ON THE APPROXIMATE DISTRIBUTION OF RATIOS

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The purpose of this paper is to apply Cramer's theorem of asymptotic expansion[1] and Berry's theorem[2] to study the approximate distribution of ratios of the following two types:

(I) $$Z = \frac{1}{n}(Y_1 + \cdots + Y_n) \Big/ \frac{1}{m}(\bar{X}_1 + \cdots + \bar{X}_m) = \bar{Y}/\bar{X},$$

(II) $$Z = Y \Big/ \frac{1}{m}(X_1 + \cdots + X_m) = Y/\bar{X}.$$

In (I) the X_i, Y_j are independent, the Y_j are equi-distributed,[3] and the X_i are equi-distributed and positive. In (II) $X_1, \cdots, X_n$, Y are independent and positive, and the X_i are equi-distributed.

1. The ratio (I). Assume that (I1) the absolute kth moment of X_i and that of Y_j are finite and positive, where k is a fixed integer ≥ 3,
(I2) the distribution of X_i and that of Y_j are non-singular.

Let

$$\xi = \epsilon(X_i), \qquad \eta = \epsilon(Y_j), \qquad \sigma^2 = \epsilon(X_i^2) - \xi^2, \qquad \tau^2 = \epsilon(Y_j^2) - \eta^2$$

and

$$U = \frac{\sqrt{m}}{\sigma}(\bar{X} - \xi), \qquad V = \frac{\sqrt{n}}{\tau}(\bar{Y} - \eta).$$

Let $F(x)$, $G(x)$ and $H(x)$ be respectively the distribution functions of Z, U and V. Let

$$b = \left(\frac{\sigma^2 x^2}{m} + \frac{\tau^2}{n}\right)^{\frac{1}{2}}, \qquad u = \frac{\xi n - \eta}{b}.$$

Then the relation $Z \leq x$ is equivalent to

$$-\frac{x\sigma U}{b\sqrt{m}} + \frac{\tau V}{b\sqrt{n}} \leq u.$$

[1] H. Cramér. *Random Variables and Probability Distributions* (1937), Chap. 7.

[2] A. C. Berry. "The accuracy of the Gaussian approximation to the sum of independent variates", *Trans. Amer. Math. Soc.*, Vol. 49 (1941), pp. 122–136.

[3] The Y_j are said to be equi-distributed if all Y_j have the same distribution function.

For simplicity we shall assume $x > 0$; the results are, however, general. Then the distribution functions of $-\frac{x\sigma U}{b\sqrt{m}}$ and $\frac{\tau V}{b\sqrt{n}}$ are

$$Pr\left\{-\frac{x\sigma U}{b\sqrt{m}} < y\right\} = 1 - G\left(-\frac{b\sqrt{m}y}{\sigma x}\right), \qquad Pr\left\{\frac{\tau V}{b\sqrt{n}} \le y\right\} = H\left(\frac{b\sqrt{n}y}{\tau}\right).$$

Hence, by the theorem of convolution,

$$F(x) = \int_{-\infty}^{\infty}\left\{1 - G\left(-\frac{b\sqrt{m}(u-y)}{\sigma x}\right)\right\} dH\left(\frac{b\sqrt{n}y}{\tau}\right). \tag{1}$$

Here we recall the theorems of Cramér and Berry: Under the conditions (I1) and (I2)

$$G(x) = \Phi(x) + \sum_{\nu=1}^{k-3}\frac{P_\nu(x)}{m^{\nu/2}} + \frac{D_k}{m^{\frac{1}{2}(k-2)}}, \tag{2}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy, \qquad P_\nu(x) = \sum_{j=1}^{\nu} c_{j\nu}\Phi^{(\nu+2j)}(x),$$

and $|D_k|$ is less than a positive number which depends only on k and the distribution of X_i. If $k = 3$, condition (I2) may be removed.[4]

Analogously,

$$H(x) = \Phi(x) + \sum_{\nu=1}^{k-3}\frac{Q_\nu(x)}{n^{\nu/2}} + \frac{D'_k}{m^{\frac{1}{2}(k-2)}}, \tag{3}$$

where

$$Q_\nu(x) = \sum_{j=1}^{\nu} d_{j\nu}\Phi^{(\nu+2j)}(x).$$

In the sequel we shall use the letter Δ_k to denote an unspecified quantity such that $|\Delta_k|$ is less than a positive number which depends only on k, the distribution of X_i and the distribution of Y_j.

Using (2) we have

$$1 - G(-x) = \Phi(x) + \sum_{\nu=1}^{k-3}\frac{(-1)^\nu P_\nu(x)}{m^{\nu/2}} + \frac{D_k}{m^{\frac{1}{2}(k-2)}} \tag{4}$$

and this making this substitution in (1) we get

$$F(x) = \int_{-\infty}^{\infty}\Phi\left(\frac{b\sqrt{m}(u-y)}{\sigma x}\right) dH\left(\frac{b\sqrt{n}y}{\tau}\right) + \sum_{\nu=1}^{k-3}\frac{(-1)^\nu}{m^{\nu/2}}\int_{-\infty}^{\infty} P_\nu\left(\frac{b\sqrt{m}(u-y)}{\sigma x}\right) dH\left(\frac{b\sqrt{n}y}{\tau}\right) + \frac{\Delta_k}{m^{\frac{1}{2}(k-2)}},$$

[4] This last assertion constitutes Berry's theorem.

and so by partial integration,

$$F(x) = \int_{-\infty}^{\infty} H\left(\frac{b\sqrt{n}(u - y)}{\tau}\right) d\Phi\left(\frac{b\sqrt{my}}{\sigma x}\right)$$
$$+ \sum_{\nu=1}^{k-3} \frac{(-1)^\nu}{m^{\nu/2}} \int_{-\infty}^{\infty} H\left(\frac{b\sqrt{n}(u - y)}{\tau}\right) dP_\nu\left(\frac{b\sqrt{my}}{\sigma x}\right) + \frac{\Delta_k}{m^{\frac{1}{2}(k-2)}}.$$

Making the transformation $y = \sigma xv/b\sqrt{m}$ and writing

$$\alpha = \frac{b\sqrt{n}}{\tau}, \qquad \beta = \frac{\sigma\sqrt{n}\,x}{\tau\sqrt{m}} \tag{5}$$

we get

$$F(x) = \int_{-\infty}^{\infty} H(du - \beta v)\Phi'(v)\,dv + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu}{m^{\nu/2}} \int_{-\infty}^{\infty} H(\alpha u - \beta y)P_\nu'(v)\,dv + \frac{\Delta_k}{m^{\frac{1}{2}(k-2)}}$$
$$= I_0 + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu}{m^{\nu/2}} I_\nu + \frac{\Delta_k}{m^{\frac{1}{2}(k-2)}}.$$

For I_0 we use (3) and obtain

$$I_0 = \int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)\Phi'(v)\,dv + \sum_{\nu=1}^{k-3} \frac{1}{n^{\nu/2}} \int_{-\infty}^{\infty} Q_\nu(\alpha u - \beta v)\Phi'(v)\,dv + \frac{\Delta_k}{n^{\frac{1}{2}(k-2)}}.$$

For I_ν we use (3) with k replaced by $k - \nu$. Thus

$$I_\nu = \int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)P_\nu'(v)\,dv + \sum_{\mu=1}^{k-3-\nu} \frac{1}{n^{\mu/2}} \int_{-\infty}^{\infty} Q_\mu(\alpha u - \beta v)P_\nu'(v)\,dv + \frac{\Delta_k}{n^{\frac{1}{2}(k-2-\nu)}}.$$

Combining these results we get

$$F(x) = \int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)\Phi'(v)\,dv + \sum_{\nu=1}^{k-3} \frac{1}{n^{\nu/2}} \int_{-\infty}^{\infty} Q_\nu(\alpha u - \beta v)\Phi'(v)\,dv \tag{6}$$
$$+ \sum_{\nu=1}^{k-3} \frac{(-1)^\nu}{m^{\nu/2}} \int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)P_\nu'(v)\,dv$$
$$+ \sum_{\nu=1}^{k-3} \sum_{\mu=1}^{k-3-\nu} \frac{(-1)^\nu}{m^{\nu/2}\,n^{\mu/2}} \int_{-\infty}^{\infty} Q_\mu(\alpha u - \beta u)P_\nu'(v)\,dv + R_k,$$

where

$$R_k = \frac{\Delta_k}{m^{\frac{1}{2}(k-2)}} + \frac{\Delta_k}{n^{\frac{1}{2}(k-2)}} + \sum_{\nu=1}^{k-3} \frac{\Delta_k}{m^{\nu/2}\,n^{\frac{1}{2}(k-2-\nu)}} = \Delta_k\left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}}\right)^{k-2}.$$

Now by (5), $\alpha > 0$ and $\alpha^2 - \beta^2 = 1$. For such values of α and β, however, it follows easily from the theorem of convolution that

$$\int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)\Phi'(v)\,dv = \Phi'(u).$$

As differentiation under the integration sign is justified by the boundedness of the derivatives of Φ we have

$$\alpha^p \int_{-\infty}^{\infty} \Phi^{(p)}(\alpha u - \beta v)\Phi'(v)\,dv = \Phi^{(p)}(u).$$

Repeated partial integration then gives

$$\int_{-\infty}^{\infty} \Phi^{(p)}(\alpha u - \beta v)\Phi^{(q)}(v)\,dv = \beta^{q-1}\int_{-\infty}^{\infty} \Phi^{(p+q-1)}(\alpha u - \beta v)\Phi'(v)\,dv$$

$$= \frac{\beta^{q-1}}{\alpha^{p+q-1}}\,\Phi^{(p+q-1)}(u).$$

Hence

$$\int_{-\infty}^{\infty} Q_\nu(\alpha u - \beta v)\Phi'(v)\,dv = \sum_{j=1}^{\nu} d_{j\nu}\int_{-\infty}^{\infty} \Phi^{(\nu+2j)}(\alpha u - \beta v)\Phi'(v)\,dv$$

$$= \sum_{j=1}^{\nu} \frac{d_{j\nu}}{\alpha^{\nu+2j}}\,\Phi^{(\nu+2j)}(u),$$

$$\int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)P'_\nu(v)\,dv = \sum_{j=1}^{\nu} c_{j\nu}\int_{-\infty}^{\infty} \Phi(\alpha u - \beta v)\Phi^{(\nu+2j+1)}(v)\,dv$$

$$= \sum_{j=1}^{\nu} \frac{\beta^{\nu+2j}c_{j\nu}}{\alpha^{\nu+2j}}\,\Phi^{(\nu+2j)}(u),$$

$$\int_{-\infty}^{\infty} Q_\mu(\alpha u - \beta v)P'_\nu(v)\,dv = \sum_{i=1}^{\mu}\sum_{j=1}^{\nu} d_{i\mu}c_{j\nu}\int \Phi^{(\mu+2i)}(\alpha u - \beta v)\Phi^{(\nu+2j+1)}(v)\,dv$$

$$= \sum_{i=1}^{\mu}\sum_{j=1}^{\nu} d_{i\mu}c_{j\nu}\,\frac{\beta^{\nu+2j}}{\alpha^{\mu+\nu+2i+2j}}\,\Phi^{(\mu+\nu+2i+2j)}(u).$$

Making all these substitutions in (6) we obtain the final result

$$F(x) = \Phi(u) + \sum_{\nu=1}^{k-3}\frac{(-1)^\nu}{m^{\nu/2}}\sum_{j=1}^{\nu}\frac{\beta^{\nu+2j}c_{j\nu}}{\alpha^{\nu+2j}}\Phi^{(\nu+2j)}(u) + \sum_{\nu=1}^{k-3}\frac{1}{n^{\nu/2}}\sum_{j=1}^{\nu}\frac{d_{j\nu}}{\alpha^{\nu+2j}}\Phi^{(\nu+2j)}(u)$$

$$+ \sum_{\nu=1}^{k-3}\sum_{\mu=1}^{k-3-\nu}\frac{(-1)^\nu}{m^{\nu/2}n^{\mu/2}}\sum_{i=1}^{\mu}\sum_{j=1}^{\nu} d_{i\mu}c_{j\nu}\frac{\beta^{\nu+2j}}{\alpha^{\mu+\nu+2i+2j}}\phi^{(\mu+\nu+2i+2j)}(u)$$

$$+ \Delta_k\left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}}\right)^{k-2}.$$

If $k = 3$, the result remains true without the condition (I2).

2. The ratio (II). Here we make the following assumptions:

(II1) The kth moment of X_i is finite and positive, where k is a fixed integer $\geq k$, $\epsilon(X_i) = 1$,[5] $\epsilon(X_i^2) - 1 = \sigma^2$.

(II2) The distribution of X_i is non-singular.

[5] As the case $\epsilon(X_i) = 0$ is excluded, there is no loss of generality in this assumption.

Let $U = \sqrt{m}(\bar{X} - 1)/\sigma$, and $F(x)$, $G(x)$ and $H(x)$ be respectively the distribution functions of Z, U and Y. Then

$$F(x) = Pr\left\{Y - \frac{\sigma x U}{\sqrt{m}} \leq x\right\}$$

Because of the positiveness of X_i and Y we may always assume $x > 0$. Then, by the theorem of convolution,

$$F(x) = \int_{-\infty}^{\infty}\left\{1 - G\left(-\frac{\sqrt{m}(x-y)}{\sigma x}\right)\right\} dH(y).$$

Using (4) we have

$$F(x) = \int_{-\infty}^{\infty}\left\{\Phi\left(\frac{\sqrt{m}(x-y)}{\sigma x}\right) + \sum_{\nu=1}^{k-3}\frac{(-1)^{\nu}}{m^{\nu/2}} P_{\nu}\left(\frac{\sqrt{m}(x-y)}{\sigma x}\right)\right\} dH(y) + \frac{A_k}{m^{\frac{1}{2}(k/2)}},$$

where, as throughout the rest of this paper, A_k represents an unspecified quantity such that $|A_k|$ is less than a positive number depending only on k, the distribution of X_i and the distribution of Y. By partial integration we get

$$(7)\quad F(x) = \int_{-\infty}^{\infty} H(x-y)\, d\left\{\Phi\left(\frac{\sqrt{m}\,y}{\sigma x}\right) + \sum_{\nu=1}^{k-3}\frac{(-1)^{\nu}P_{\nu}\left(\frac{\sqrt{m}\,y}{\sigma x}\right)}{m^{\nu/2}}\right\} + \frac{A_k}{m^{\frac{1}{2}(k-2)}}$$

$$= \int_{-\infty}^{\infty} H\left(x - \frac{\sigma x z}{\sqrt{m}}\right)\left(\Phi'(z) + \sum_{\nu=1}^{k-3}\frac{(-1)^{\nu}P'_{\nu}(z)}{m^{\frac{1}{2}(k-2)}}\right) dz + \frac{A_k}{m^{\frac{1}{2}(k-2)}}.$$

An interesting special case is the following: Suppose that (II3) $H^{(k-2)}(x)$ exists and is continuous for all $x \geq 0$; (II4) the functions

$$\xi_{\nu}(x) = x^{\nu}H^{(\nu)}(x) \qquad (\nu = 1, \cdots, k-3)$$

are bounded, i.e.

$$\xi_{\nu}(x) = A_k;$$

(II3) there is a positive constant $c < 1$ such that

$$x^{k-2}H^{(k-2)}(y) = A_k$$

for all $x \geq 0$ and $(1-c)x \leq y \leq (1+c)x$. Under these conditions we have

$$H\left(x - \frac{\sigma x z}{\sqrt{m}}\right) = \sum_{\nu=0}^{k=3}\frac{(-1)^{\nu}\sigma^{\nu}x^{\nu}z^{\nu}H^{(\nu)}(x)}{\nu!\, m^{\nu/2}}$$

$$+ \frac{(-1)^{k-2}\sigma^{k-2}x^{k-2}z^{k-2}}{(k-2)!\, m^{\frac{1}{2}(k-2)}} H^{(k-2)}\left(x + \frac{\delta\sigma x z}{\sqrt{m}}\right) \quad (|\delta| \leq 1),$$

and so, for $|z| \leq \frac{c\sqrt{m}}{\sigma}$ we have

$$(8)\qquad H\left(x - \frac{\sigma x z}{\sqrt{m}}\right) = \sum_{\nu=0}^{k-3}\frac{(-1)^{\nu}\sigma^{\nu}\xi_{\nu}z^{\nu}}{\nu!\, m^{\nu/2}} + \frac{A_k z^{k-2}}{m^{\frac{1}{2}(k-2)}}.$$

Separate now the integral in (7) into two parts:

$$I_1 = \int_{|z| \le c\sqrt{m}/\sigma}, \qquad I_2 = \int_{|z| > c\sqrt{m}/\sigma}.$$

Now

$$|I_2| \le \int_{|z| > c\sqrt{m}/\sigma} \left| \Phi'(z) + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu P'_\nu(z)}{m^{\nu/2}} \right| dz.$$

Evidently this last integral is exponially small and so is $A_k/m^{\frac{1}{2}(k-2)}$. By (8),

$$I_1 = \int_{|z| \le c\sqrt{m}/\sigma} \left(\sum_{\nu=0}^{k-3} \frac{(-1)^\nu \sigma^\nu \xi_\nu z^\nu}{\nu! m^{\nu/2}} \right) \left(\Phi'(z) + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu P'_\nu(z)}{m^{\nu/2}} \right) dz + \frac{A_k}{m^{\frac{1}{2}(k-2)}}$$

$$= \int_{-\infty}^{\infty} \left(\sum_{\nu=0}^{k-3} \frac{(-1)^\nu \sigma^\nu \xi_\nu z^\nu}{\nu! m^{\nu/2}} \right) \left(\Phi'(z) + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu P'_\nu(z)}{m^{\nu/2}} \right) dz + \frac{A_k}{m^{\frac{1}{2}(k-2)}}.$$

Combining these results we obtain

$$F(x) = \int_{-\infty}^{\infty} \left(\sum_{\nu=0}^{k-3} \frac{(-1)^\nu \sigma^\nu \xi_\nu z^\nu}{\nu! m^{\nu/2}} \right) \left(\Phi'(z) + \sum_{\nu=1}^{k-3} \frac{(-1)^\nu}{m^{\nu/2}} \sum_{j=1}^{\nu} c_{j\nu} \Phi^{(\nu+2j+1)}(z) \right) dz + \frac{A_k}{m^{\frac{1}{2}(k-2)}}$$

$$\doteq \sum_{\nu=0}^{k-3} \frac{d_\nu \xi_\nu}{m^{\nu/2}} I_{\nu 1} + \sum_{\nu=0}^{k-3} \sum_{\mu=1}^{k-3} \sum_{j=1}^{\mu} \frac{d_{j\mu\nu} \xi_\nu}{m^{\frac{1}{2}(\mu+\nu)}} \Phi_{\nu, \mu+2j+1} + \frac{A_k}{m^{\frac{1}{2}(k-2)}}$$

$$= \sum_1 + \sum_2 + \frac{A_k}{m^{\frac{1}{2}(k-2)}},$$

where

$$I_{\alpha\beta} = \int_{-\infty}^{\infty} z^\alpha \Phi^{(\beta)}(z)\, dz.$$

Now the following facts can easily be established by means of partial integration:

(9) $$I_{\alpha\beta} = 0 \quad \text{when} \quad \alpha - \beta \text{ is even,}$$

(10) $$I_{\alpha\beta} = 0 \quad \text{when} \quad \beta - \alpha > 1.$$

By (9), the non-vanishing terms in $\sum_1$ are the even terms and the non-vanishing terms in $\sum_2$ are those for which $\mu + \nu$ is even. Hence

$$\sum_1 = \sum_{\nu=0}^{[\frac{1}{2}(k-3)]} \frac{e_\nu \xi_{2\nu}}{m^\nu},$$

$$\sum_2 = \sum_{\nu=0}^{[\frac{1}{2}(k-3)]} \sum_{\mu=1}^{[\frac{1}{2}(k-3)]} \sum_{j=1}^{2\mu} \frac{e_{j\mu\nu} \xi_{2\nu}}{m^{\mu+\nu}} I_{2\nu, 2\mu+2j+1} + \sum_{\nu=0}^{[\frac{1}{2}(k-4)]} \sum_{\mu=0}^{[\frac{1}{2}(k-4)]} \sum_{j=1}^{2\mu+1} \frac{e'_{j\mu\nu} \xi_{2\nu+1}}{m^{\mu+\nu+1}} I_{2\nu+1, 2\mu+2j+2}.$$

Using (10) to reduce $\sum_2$ further we get

$$\sum_2 = \sum_{\nu=2}^{[\frac{1}{2}(k-3)]} \sum_{\mu=1}^{\nu-1} \sum_{j=1}^{2\mu} \frac{e_{j\mu\nu}\,\xi_{2\nu}}{m^{\mu+\nu}} I_{2\nu,\,2\mu+2j+1} + \sum_{\nu=1}^{[\frac{1}{2}(k-4)]} \sum_{\mu=0}^{\nu-1} \sum_{j=1}^{2\mu+1} \frac{e'_{j\mu\nu}\,\xi_{2\nu+1}}{m^{\mu+\nu+1}} I_{2\nu+1,\,2\mu+2j+2}$$

$$= \sum_{\nu=0}^{[\frac{1}{2}(k-7)]} \sum_{\mu=0}^{\nu} \frac{g_{\mu\nu}\,\xi_{2\nu+4}}{m^{\mu+\nu+3}} + \sum_{\nu=0}^{[\frac{1}{2}(k-6)]} \sum_{\mu=0}^{\nu} \frac{g'_{\mu\nu}\,\xi_{2\nu+3}}{m^{\mu+\nu+2}}$$

$$= \sum_{\alpha=0}^{[\frac{1}{2}(k-9)]} \frac{1}{m^{\alpha+3}} \sum_{\beta=[\frac{1}{2}(\alpha+1)]}^{\alpha} h_{\alpha\beta}\,\xi_{2\beta+4} + \sum_{\alpha=0}^{[\frac{1}{2}(k-6)]} \frac{1}{m^{\alpha+2}} \sum_{\beta=[\frac{1}{2}(\alpha+1)]}^{\alpha} h'_{\alpha\beta}\,\xi_{2\beta+3} + \frac{A_k}{m^{\frac{1}{2}(k-2)}}$$

$$= \sum_{i=3}^{[\frac{1}{2}(k-3)]} \frac{1}{m^i} \sum_{j=[\frac{1}{2}(i-2)]}^{i-3} l_{ij}\,\xi_{2j+4} + \sum_{i=2}^{[\frac{1}{2}(k-3)]} \frac{1}{m^i} \sum_{j=[\frac{1}{2}(i-1)]}^{i-2} l'_{ij}\,\xi_{2j+3} + \frac{A_k}{m^{\frac{1}{2}(k-2)}}.$$

Hence

$$\sum_1 + \sum_2 = \xi_0 + \frac{e_1\xi_2}{m} + \frac{e_2\xi_4 + l'_{20}\xi_3}{m^2} + \sum_{\nu=3}^{[\frac{1}{2}(k-3)]} \frac{1}{m^\nu} \cdot\left(e_\nu\,\xi_{2\nu} + \sum_{\mu=[\frac{1}{2}(\nu-2)]}^{\nu-3} l_{\mu\nu}\,\xi_{2\nu+4} + \sum_{\mu=[\frac{1}{2}(\nu-2)]}^{\nu-2} l'_{\mu\nu}\,\xi_{2\nu+3}\right) + \frac{A_k}{m^{\frac{1}{2}(k-2)}}$$

$$= \xi_0 + \sum_{\nu=1}^{k-3} \frac{1}{m^\nu} \sum_{j=\nu+1}^{2\nu} p_{j\nu}\,\xi_j + \frac{A_k}{m^{\frac{1}{2}(k-2)}}.$$

Hence

$$F(x) = \xi_0 + \sum_{\nu=1}^{k-3} \frac{1}{m^\nu} \sum_{j=\nu+1}^{2\nu} p_{j\nu}\,\xi_j + \frac{p_k}{m^{\frac{1}{2}(k-2)}}.$$

Our final conclusion is: Under the conditions (II1)–(II5) formula (11) is true; if $k = 3$, (11) remains true without the condition (II2).