THE EXPECTED VALUE AND VARIANCE OF THE RECIPROCAL AND OTHER NEGATIVE POWERS OF A POSITIVE BERNOULLIAN VARIATE[1]

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1. Introduction. The expected value of the reciprocal of a Bernoullian variate appears in certain problems of random sampling wherein both practical considerations and mathematical necessity make zero an inadmissible value of the variate. This special condition excluding zero is necessary from a practical standpoint because statistics can not be calculated from an empty class. It is a necessary condition, in the mathematical sense, for the expected value, and variances involving it, to be finite. When subject to this condition the Bernoullian variate will be designated the *positive* Bernoullian variate.

There appears to be no simple expression for the expected value of the reciprocal such as there is for the expected value of positive integral powers of the positive Bernoullian variate. This paper presents in (15) a factorial series, which can be computed conveniently to any desired number of terms by means of the recursion relation (18). Upper and lower bounds on the remainder may be computed readily from (20), (21), (23), (24), and (26) and the approximation may be improved by adding an estimate of the remainder taken between these bounds. A factorial series for the expected value of negative integral powers is given in (34). A factorial series for the expected value of the reciprocal of the positive hypergeometric variate is given in (53). Series for the variances follow directly from the series for expected values.

A simple example of the sampling problems in which this expected value appears is presented by the following instance of estimates derived from samples of variable size:

An infinite population consists of items of two kinds or classes, A and B. Lots of N items each are drawn at random. In such lots the number of items, x', that are of class A is an ordinary Bernoullian variate. Next, every lot composed entirely of items of class B is discarded. This excludes all lots for which $x' = 0$. From each remaining lot the $N - x'$ items of class B are set aside, leaving a sample composed entirely of items of class A. The number of such items, x, varies from sample to sample. It will be designated a positive Bernoullian variate since $x = x'$ if $x' > 0$ and x does not exist if $x' \leq 0$. Finally, let there be associated with each item in class A a particular value of a variable, y, the variance of which in A is σ^2. Then if the mean value of y is computed for each sample, the error variance of such means is $E(\sigma^2/x) = \sigma^2 E(1/x)$.

Instances similar to that just described occur in the design of sampling surveys from which statistics are to be obtained separately for each of several classes

[1] Developed from a section of a paper presented to the Washington meeting of the Institute of Mathematical Statistics on June 18, 1943.

of the population, i.e., each statistic is to be computed from some part of the sample instead of all of it. They also occur in certain sampling problems in which some of the items drawn for a sample turn out to be blanks.

A related problem concerning the error variance of the proportion of males among infants born in any one year was considered by G. Bohlmann in a paper on approximations to the expected value and standard error of a function [1]. His approach to the problem was to expand the function in a Taylor series and take the expected value of each term. The conditions under which the resulting series converges were developed for certain functions of a Bernoullian variate. The present paper provides a different and, in certain respects, superior approach to the problem employing a method due to Stirling [2]. While the method is applied to the reciprocal and negative powers it is also applicable to certain other functions of a Bernoullian variate.

2. The positive Bernoullian variate. Let x be a random variate defined by a Bernoullian probability function subject to the special condition $x > 0$. The probability of x in n is

$$P(x) = \binom{n}{x} p^x q^{n-x}/(1 - q^n) \tag{1}$$

where x and n are integers, $1 \leq x \leq n$, and

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}. \tag{2}$$

The probabilities p and q are constants, $0 < p = 1 - q < 1$.

The divisor $1 - q^n$ arises from the condition excluding zero. (Bohlmann omits this factor, assuming that q^n is negligible, an assumption that is not always valid. In fact, $q^n \sim e^{-np}$.) An extension of this condition to exclude all values of x less than a specified constant will be considered in a later section.

Throughout this paper summation is understood to be from $x = 1$ to $x = n$ unless it is shown otherwise.

3. Expected values and moments. The expected values of x and its positive integral powers are

$$E(x) = np/(1 - q^n) \tag{3}$$

$$E(x^2) = (npq + n^2p^2)/(1 - q^n) \tag{4}$$

and, in general

$$E(x^i) = \nu_i/(1 - q^n) = \sum_j \mathfrak{S}_i^j\, j! \binom{n}{j} \frac{p^j}{1 - q^n}, \qquad i > 0 \tag{5}$$

where ν_i is the ith moment about zero of an ordinary Bernoullian variate with the same n and p and the $\mathfrak{S}_i^j$ are the Stirling numbers of the second kind (see Table 1).

The moments about $E(x)$ are somewhat more complicated than the corresponding moments of the ordinary Bernoullian variate. For example, the variance

$$E\{(x - E(x))^2\} = \frac{npq}{1 - q^n} - \frac{n^2p^2q^n}{(1 - q^n)^2} \tag{6}$$

and the third moment

$$E\{(x - E(x))^3\} = \frac{(q - p)npq}{1 - q^n} - \frac{3n^2p^2q^{n+1}}{(1 - q^n)^2} + \frac{n^3p^3q^n(1 + q^n)}{(1 - q^n)^3}. \tag{7}$$

The moments about np, the first moment of an ordinary Bernoullian variate, are

$$E\{(x - np)^i\} = (\mu_i + (-1)^{i-1}(np)^iq^n)/(1 - q^n) \tag{8}$$

TABLE 1

Stirling numbers of the second kind, $\mathfrak{S}_i^j$

i \ j	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	3	1	0	0	0
4	1	7	6	1	0	0
5	1	15	25	10	1	0
6	1	31	90	65	15	1
7	1	63	301	350	140	21
8	1	127	966	1,709	1,050	266
9	1	255	3,025	7,770	6,951	2,646
10	1	511	9,330	34,105	42,525	22,827

where μ_i is the ith moment, about the mean, of an ordinary Bernoullian variate with the same values n and p.

The expected value of the reciprocal is

$$E\left(\frac{1}{x}\right) = \frac{1}{1 - q^n}\left\{\frac{1}{1}npq^{n-1} + \frac{1}{2}\cdot\frac{1}{2}n(n - 1)p^2q^{n-2} + \cdots + \frac{1}{i}\binom{n}{i}p^iq^{n-i} + \cdots + \frac{1}{n}p^n\right\}. \tag{9}$$

This equation is not suitable for the computation of $E(1/x)$ to a satisfactory degree of approximation unless np is small, say less than 5 for most purposes. The number of terms necessary to obtain a computed value with four significant figures, for example, may be estimated to be approximately $8\sqrt{npq/(-q^n)}$.

Expressed as a function of q, $E(1/x)$ becomes

$$E\left(\frac{1}{x}\right) = \frac{1}{1 - q^n}\Sigma\frac{q^{x-1} - q^n}{n - x + 1} \tag{10}$$

a series which may be convenient for small values of q.

$E(1/x)$ may be expanded in a power series by Taylor's Theorem. It may also be expanded in a finite series of expected values of powers, either in $E(x)$, $E(x^2), \cdots$ or in $E(x-c)$, $E(x-c)^2, \cdots$ c being any positive constant. The second of these three series may be obtained by expanding $\frac{1}{x}\left(1-\frac{x}{c}\right)^t$ and taking expected values, and the third by dividing out $\frac{1}{x} = \frac{1}{c+(x-c)}$ and taking expected values. For all three expansions, however, the terms become progressively more complicated and laborious to compute. A simpler and more convenient series for actual computations may be obtained by expanding $1/x$ in a factorial series.

4. Expansion of $E(1/x)$ in a series of inverse factorials. It is easy to prove by induction that, $x > 0$,

$$(11) \quad \frac{1}{x} = \frac{0!}{x+1} + \frac{1!}{(x+1)(x+2)} + \cdots + \frac{(i-1)!x!}{(x+i)!} + \cdots + \frac{(t-1)!x!}{(x+t)!} + R_t(x)$$

where

$$(12) \qquad R_t(x) = t!(x-1)!/(x+t)!$$

is the remainder after the first t terms. This is, of course, an expansion in Beta functions. It is also a simple special case of the expansion of a function in a "faculty series" or series of inverse factorials [3] with an exact expression for the remainder.

Let

$$(13) \quad s_i = \Sigma \binom{n+i}{x+i} \frac{p^{x+i} q^{n-x}}{1-q^n} = \frac{1}{1-q^n}\left(1 - \sum_{x=0}^{i} \binom{n+i}{x} p^x q^{n+i-x}\right).$$

Then, since

$$(14) \qquad \Sigma \frac{x!}{(x+1)!}\binom{n}{x} p^x q^{n-x} = \frac{n!\,s_i\,(1-q^n)}{(n+i)!\,p^i}$$

the expected value of (11) is

$$(15) \quad E\left(\frac{1}{x}\right) = \frac{0!\,s_i}{(n+1)p} + \frac{1!\,s_2}{(n+1)(n+2)p^2} + \cdots + \frac{(i-1)!\,n!\,s_i}{(n+1)!\,p^i} + \cdots + \frac{(t-1)!\,n!\,s_t}{(n+t)!\,p^t} + \Sigma R_t(x)P(x).$$

When developed as infinite series, both (11) and (15) are convergent since the remainders $R_t(x) \to 0$ as $t \to \infty$.

For computing purposes it is convenient to write

$$(16) \qquad E\left(\frac{1}{x}\right) = \sum_{i=1}^{t} u_i + E(R_t(x))$$

in which, since

$$(17) \qquad s_i = s_{i-1} - q\binom{n+i-1}{i}\frac{p^i q^{n-1}}{1-q^n},$$

the following recursion relation exists between u_i and u_{i-1}

$$(18) \qquad u_i = \frac{(i-1)!\,n!\,s_i}{(n+i)!\,p^i} = \frac{(i-1)u_{i-1} - k/i}{(n+i)p}, \qquad i > 1; \qquad u_1 = \frac{1-k}{(n+1)p}$$

where

$$(19) \qquad k = npq^n/(1-q^n) \sim np/(e^{np}-1).$$

This reduces the computing of the u_i to a simple repetitive procedure. The computing is still simpler in those problems in which, for the degree of precision desired, k is negligible.

An estimate of $E(R_t(x))$ should be added to the sum in (16) to improve the approximation. To determine a suitable estimate, a lower bound for the expected value of the remainders may be computed from one of the following inequalities:

$$(20) \qquad \begin{aligned} E(R_t(x)) &= \Sigma \frac{t}{x} \frac{(t-1)!\,x!}{(x+t)!} P(x) \\ &= \Sigma t\left(\frac{1}{m} - \frac{x-m}{m^2} + \frac{(x-m)^2}{m^2x}\right) \frac{(t-1)!\,x!}{(x+t)!} P(x) \\ &> \frac{1}{m} tu_t - \frac{1}{m^2} t(t-1)u_{t-1} + \frac{m+t}{m^2} tu_t, \qquad m \neq 0 \end{aligned}$$

which is maximized by setting $m = \{(t-1)u_{t-1} - tu_t\}/u_t$, whence

$$(21) \qquad E(R_t(x)) > tu_t^2/\{(t-1)u_{t-1} - tu_t\}, \qquad t > 1.$$

Also, since when $m = E(x)$

$$(22) \qquad \Sigma(x-m) \frac{(t-1)!\,x!}{(x+t)!} P(x) < \Sigma(x-m)P(x) = 0,$$

a simpler inequality is

$$(23) \qquad E(R_t(x)) > tu_t(1-q^n)/np.$$

Further, if only the first $c < n$ terms in (20) are taken,

$$(24) \qquad E(R_t(x)) > \sum_{x=1}^{c} \frac{t!\,(x-1)!}{(x+t)!} P(x) = \sum_{x=1}^{c} v_x$$

where

$$(25) \qquad v_1 = \frac{k}{(t+1)q} \quad \text{and} \quad v_x = \frac{(x-1)(n-x+1)p}{x(x+t)q} v_{x-1}.$$

An upper bound may be computed from

$$(26) \qquad E(R_t(x)) < \begin{cases} tu_t & (26.1) \\ \frac{1}{2} tu_t + \frac{1}{2} v_1 & (26.2) \\ \frac{1}{3} tu_t + \frac{2}{3} v_1 + \frac{1}{6} v_2 & (26.3) \\ \cdots & \\ \frac{1}{j} tu_t + \sum_{x=1}^{j-1} \left(\frac{1}{x} - \frac{1}{j}\right) v_x & (26.j) \end{cases}$$

the choice among which may be governed by computing convenience. Taken with (16), these inequalities provide lower and upper bounds for $E(1/x)$.

5. Examples. Two examples will serve to illustrate the factorial series (15).

EXAMPLE 1

Computation of $E(1/x)$ for $n = 100$ and $p = 0.1$

$np = 10 \quad k = .000,265,621 \quad E(1) = .111,527$

t	*Binomial sum of t terms*	*Sum of t terms*	*Factorial series lower bounds**		*Upper bound***
1	.000,295	.098,984	.099,647		.132,167
2	.001,107	.108,675	.109,006	(.111,034)	.115,247
3	.003,071	.110,548	.110,752	(.111,313)	.112,498
4	.007,039	.111,082	.111,223	(.111,381)	.111,852
5	.013,813	.111,280	.111,385	(.111,452)	.111,657
6	.023,743	.111,370	.111,452	(.111,478)	.111,587
7	.036,442	.111,416	.111,483	(.111,489)	.111,556
8	.050,796	.111,444	.111,500	(.111,497)	.111,544
9	.065,287	.111,461	.111,509	(.111,503)	.111,537
10	.078,474	.111,472	.111,514	(.111,508)	.111,534
11	.089,372	.111,481	.111,518	(.111,511)	.111,532
12	.097,604	.111,487	.111,520		.111,530
13	.103,320	.111,492	.111,521		.111,529
14	.106,985	.111,495	.111,523		.111,529
15	.109,164	.111,498	.111,524		.111,529
16	.110,369	.111,501	.111,524		.111,528
17	.110,992	.111,503	.111,525		.111,528
18	.111,294	.111,505	.111,525,4		.111,527,5
19	.111,431	.111,506	.111,525,6		.111,527,3
20	.111,489	.111,508	.111,525,8		.111,527,1
...					
24	.111,526				
...					
100	.111,527 (end of series)				

* Sum of t terms plus lower bound for $E(R(x))$ from (24) with $c = 3$. Numbers in parentheses are calculated from (21).

** Sum of t terms plus upper bound on $E(R(x))$ from (26.3).

EXAMPLE 2

Computation of $E(1/x)$ for $n = 1000$ and $p = 0.3$

$np = 300 \quad k = 9.7 \times 10^{-14}$

t	*Sum of t terms*	*Factorial series upper and lower bounds**
1	.003,330,003,330	
2	.003,341,081,185	.003,346,7 .003,341,0 (.003,341,155,4)

* Computed as in Example 1.

t	*Sum of t terms*	*Factorial series upper and lower bounds**
3	.003,341,154,817	.003,341,211 .003,341,155
4	.003,341,155,549	.003,341,156,29 .003,341,155,56
5	.003,341,155,559	.003,341,155,58 .003,341,155,57

For the binomial series, the sum of the largest eight terms of (9), not the first eight terms, is approximately .0007 which is less than 1/4 of the value of $E(1/x)$.

In the first example the value of np is almost small enough to make computation by (9) convenient. In the second example about 120 terms of (9) must be computed to obtain an approximation to four significant figures but only four terms of the factorial series are needed to obtain seven significant figures. It is evident that as np increases, the number of terms of (16) required to obtain an approximation to a given number of significant figures decreases. The opposite is true of (9) as n increases, or as p approaches a value near 1/2.

6. Extending the special condition. In some sampling problems all values of x less than a specified value, g, and greater than another specified value, h, are inadmissible. Then the probability of x in n is

$$P(x \mid g, h) = \binom{n}{x} p^x q^{n-x}/s_{0,g,h}, \qquad g \leq x \leq h, \tag{27}$$

where

$$s_{0,g,h} = \sum_{x=g}^{h} \binom{n}{x} p^x q^{n-x}. \tag{28}$$

With this new condition, $E(1/x)$ is given by (15) if s_i is replaced by

$$s_{i,g,h} = \sum_{x=g}^{h} \binom{n+i}{x+i} \frac{p^{x+i} q^{n-x}}{s_{0,g,h}} \tag{29}$$

and the summation in the remainder term is from g to h. Also since

$$s_{i,g,h} = s_{i-1,g,h} - \frac{q}{s_{0,g,h}} \left\{ \binom{n+i-1}{g+i-1} p^{g+i-1} q^{n-g} - \binom{n+i-1}{h+i} p^{h+i} q^{n-h-1} \right\} \tag{30}$$

a recursion relation similar to (18) may be used in computing

$$(31)\quad \begin{aligned} u_{i,g,h} &= \frac{(i-1)!\,n!\,s_{i,g,h}}{(n+i)!\,p^i} \\ &= \frac{(i-1)u_{i-1,g,h} - (i-1)!\,\{k_g/(g+i-1)! + k_{h+1}/(h+i)\}}{(n+i)p} \end{aligned}$$

where

$$(32)\qquad k_g = \frac{n!\,p^g\,q^{n-g+1}}{(n-g)!\,s_{0,g,h}}$$

$$(33)\qquad k_h = \frac{n!\,p^h\,q^{n-h+1}}{(n-h)!\,s_{0,g,h}}.$$

The inequalities (20) to (23) inclusive and (26) are applicable to this extension on substitution of $u_{i,g,h}$ for u_i.

7. Expansion of $E(x^{-a})$ in a factorial series. Equation (11) may be extended to other negative integral powers of x. If a is a positive integer

$$(34)\quad \begin{aligned} E(x^{-a}) = \Sigma \frac{1}{x^a} P(x) &= \frac{b_{1,a}\,s_1}{(n+1)p} + \frac{b_{2,a}\,s_2}{(n+1)(n+2)p^2} \\ &\quad + \cdots + \frac{b_{t,a}\,s_t\,n!}{(n+t)!\,p^t} + \Sigma R_t'(x)P(x) \end{aligned}$$

where

$$(35)\qquad R_t'(x) = \sum_{j=1}^{a} b_{t+1,j}\,\frac{x^{j-1}\,x!\,P(x)}{(x+t)!\,x^a}$$

and the $b_{i,j}$ are the absolute values of the Stirling numbers of the first kind (see Table 2) formed by the recursion relation

$$(36)\quad b_{i,j} = b_{i-1,j-1} + (i-1)b_{i-1,j}, \qquad b_{i,j} = 0 \text{ if } j > i \text{ or } j < 1.$$

It is evident that

$$(37)\qquad \sum_{j=1}^{i} b_{i,j} = i!$$

$$(38)\qquad b_{i,1} = (i-1)! \text{ and } b_{i,j} < i! \text{ if } j > 1,$$

whence

$$(39)\quad \begin{aligned} R_t'(1) &= \frac{1}{t+1} P(1) \\ R_t'(x) &< \frac{((t+1)! - t!)x!P(x)}{2(x+t)!}, \qquad x > 1 \\ &< \frac{1}{(t+1)} P(x). \end{aligned}$$

Hence $R'_t(x) \to 0$ and $E(R'_t(x)) \to 0$ as $t \to \infty$ and the sum of the first t terms of (34) converges to $E(x^{-a})$ as $t \to \infty$.

The following recursion relation corresponding to (18) provides a simple procedure for computing:

$$u_{i,a} = b_{i,a} u_i/(i-1)! = b_{i,a} \frac{(u_{i-1,a}/b_{i-1,a}) - k/i!}{(n+1)p}. \tag{40}$$

The computing procedure, then, follows a cycle of four simple operations:

1. Divide $\{k/(i-1)!\}$ by i.
2. Subtract the quotient from $\{u_{i-1,a}/b_{i-1,a}\}$.
3. Divide the difference by $\{(n+i+1)p\} + p$. The quotient is $u_{i,a}/b_{i,a}$.
4. Multiply this quotient by $b_{i,a}$.

TABLE 2

*Absolute values of Stirling numbers of the first kind, $b_{i,j}$**

i \ j	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	2	3	1	0	0	0
4	6	11	6	1	0	0
5	24	50	35	10	1	0
6	120	274	225	85	15	1
7	720	1,764	1,624	735	175	21
8	5,040	13,068	13,132	6,769	1,960	322
9	40,320	109,584	118,124	67,284	22,449	4,536
10	362,880	1,026,576	1,172,700	723,680	269,325	63,273

* These numbers are also known as differential coefficients of zero [4].

The expressions in braces are quantities obtained in the preceding cycle.

The $u_{i,a}$ may also be calculated from (18), or checked by such a calculation.

A lower bound for $E(R'(x))$ after t terms may be calculated from the first c terms of

$$\begin{aligned} E(R'(x)) &= \sum R'_t(x)P(x) > \sum_{x=1}^{c} R'_t(x)P(x) \\ &= \sum_{x=1}^{c} \sum_{j=1}^{a} \frac{b_{t+1,j}\, n!\, p^x q^{n-x}}{x^{a-j+1}(x+t)!\,(n-x)!\,(1-q^n)} \end{aligned} \tag{41}$$

or from an inequality similar to (23)

$$E(R'(x)) > \frac{u_t}{(t-1)!} \sum_{j=1}^{a} \frac{b_{t+1,j}}{(E(x))^{a-j+1}} \tag{42}$$

which may also be written

$$E(R'(x)) > \frac{u_t}{(t-1)!(E(x))^{a+1}}\left\{(E(x)+t)(E(x)+t-1)\cdots E(x) - \sum_{j=a+1}^{t+1} b_{t+1,j}(E(x))^j\right\}. \tag{43}$$

An upper bound may be calculated from

$$E(R'(x)) < \frac{u_t}{(t-1)!}\sum_{j=1}^{a} b_{t+1,j} < t(t+1)u_t \tag{44}$$

or

$$\begin{aligned} E(R'(x)) &< \sum_{x=1}^{c} R'(x)P(x) + \sum_{x=c+1}^{n}\sum_{j=1}^{a} b_{t+1,j}\frac{x!P(x)}{(x+t)!c^{a-j+1}} \\ &< \sum_{x=1}^{c} R'(x)P(x) + \frac{u_t}{(t-1)!}\sum_{j=1}^{a}\frac{b_{t+1,j}}{c^{a-j+1}} = \sum_{x=1}^{c} R'(x)P(x) \\ &\quad + \frac{u_t}{(t-1)!c^{a+1}}\left\{(c+t)(c+t-1)\cdots c - \sum_{j=a+1}^{t+1} b_{t+1,j}c^j\right\}. \end{aligned} \tag{45}$$

8. The positive hypergeometric variate. The theory of sampling without replacement from a finite population rests on the hypergeometric variate. Its probability function is

$$P(x \mid N, M, n) = \binom{M}{x}\binom{N-M}{n-x}\bigg/\binom{N}{n}. \tag{46}$$

In applications to finite sampling, N is the number of items in the population, M is the number of them that are of a certain kind, n is the number of items drawn for the sample, and x is the number of items of the designated kind in the sample.

As in the case of the Bernoullian variate, it is necessary to exclude zero in defining the expected value of $1/x$. The probability function of the positive hypergeometric variate, then, is

$$P_H(x) = P(x \mid N, M, n)/s_0, \qquad x > 0 \tag{47}$$

where

$$s_0 = 1 - P(0 \mid N, M, n). \tag{48}$$

Throughout this section the notation will have reference to (47) instead of (1). The expected values of positive integral powers of x are

$$E(x) = Mn/(Ns_0) \tag{49}$$

$$E(x^2) = \frac{1}{s_0}\left\{\frac{M(M-1)n(n-1)}{N(N-1)} + \frac{Mn}{N}\right\} \tag{50}$$

and, in general,

$$E(x^i) = \sum_{j=1}^{i} \mathfrak{S}_i^j E(x!/(x-j)!) \tag{51}$$

where the $\mathfrak{S}_i^j$ are the Stirling numbers of the second kind and

$$E\left(\frac{x!}{(x-j)!}\right) = \frac{M!n!(N-j)!}{(M-j)!(n-j)!N!s_0}. \tag{52}$$

The factorial series corresponding to (16) is

$$E\left(\frac{1}{x}\right) = \sum \frac{1}{x} P_H(x) = \sum_{i=1}^{t} u_i + E(R_t(x)) \tag{53}$$

where

$$u_i = \Sigma \frac{(i-1)!x!}{(x+i)!} P_H(x) \tag{54}$$

and

$$E(R_t(x)) = \Sigma \frac{t!(x-1)!}{(x+t)!} P_H(x). \tag{55}$$

The u_i may be computed from

$$\begin{aligned} u_1 &= \frac{(N+1)s_1}{(M+1)(n+1)s_0} \\ &= \frac{1}{s_0}\left\{\frac{N+1}{(M+1)(n+1)} - \frac{(N-M)!(N-n)!}{N!(N-M-n-1)!(M+1)(n+1)}\right\} \end{aligned} \tag{56}$$

and the recursion relation

$$u_i = \frac{(N+i)s_i}{(M+i)(n+i)s_{i-1}} u_{i-1} \tag{57}$$

where

$$s_i = 1 - \sum_{x=0}^{i} P(x \mid N+i, M+i, n+i). \tag{58}$$

The computing is quite simple in those instances in which $1 - s_t$ is negligible.

Corresponding to (26), an upper bound for the expected value of the remainders after t terms may be computed from

$$E(R_t(x)) < \begin{cases} tu_t & (59.1) \\ \frac{1}{2}tu_t + \frac{1}{2}P_H(1)/(t+1) & (59.2) \\ \frac{1}{3}tu_t + \frac{2}{3}\frac{P_H(1)}{t+1} + \frac{1}{6}\frac{P_H(2)}{(t+1)(t+2)} & (59.3) \\ \cdots & \\ \frac{1}{j}tu_t + t!\sum_{x=1}^{j-1}\left(\frac{1}{x} - \frac{1}{j}\right) P_H(x) \frac{x!}{(x+t)!}. & (59.j) \end{cases} \tag{59}$$

A lower bound for the expected value of the remainders may be computed from one of the following inequalities corresponding to (23), (21) and (24)

(60) $$E(R_t(x)) > tu_tNs_0/(Mn)$$

(61) $$E(R_t(x)) > tu_t^2/\{(t-1)u_{t-1} - tu_t\}$$

(62) $$E(R_t(x)) > \sum_{x=1}^{j} \frac{t!(x-1)!}{(x+t)!} P_H(x).$$

The expected values of other negative integral powers of the positive hypergeometric variate may be calculated from

(63) $$E(x^{-a}) = \sum_{i=1}^{t} b_{i,a}u_i/(i-1)! + E(R_t'(x))$$

where

(64) $$R_t'(x) = \sum_{j=1}^{a} b_{t+1,j} \frac{x^{j-1}x!P_H(x)}{x^a(x+t)!}.$$

With $P_H(x)$ substituted for $P(x)$, (39), (42), (43), (44), and (45) provide lower and upper bounds for $E(R_t'(x))$ for the positive hypergeometric variate. Also, corresponding to (41)

(65) $$E(R_j'(x)) > \sum_{x=1}^{c} R_t'(x)P_H(x).$$

9. Variance and moments of $1/x$ and x^{-a}. The variance of $1/x$, which is $E(1/x^2) - (E(1/x))^2$, may be calculated from (16) and (34), with $a = 2$, for the positive Bernoullian variate, and from (53) and (63), with $a = 2$, for the positive hypergeometric variate. Likewise, the variance of x^{-a} and the moments of $1/x$ and x^{-a} about $E(1/x)$ may be computed by the usual formulae.

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[2] The writer is indebted to Dr. Felix Bernstein for the reference to Bohlman.