.0113
8	.1740	40	.0063
9	.1357	50	.0040
10	.1088	60	.0028
11	.0891		
12	.0743		

It will be noted that this function exhibits a more or less singular behavior in $n - 1$ points, the points

$$\delta^2/s^2 = \frac{2n}{n-1}\left(1 - \cos\frac{\mu\pi}{n}\right),$$

$$\mu = 1, \cdots, n - 1,$$

which is particularly well marked for the smaller values of n.

For odd values of n these singularities are of the order $\frac{1}{2}(n - 4)$ (i.e. the $\frac{1}{2}(n - 3)$ derivative becomes infinite of order $\frac{1}{2}$). This was shown loc. cit.[1], p. 390.

Thus for $n = 3$ the function has infinities of order $\frac{1}{2}$; for $n = 5$ the direction and for $n = 7$ the curvature have infinities of the same order.

For even values of n these singularities are also of order $\frac{1}{2}(n - 4)$ (i.e., the $\frac{1}{2}(n - 4)$ derivative has an ordinary discontinuity) when μ is odd, but a logarithmic factor must be added to this (i.e. the $\frac{1}{2}(n - 4)$ derivative becomes logarithmically infinite) when μ is even. Proofs of these statements will appear later.

Thus for $n = 4$ the function has an ordinary discontinuity at $\mu = 1, 3$ and a logarithmic infinity at $\mu = 2$; for $n = 6$ the direction has corresponding singularities at $\mu = 1, 3, 5$ and at $\mu = 2, 4$ respectively.

For $n \geqq 8$ (both even and odd) these phenomena would probably be much less easily recognized by mere inspection.