NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

NOTE ON RUNS OF CONSECUTIVE ELEMENTS

By J. Wolfowitz

Columbia University

In my paper [1] I did not derive the asymptotic distribution of $W(R)$, an omission which I wish to correct in this note.

Let the stochastic variable $R = (x_1, \cdots, x_n)$ be a permutation of the first n positive integers, where each permutation has the same probability $\frac{1}{n!}$. A subsequence $x_{i+1}, x_{i+2}, \cdots, x_{i+l}$, is called a run of consecutive elements of length l if:

a) when l' is any integer such that $1 \leq l' < l$,

$$|x_{i+l'} - x_{i+l'+1}| = 1$$

b) when $i > 0, |x_i - x_{i+1}| > 1$

c) when $i + l < n, |x_{i+l} - x_{i+l+1}| > 1$.

Let $W(R)$ be the total number of runs in R. Then $n - W(R)$ is a stochastic variable which, it will be shown, has in the limit the Poisson distribution with mean value 2. More precisely, if $p(w)$ is the probability that $n - W(R) = w$, then

$$\lim_{n \to \infty} p(w) = \frac{2^w}{e^2 \cdot w!}. \tag{1}$$

Proof: Define stochastic variables $y_i (i = 1, 2, \cdots, n)$, as follows: $y_i = 1$ if x_i is the first element of a run of length 2, $y_i = 0$ otherwise. It is easy to see that the probability that $x_i (i = 1, 2, \cdots, n)$ be the initial element of a run of length greater than two is $O\left(\frac{1}{n^2}\right)$ and hence that the probability of the occurrence of a run of length greater than two is $O\left(\frac{1}{n}\right)$. Hence the limiting distribution of $n - W(R)$ is the same as that of

$$y = \sum_{i=1}^{n} y_i,$$

provided either exists.

The y_i are dependent stochastic variables and almost all (i.e., all with the exception of a fixed number) have the same marginal distribution. We now wish to consider the expression

$$E(y_{i_1}^{\alpha_1} y_{i_2}^{\alpha_2} \cdots y_{i_k}^{\alpha_k})$$

(where the symbol E denotes the expectation) for any set of fixed positive integers $k, \alpha_1, \cdots, \alpha_k$, and for all k-tuples $i_1, i_2, \cdots, i_k$, with no two elements equal. Now

$$E(y_{i_1}^{\alpha_1} y_{i_2}^{\alpha_2} \cdots y_{i_k}^{\alpha_k}) = E(y_{i_1} y_{i_2} \cdots y_{i_k})$$

is the probability that $y_{i_1} = 1, y_{i_2} = 1, \cdots, y_{i_k} = 1$, simultaneously. This probability is either zero (for example, when $| i_2 - i_1 | = 1, | i_3 - i_2 | = 1$, etc. or when $i_1 = n$, etc.) or $\left(\frac{2}{n}\right)^k + O\left(\frac{1}{n^{k+1}}\right)$. Moreover, the ratio of the number of k-tuples $i_1, i_2, \cdots, i_k$ for which the probability is zero to the number of k-tuples for which the probability is $\left(\frac{2}{n}\right)^k + O\left(\frac{1}{n^{k+1}}\right)$ is $O\left(\frac{1}{n}\right)$. Let $Z_i (i = 1, \cdots, n)$ be independent stochastic variables each with the same distribution such that the probability that $Z_i = 1$ is $2/n$ and the probability that $Z_i = 0$ is $(n - 2)/n$. It follows readily that the limit, as $n \to \infty$, of the jth moment ($j = 1, 2, \cdots$, ad inf.) of y about the origin, is the same as the limit of the same moment of Z, where

$$Z = \sum_{i=1}^{n} Z_i .$$

Since the Z_i are independently distributed, and since each can take only the values 0 and 1, the probability of the value 1 being $2/n$, the jth moment of Z about the origin approaches, as $n \to \infty$,

$$\mu_j = e^{-2} \sum_{i=1}^{\infty} \frac{i^j 2^i}{i!},$$

which is the jth moment about the origin of the Poisson distribution with mean value 2. By the preceding paragraph, μ_j is also the limit of the jth moment of y about the origin. Now von Mises [2] has proved that if the jth moment ($j = 1, 2, \cdots$, ad inf.) of a chance variable X_n, ($n = 1, 2, \cdots$, ad inf.), approaches, as $n \to \infty$, the jth moment of a Poisson distribution, then the distribution of X_n approaches the Poisson distribution with corresponding mean value. From this it follows that y has in the limit the distribution (1). We have already shown that y and $n - W(R)$ have the same limiting distribution, so that the required result follows.

REFERENCES

[1] J. Wolfowitz, *Annals of Math. Stat.*, Vol. 13 (1942), p. 247.
[2] R. v. Mises, *Zeitschrift für die angewandte Math. und Mechanik*, Vol. 1 (1921), p. 298.

NOTE ON CONSISTENCY OF A PROPOSED TEST FOR THE PROBLEM OF TWO SAMPLES

By Albert H. Bowker

Columbia University

Certain tests for the hypothesis that two samples are from the same population assume nothing about the distribution function except that it is continuous. Since the power functions of these tests have not been obtained, optimum