ON THE POWER FUNCTIONS OF THE E^2-TEST AND THE T^2-TEST

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1. The general linear hypothesis. Every linear hypothesis about a p-variate normal population or several such populations having common variances and covariances is reducible to the following canonical form [4]: The sample distribution, when nothing whatever has been discarded from the whole sample, being

$$(1)\quad (2\pi)^{-\frac{1}{2}p(m+n)} \mid \alpha_{ij} \mid^{\frac{1}{2}(m+n)} \exp \left\{ -\tfrac{1}{2} \sum_{i,j=1}^{p} \alpha_{ij} \sum_{r=1}^{m} (y_{ir} - \eta_{ir})(y_{jr} - \eta_{jr}) - \tfrac{1}{2} \sum_{i,j=1}^{p} \alpha_{ij} \sum_{s=1}^{n} z_{is} z_{js} \right\} \Pi\, dy\, dz \qquad (n \geq p),$$

where the η_{ir} and the α_{ij} are unknown, the hypothesis to be tested is

$$H: \quad \eta_{ir} = 0 \qquad (i = 1, \cdots, p; r = 1, \cdots, n_1, n_1 \leq m).$$

It is clear that the y_{ir} $(i = 1, \cdots, p;\ r = n_1+1, \cdots, m)$ can have no use. Also, the only useful quantities supplied by the set z_{is} are the statistics

$$b_{ij} = \sum_{s=1}^{n} z_{is} z_{js},$$

because the remaining quantities may be regarded as a set of angles which are independent of y_{ir} and the b_{ij} and which has a known distribution free from any unknown parameter in (1), [2]. After discarding the irrelevant y's and the angles there results the reduced sample distribution

$$K \mid \alpha_{ij} \mid^{\frac{1}{2}(n_1+n)} \mid b_{ij} \mid^{\frac{1}{2}(n-p-1)} \exp \left\{ -\tfrac{1}{2} \sum_{i,j=1}^{p} \alpha_{ij} \cdot \sum_{r=1}^{n_1} (y_{ir} - \eta_{ir})(y_{jr} - \eta_{jr}) - \tfrac{1}{2} \sum_{i,j=1}^{p} \alpha_{ij} b_{ij} \right\} \Pi\, dy\, db.$$

Hereafter the indices i, j and r shall have the following ranges:

$$i, j = 1, \cdots, p, \qquad r = 1, \cdots, n_1,$$

and the convention that repetition of an index indicates summation will be adopted. Writing

$$a_{ij} = y_{ir} y_{jr}, \qquad c_{ij} = a_{ij} + b_{ij},$$

we obtain the distribution of the y_{ir} and the c_{ij}:

$$(2)\quad K \mid \alpha_{ij} \mid^{\frac{1}{2}(n_1+n)} \mid c_{ij} - a_{ij} \mid^{\frac{1}{2}(n-p-1)} \exp\, (-\tfrac{1}{2}\alpha_{ij} c_{ij} + \alpha_{ij} y_{ir} \eta_{jr} - \tfrac{1}{2}\alpha_{ij} \eta_{ir} \eta_{jr}) \Pi\, dy\, dc.$$

In the remaining two sections of this paper we deal exclusively with the special cases $p = 1$ and $n_1 = 1$. According as $p = 1$ or $n_1 = 1$ we shall drop the indices i and j or the index r.

The case $p = 1$. When $p = 1$, (2) reduces to

$$K\alpha^{\frac{1}{2}(n_1+n)}(c - y_r y_r)^{\frac{1}{2}n} \exp\left(-\tfrac{1}{2}\alpha c + \alpha y_r \eta_r - \tfrac{1}{2}\alpha\eta_r\eta_r\right) dc\Pi\, dy.$$

Putting $y_r = c^{\frac{1}{2}}x_r$ we obtain

$$(3) \quad K\alpha^{\frac{1}{2}(n_1+n)}c^{\frac{1}{2}(n_1+n)-1}(1 - x_r x_r)^{\frac{1}{2}n-1}\exp\left(-\tfrac{1}{2}\alpha c + \alpha c^{\frac{1}{2}}x_r\eta_r - \tfrac{1}{2}\alpha\eta_r\eta_r\right) dc\Pi dx.$$

The hypothesis H is now

$$H'\colon\ \eta_r = 0 \quad (r = 1, \cdots, n_1).$$

If w is any critical region for the rejection of H', denote by $w(c)$ the cross section of w for every fixed c. Then the power function of w is

$$(4) \quad \begin{aligned} \beta_w(\eta, \alpha) &= \beta_w(\eta_1, \cdots, \eta_{n_1}, \alpha) \\ &= K\alpha^{\frac{1}{2}(n_1+n)} e^{-\frac{1}{2}\alpha\eta_r\eta_r} \int_0^\infty c^{\frac{1}{2}(n_1+n)-1} e^{-\frac{1}{2}\alpha c}\, dc \int_{w(c)} (1 - x_r x_r)^{\frac{1}{2}n-1} e^{\alpha c^{\frac{1}{2}} x_r \eta_r}\, \Pi\, dx. \end{aligned}$$

It is known [3] that, in order to have

$$(5) \quad \beta_w(0, \alpha) = \epsilon$$

for all α, it is necessary and sufficient that

$$(6) \quad \int_{w(c)} (1 - x_r x_r)^{\frac{1}{2}n-1} \Pi\, dx = A\epsilon,$$

where A is a constant.

The E^2-test is the test based on the critical region

$$w_0\colon\ x_r x_r = c^{-\frac{1}{2}} y_r y_r = E^2 \geq \text{const.}$$

The author has proved [3] that of all the critical regions which satisfy (5) and whose power function is a function of $\alpha\eta_r\eta_r$ alone, the region w_0 is the uniformly most powerful one. This result is generalized by Wald [7], who proved that, of all the regions satisfying (5); the surface integral

$$\gamma_w(\alpha, \lambda) = \int_{\eta_r\eta_r=\lambda} \beta_w(\eta, \alpha)\, dA$$

is maximum when w is w_0. The author gives here another proof of Wald's theorem which is easier as it dispenses with the somewhat intricate Lemma 1 of Wald. From (4) we have

$$\begin{aligned} \gamma_w(\alpha, \lambda) = K\alpha^{\frac{1}{2}(n_1+n)} &\int_0^\infty c^{\frac{1}{2}(n_1+n)-1} e^{-\frac{1}{2}\alpha c}\, dc \\ &\cdot \int_{w(c)} (1 - x_r x_r)^{\frac{1}{2}n-1} \Pi\, dx \int_{\eta_r\eta_r=\lambda} \exp\left(-\tfrac{1}{2}\alpha\eta_r\eta_r + \alpha c^{\frac{1}{2}} x_r \eta_r\right) dA. \end{aligned}$$

By means of a rotation in the space of $(\eta_1, \cdots, \eta_{n_1})$ we can obtain

$$\int_{\eta_r\eta_r=\lambda} \exp\left(-\tfrac{1}{2}\alpha\eta_r\eta_r + \alpha c^{\frac{1}{2}} x_r\eta_r\right) dA$$

$$= \int_{\zeta_r\zeta_r=\lambda} \exp\left(-\tfrac{1}{2}\alpha\zeta_r\zeta_r + \alpha c^{\frac{1}{2}}(x_r x_r)^{\frac{1}{2}}\zeta_1\right) dA = \sum_{k=0}^{\infty} a_k \alpha^{2k}(c x_r x_r)^k,$$

where a_k depends only on α, k and λ. Hence

$$(7)\quad \gamma_w(\alpha, \lambda) = \sum_{k=0}^{\infty} b_k \int_0^{\infty} c^{\frac{1}{2}(n_1+n)-1} e^{-\frac{1}{2}\alpha c}\, dc \int_{w(c)} (x_r x_r)^k (1 - x_r x_r)^{\frac{1}{2}n-1} \Pi\, dx,$$

where b_k depends only on k, α and λ. Since $w(c)$ satisfies (6), it follows from a lemma of Neyman and Pearson [5] that

$$\int_{w(c)} (x_r x_r)^k (1 - x_r x_r)^{\frac{1}{2}n-1} \Pi\, dx$$

is maximum, for all c and k, when $w(c)$ is the region $x_r x_r \geq$ const., i.e. when w is itself the region $x_r x_r \geq$ const. This proves Wald's theorem.

Still another optimum property of the E^2-test may be established on using the volume integral instead of the surface integral. This is stated in the following theorem.

THEOREM 1. *Let S be any linear set and let*

$$\varphi_w(\alpha, S) = \int_{\eta_r\eta_r \in S} \beta_w(\eta, \alpha) \Pi\, d\eta.$$

Of all the regions satisfying (5), *the region w_0 has the maximum $\varphi_w(\alpha, S)$.*

For, by the same computation which leads to (7), we easily obtain

$$\varphi_w(\alpha, S) = \sum_{k=0}^{\infty} c_k \int_0^{\infty} c^{\frac{1}{2}(n_1+n)-1} e^{-\frac{1}{2}\alpha c}\, dc \int_{w(c)} (x_r x_r)^k (1 - x_r x_r)^{\frac{1}{2}n-1} \Pi\, dx,$$

where c_k depends only on k, α and S. Hence the result follows.

This theorem also contains my previous result as a consequence. For, writing

$$\beta_w(\eta, \alpha) = f(\alpha\eta_r\eta_r), \qquad \beta_{w_0}(\eta, \alpha) = f_0(\alpha\eta_r\eta_r),$$

we have

$$0 \leq \int_{\eta_r\eta_r \in S} (f_0(\alpha\eta_r\eta_r) - f(\alpha\eta_r\eta_r)) \Pi\, d\eta = \frac{\pi^{\frac{1}{2}n_1}}{\Gamma(\frac{1}{2}n_1)} \int_S t^{\frac{1}{2}n_1-1}(f_0(\alpha t) - f(\alpha t))\, dt.$$

Since S is arbitrary, we must have $f(\alpha t) \leq f_0(\alpha t)$.

The case $n_1 = 1$. When $n_1 = 1$, (2) and H become respectively

$$(8)\quad K \,|\, \alpha_{ij} \,|^{\frac{1}{2}(n+1)} \,|\, c_{ij} - y_i y_j \,|^{\frac{1}{2}(n-p-1)} \exp\left(-\tfrac{1}{2}\alpha_{ij}c_{ij} + \alpha_{ij}y_i\eta_j - \tfrac{1}{2}\alpha_{ij}\eta_i\eta_j\right) \Pi\, dy\, dc,$$

$$H'': \quad \eta_i = 0 \qquad (i = 1, \cdots, p).$$

There is a unique real matrix

$$\mathbf{T} = \begin{bmatrix} t_{11} & & & \\ t_{12} & t_{22} & & \\ \cdots & \cdots & \cdots & \cdots \\ t_{1p} & t_{2p} & \cdots & t_{pp} \end{bmatrix} \quad (t_{ii} > 0; \text{ zeros above the principal diagonal})$$

such that $[c_{ij}] = \mathbf{TT}'[2]$. Introducing the new variables $x_1, \cdots, x_p$ by means of the transformation

$$(9) \qquad [y_1, \cdots, y_p] = [x_1, \cdots, x_p]\mathbf{T}'$$

with the Jacobian $|\mathbf{T}| = |c_{ij}|^{\frac{1}{2}}$ we obtain the distribution

$$(10) \qquad \begin{aligned} f(x, c)\Pi\, dx\, dc = K\,|\alpha_{ij}|^{\frac{1}{2}(n+1)}\,|c_{ij}|^{\frac{1}{2}(n-p)}(1 - x_i x_i)^{\frac{1}{2}(n-p-1)} \\ \cdot \exp\left(-\tfrac{1}{2}\alpha_{ij}c_{ij} + \alpha_{ij}t_{ki}x_k\eta_j - \tfrac{1}{2}\alpha_{ij}\eta_i\eta_j\right)\Pi\, dx\, dc \end{aligned}$$

$$(k = 1, \cdots, p; \quad t_{ki} = 0 \quad \text{when} \quad k > i).$$

If w is any region, we write

$$\beta_w(\eta, \alpha) = \beta_w(\eta_1, \cdots, \eta_p, \alpha_{11}, \alpha_{12}, \cdots, \alpha_{pp}) = \int_w f(x, c)\Pi\, dx\, dc,$$

so that $\beta_w(\eta, \alpha)$ is the power function if w serves as a critical region for rejecting H''. We have, symbolically,

$$w = D \times w(c),$$

where D is the set of points (c_{ij}) for which $[c_{ij}]$ is positive definite and $w(c)$ is the cross section of w for fixed c_{ij}. Then

$$\begin{aligned} \beta_w(\eta, \alpha) = K\,|\alpha_{ij}|^{\frac{1}{2}(n+1)}\,e^{-\frac{1}{2}\alpha_{ij}\eta_i\eta_j}\int_D |c_{ij}|^{\frac{1}{2}(n-p)}\,e^{-\frac{1}{2}\alpha_{ij}c_{ij}}\,\Pi\, dc \\ \cdot \int_{w(c)} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)}\,e^{\alpha_{ij}t_{ki}x_r\eta_j}\,\Pi\, dx. \end{aligned}$$

It is known [6] that, in order to have

$$(11) \qquad \beta_w(0, \alpha) = \epsilon$$

for all α_{ij}, it is necessary and sufficient that

$$(12) \qquad \begin{aligned} \int_{w(c)} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)}\,\Pi\, dx = B\epsilon, \\ \text{where } B = \int_{x_i x_i \le 1} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)}\,\Pi\, dx. \end{aligned}$$

The T^2-test is the test based on the critical region

$$w_0: \quad x_i x_i = c^{ij}y_i y_j = T^2/(1 + T^2) \ge \text{const.}, \quad \text{or} \quad T^2 \ge \text{const.},$$

where c^{ij} is the general element of $[c_{ij}]^{-1}$ and T^2 is, except for a constant factor, Hotelling's generalization of "Student's" ratio.

In order to establish an optimum property of T^2 analogous to that of E^2 given in Theorem 1, we define, for any linear set S and any region R in the sample space,

$$\psi_R(S) = \int_{\alpha_{ij}\eta_i\eta_j \epsilon S} \beta_R(\eta, \alpha)\Pi\, d\eta\, d\alpha.$$

$\Psi_R(S)$ does not necessarily have a finite value, and it is this fact which renders the following theorem less satisfactory than Theorem 1.

THEOREM 2. *Let ρ_p be the smallest latent root of $[c_{ij}]$ and let E be any subset of D in which ρ_p is at least equal to a fixed positive constant. Of all the critical regions w which satisfy* (11), *the region w_0 has the maximum $\psi_{wE}(S)$.*

In order to prove this theorem we need the following two lemmas.

LEMMA 1. *If c is a positive constant, the integral*

$$I = \int_{\rho_p \geq c} |c_{ij}|^{-(p+\frac{1}{2})}\Pi\, dc$$

has a finite value.

PROOF. Let $\rho_1, \cdots, \rho_p$ be the latent roots of $[c_{ij}]$ in the descending order of magnitude. From a known theorem [1] we get

$$I = C\int_{c \leq \rho_p \leq \cdots \leq \rho_1 < \infty} (\rho_1 \cdots \rho_p)^{-(p+\frac{1}{2})} \prod_{i<j} (\rho_i - \rho_j)\Pi\, d\rho$$
$$\leq C \int_c^\infty \cdots \int_c^\infty \left(\prod_{i=1}^p \rho_i^{-(i+\frac{1}{2})}\right) d\rho_1 \cdots d\rho_p .$$

Hence I is finite.

LEMMA 2.

$$(13) \quad \psi_{wE}(S) = \sum_{k=0}^\infty g_k \int_E |c_{ij}|^{-(p+\frac{1}{2})}\Pi\, dc \int_{w(c)} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)}(x_i x_i)^k \Pi\, dx$$

and $\psi_{wE}(S)$ is finite, where g_k depends only on k and S.

PROOF. Let Δ be the set of points (α_{ij}) for which $[\alpha_{ij}]$ is positive definite. By (8), we have

$$\psi_{wE}(S) = K \int_{wE} |c_{ij} - y_i y_j|^{\frac{1}{2}(n-p-1)}\Pi\, dy\, dc \int_\Delta |\alpha_{ij}|^{\frac{1}{2}(n+1)} e^{-\frac{1}{2}c_{ij}\alpha_{ij}} J\,\Pi\, d\alpha,$$

where

$$J = \int_{\alpha_{ij}\eta_i\eta_j \epsilon S} \exp\left(-\tfrac{1}{2}\alpha_{ij}\eta_i\eta_j + \alpha_{ij}y_i\eta_j\right)\Pi\, d\eta.$$

There is a real non-singular matrix $\mathbf{G} = [g_{ij}]$ such that $[\alpha_{ij}] = \mathbf{GG}'$. Using the transformation

$$[\eta_1, \cdots, \eta_p]\mathbf{G} = [\zeta_1, \cdots, \zeta_p],$$

whose Jacobian is $| \mathbf{G} |^{-1} = | \alpha_{ij} |^{-\frac{1}{2}}$, we have

$$J = | \alpha_{ij} |^{-\frac{1}{2}} \int_{\zeta_i \zeta_i \epsilon S} \exp(-\tfrac{1}{2}\zeta_i \zeta_i + g_{ji} \zeta_i y_j) \Pi \, d\zeta.$$

This is reducible by means of a rotation to

$$(14) \qquad \begin{aligned} J &= | \alpha_{ij} |^{-\frac{1}{2}} \int_{\tau_i \tau_i \epsilon S} \exp(-\tfrac{1}{2}\tau_i \tau_i + (\alpha_{ij} y_i y_j)^{\frac{1}{2}} \tau_1) \Pi \, d\tau \\ &= | \alpha_{ij} |^{-\frac{1}{2}} \sum_{k=0}^{\infty} d_k (\alpha_{ij} y_i y_j)^k, \end{aligned}$$

where

$$d_k = \frac{1}{(2k)!} \int_{\tau_i \tau_i \epsilon S} \tau_1^{2k} e^{-\frac{1}{2}\tau_i \tau_i} \Pi \, d\tau \leq \frac{1}{(2k)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \tau_1^{2k} e^{-\frac{1}{2}\tau_i \tau_i} \, d\tau_1 \cdots d\tau_p = \frac{(2\pi)^{\frac{1}{2}p}}{2^k k!}$$

and d_k depends only on k and S. Hence

$$\int_\Delta | \alpha_{ij} |^{\frac{1}{2}(n+1)} e^{-\frac{1}{2}c_{ij}\alpha_{ij}} J \, \Pi \, d\alpha = \sum_{k=0}^{\infty} d_k I_k ,$$

where

$$(15) \qquad I_k = \int_\Delta | \alpha_{ij} |^{\frac{1}{2}n} (\alpha_{ij} y_i y_j)^k e^{-\frac{1}{2}\alpha_{ij} c_{ij}} \Pi \, d\alpha.$$

Now

$$I_k = \frac{d^k}{dt^k} f(t) \Big|_{t=0},$$

where

$$\begin{aligned} f(t) = \int_\Delta | \alpha_{ij} |^{\frac{1}{2}n} e^{-\frac{1}{2}(c_{ij} - 2t y_i y_j)\alpha_{ij}} \Pi \, d\alpha &= K_1 | c_{ij} - 2t y_i y_j |^{-\frac{1}{2}(n+p+1)} \\ &= K_1 | c_{ij} |^{-\frac{1}{2}(n+p+1)} (1 - 2t c^{ij} y_i y_j)^{-\frac{1}{2}(n+p+1)} \end{aligned}$$

Hence

$$(16) \qquad I_k = e_k | c_{ij} |^{-\frac{1}{2}(n+p+1)} (c^{ij} y_i y_j)^k,$$

where

$$e_k = \frac{K_1 2^k \Gamma\left(\frac{n+p+1}{2} + k\right)}{k! \Gamma\left(\frac{n+p+1}{2}\right)}.$$

Hence

$$\psi_{wE}(S) = K \sum_{k=0}^{\infty} d_k e_k \int_{wE} | c_{ij} |^{-\frac{1}{2}(n+p+1)} | c_{ij} - y_i y_j |^{\frac{1}{2}(n-p-1)} (c^{ij} y_i y_j)^k \Pi \, dy \, dc$$

$$= \sum_{k=0}^{\infty} g_k \int_E | c_{ij} |^{-(p+\frac{1}{2})} \Pi \, dc \int_{w(c)} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)} (x_i x_i)^k \Pi \, dx,$$

where $g_k = K_1 d_k e_k$ depends only on k and S.

Now

$$\int_{w(c)} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)} (x_i x_i)^k \Pi \, dx \leq \int_{x_i x_i \leq 1} \Pi \, dx,$$

$$\int_E | c_{ij} |^{-(p+\frac{1}{2})} \Pi \, dc \leq \int_{\rho_p \geq c > 0} | c_{ij} |^{-\frac{1}{2}(p+\frac{1}{2})} \Pi \, dc$$

is finite by Lemma 1. Hence

$$\psi_{wE}(S) \leq \text{const.} \sum_{k=0}^{\infty} d_k e_k = \text{const.} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n+p+1}{2} + k\right)}{(k!)^2}$$

and so $\varphi_{wE}(S)$ is finite. This proves Lemma 2.

Proof of Theorem 2. Since $\psi_{wE}(S)$ is expressible as (13) and is always finite, it follows from (12) and the Neyman-Pearson Lemma that $\psi_{wE}(S)$ is maximum when w is w_0. This proves Theorem 2.

Simaika [6] proved that of all the critical regions w which satisfy the conditions

(a) $\beta_w(0, \alpha) = \epsilon$ for all α_{ij},

(b) $\beta_w(\eta, \alpha) = f(\alpha_{ij}\eta_i\eta_j)$,

w_0 is the uniformly most powerful one. Strangely enough, this result cannot be deduced as a consequence from our Theorem 2.

The difficulty in dealing with the integral $\psi_w(S)$ is that it is not always finite. In order to have a finite integral let us consider the following:

$$\Gamma_w(\theta, S) = \int_{\alpha_{ij}\eta_i\eta_j \epsilon S} e^{-\frac{1}{2}\theta_{ij}\alpha_{ij}} \beta_w(\eta, \alpha) \Pi \, d\eta \, d\alpha,$$

where $[\theta_{ij}]$ is a positive definite matrix. As an immediate consequence of Simaika's theorem we have

$$\Gamma_w(\theta, S) \leq \Gamma_{w_0}(\theta, S) \tag{17}$$

for any region w satisfying (a) and (b). Now the question arises whether (17) remains true if the condition (b) on w is removed. The following theorem answers this question in the negative.

THEOREM 3. *Let* $[\theta_{ij}]$ *be a positive definite matrix,* $[\rho_{ij}] = [c_{ij} + \theta_{ij}]^{-1}$ *and* $\lambda_1, \cdots, \lambda_p$ *be the roots of the equation* $| c_{ij} - \lambda\theta_{ij} | = 0$. *There is a function* $g = g(\lambda_1, \cdots, \lambda_p)$ *such that the region*

$$w_1 : \quad \rho_{ij} y_i y_j \geq g(\lambda_1, \cdots, \lambda_p)$$

satisfies (a) *and has the maximum* $\Gamma_w(\theta, S)$.

PROOF. From (10) and (14) we obtain

$$\Gamma_w(\theta, S) = K \sum_{k=0}^{\infty} d_k \int_w | c_{ij} - y_i y_j |^{\frac{1}{2}(n-p-1)} \Pi \, dy \, dc$$

$$\cdot \int_\Delta | \alpha_{ij} |^{\frac{1}{2}n} (\alpha_{ij} y_i y_j)^k e^{-\frac{1}{2}(c_{ij}+\theta_{ij})\alpha_{ij}} \Pi \, d\alpha.$$

Comparing the inner integral with (15) and using (16) we get

$$\Gamma_w(\theta, S) = \sum_{k=0}^{\infty} g_k \int_w | c_{ij} + \theta_{ij} |^{-\frac{1}{2}(n+p+1)} | c_{ij} - y_i y_j |^{\frac{1}{2}(n-p-1)} (\rho_{ij} y_i y_j)^k \Pi . dy \, dc$$

(18) $$= \sum_{k=0}^{\infty} g_k \int_D | c_{ij} + \theta_{ij} |^{-\frac{1}{2}(n+p+1)} | c_{ij} |^{\frac{1}{2}(n-p)} \Pi \, dc$$

$$\cdot \int_{w(c)} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)} (\gamma_{ij} x_i x_j)^k \Pi \, dx,$$

where $\gamma_{ij} x_i x_j$ is the result of applying the transformation (9) on $\rho_{ij} y_i y_j$. We shall show that, for every fixed set of c_{ij}, a unique number $g = g(\lambda_1, \cdots, \lambda_p)$ exists such that the region $\rho_{ij} y_i y_j = \gamma_{ij} x_i x_j \geq g$ satisfies (12), i.e.

(19) $$\int_{\gamma_{ij} x_i x_j \geq g} (1 - x_i x_i)^{\frac{1}{2}(n-p-1)} \Pi \, dx = B\epsilon.$$

Since $[\gamma_{ij}] = T'[c_{ij} + \theta_{ij}]^{-1} T$, the latent roots of $[\gamma_{ij}]$ are $\lambda_i/(1 + \lambda_i)$ $(i = 1, \cdots, p)$. Hence by a rotation the equation (19) is reduced to

(20) $$\int_{(\lambda_i/1+\lambda_i)\xi_i\xi_i \geq g} (1 - \xi_i \xi_i)^{\frac{1}{2}(n-p-1)} \Pi \, d\xi = B\epsilon.$$

As g increases from 0 onwards, the left member of (20) decreases steadily from B to 0. Hence there is a unique $g = g(\lambda_1, \cdots, \lambda_p)$ which satisfies (20).

For this $g(\lambda_1, \cdots, \lambda_p)$ the region w_1 satisfies (a). Hence, applying the Neyman-Pearson Lemma on (18) we obtain the result.

From Theorem 3 we learn that there actually exist other exact tests for H'' which have some optimum property not possessed by T^2, viz., the tests based on the critical regions w_1 corresponding to various values of the θ_{ij}. However, the great difficulty in numerical computation prohibits their application and the T^2-test stands out as the only test which is both simple and good.

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