But (*iii*) states that Q/σ^2 has a χ^2-distribution for *all* parameter values. This contradiction completes the proof.

We remark in closing that we have at hand a counter example of practical interest to the statement found in several statistics texts that if z is a normal variable with zero mean and v is an independent unbiased quadratic estimate of the variance of z, then $z/v^{\frac{1}{2}}$ has a t-distribution. The counter example consists of taking $z = \bar{x} - \bar{y} - \delta$ and $v = Q/k$ defined by (9). It may be shown that $z/v^{\frac{1}{2}}$ does not have a t-distribution except in the trivial case (10).

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ON MULTIPLE MATCHING WITH ONE VARIABLE DECK

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The problem of card matching has been considered by a number of writers. A complete bibliography has been given by Battin [1], who also published the most general treatment of the subject to date, dealing with the simultaneous matching of any number of decks of arbitrary composition. He considers, however, only the case in which the order of every deck is variable, all possible permutations being equally likely. Some interest attaches to the case in which all the decks but one have fixed orders in relation to one another, especially in connection with radio experiments in telepathy, where a large number of subjects simultaneously attempt to call the same target.

The simplest case is that in which the target for each trial is chosen at random, independently of the other trials. If the target is to be selected from s possibilities, and if p_i denotes the probability that the ith possibility will be chosen as the target, while m_i denotes the number of subjects who call the ith possibility, then the mean value of h, the number of correct calls is, of course,

$$M_h = \sum_{i=1}^{s} p_i m_i, \tag{1}$$

and the variance is

$$V_h = \sum_{i=1}^{s} p_i m_i^2 - M_h^2. \tag{2}$$

Evidently, the mean number of hits for a succession of trials is the sum of the means for the individual trials, and the variance is the sum of the variances.

A slightly more difficult problem is presented when the target series is a true "deck": that is, when its composition is determined in advance, only the order being left to chance. Let n denote the number of trials and $n_i (i = 1, 2, \cdots, s)$

the number of targets of the ith kind so that $\sum_{i=1}^{s} n_i = n$. Also let $m_{ij}(j = 1, 2, \cdots, n)$ denote the number of subjects who call the ith possibility on the jth trial, and let m_i denote the total number of such calls in all the n trials, so that $m_i = \sum_{j=1}^{n} m_{ij}$. Then, using the ingenious type of counting function introduced by Wilks [6] and improved by Battin [1], we define

$$\Phi = \frac{1}{N} \prod_{j=1}^{n} \left(\sum_{i=1}^{s} x_i e^{m_{ij}\theta} \right),$$

where $N = n! \Big/ \prod_{i=1}^{s} n_i!$. We also define an operator K_{n_i} such that if $u = u(x_1, x_2, \cdots, x_s)$, then $K_{n_i} u$ is the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s}$ in u. Then, if h denotes the number of hits

$$P(h = r) = \text{coefficient of } e^{r\theta} \text{ in } K_{n_i}\Phi.$$

Also,

$$E(h^p) = K_{n_i} \left. \frac{\partial^p \Phi}{\partial \theta^p} \right|_{\theta=0}.$$

It follows immediately that

$$M_h = \frac{1}{n} \sum_{i=1}^{s} n_i m_i = \sum_{i=1}^{s} p_i m_i, \tag{3}$$

where p_i is written for n_i/n. Similarly, it can be shown that

$$E(h^2) = \frac{1}{n} \sum_{i=1}^{s} \left(n_i \sum_{j=1}^{n} m_{ij}^2 \right) + \frac{1}{n(n-1)} \sum_{i=1}^{s} \left[n_i(n_i - 1) \sum_{\substack{j,k=1 \\ j \neq k}}^{n} m_{ij} m_{ik} \right] + \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{s} \left[n_i n_j \sum_{\substack{k,l=1 \\ k \neq l}}^{n} m_{ik} m_{jl} \right].$$

If we write $\lambda_{ij} = \sum_{k=1}^{n} m_{ik} m_{jk}$, it is found that

$$V_h = \frac{1}{n-1} \left[\sum_{i=1}^{s} p_i (n\lambda_{ii} - m_i^2) - \sum_{i,j=1}^{s} p_i p_j (n\lambda_{ij} - m_i m_j) \right]. \tag{4}$$

It should be noted that it is not necessary that the total number of subjects be the same for each trial.

Certain special cases are of interest. When there is just one subject for each target, as in the ordinary matching of two decks, $\lambda_{ij} = m_i$ for $i = j$, and 0 otherwise; and $\Sigma_i m_i = n$. Then, if π_i denotes m_i/n and $\rho_i = 1 - \pi_i$, and if $q_i = 1 - p_i$,

$$M_h = n\Sigma_i p_i \pi_i \quad \text{and} \quad V_h = \frac{n^2}{n-1} [\Sigma_i p_i q_i \pi_i \rho_i + \sum_{i \neq j} p_i p_j \pi_i \pi_j], \tag{5}$$

a compact expression which is equivalent to the forms previously given by Stevens [4, 5], Greenwood [2], Battin [1], and the author [3].

On the other hand, if the different kinds of targets occur with equal frequency, so that every $p_i = 1/s$ and every $q_i = (s - 1)/s$; and if $\nu_k = \Sigma_i m_{ik}$ and $\nu = \Sigma_i m_i = \Sigma_k \nu_k$, the expression (4) becomes

$$V_h = \frac{n}{s^2(n-1)} [\nu^2 - n\Sigma_k \nu_k^2 + s\Sigma_i(n\lambda_{ii} - m_i^2)]. \tag{6}$$

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