COMPACT COMPUTATION OF THE INVERSE OF A MATRIX

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1. Introduction. Among the most common applications of mathematics to practical problems are the solution of simultaneous equations, the evaluation of determinants, and the computation of the complete inverse, (or the complete adjugate), of a given matrix. Even with modern computing machines these are laborious, time-consuming jobs. For that reason there has been great interest in recent years in the development of so-called "compact" methods; that is, methods that eliminate all unnecessary detail, that use computing machines to do as much of the work as possible, and that only require copying the results needed in further analysis.

In 1935 a paper by one of the authors [1] and since then papers by the other author [2], [3], [4], [5], [6] and [7] have outlined a variety of compact methods and have applied them to actual problems. These papers, together with other recent contributions, such as those presented in [8], [9] and [10], have resulted in much improved and more compact techniques in the general field of the solution of linear simultaneous equations and allied topics, especially if the matrix is axi-symmetric. It is not generally recognized, however, that extension of these procedures (usually involving matrix factorization [7] [10]) can be used to compute the inverse (and adjugate) directly from the matrix factors without the necessity of the reduction of the unit matrix [11; 150] [2; 121] when the matrix is non-symmetric.

The present paper extends the use of compact methods in three ways.

(a) It presents a method of computing the inverse (and adjugate) of a symmetric or non-symmetric matrix by compact Gaussian methods without the formal reduction of an auxiliary identity matrix.

(b) It introduces the method of multiplication and subtraction with division—a modification of the method of multiplication and subtraction—and shows that the terms recorded in the compact solution are themselves determinants which are minors of the determinant of the matrix.

(c) It uses the method of multiplication and subtraction with division as a compact means of computing the exact value of any minor of the determinant of the matrix (whether symmetric or non-symmetric). It further shows how all cofactors of order $n - 1$ (constituting the adjugate) can be computed from a compact presentation of the calculations of the determinant of the matrix.

2. Gaussian methods and notation. Probably the method most generally used to solve simultaneous equations is the division method originated by Gauss [12]. Variations of this method are known as the Doolittle Method [13], the method of pivotal condensation [14], the method of single division [2; 104–112], and the Crout method [8]. The methods as outlined by Gauss and Doolittle are applicable only to axi-symmetric matrices (common to least squares theory) while a more general presentation, applicable to non-symmetric matrices as well, has been made by more recent authors.

The compact form of this method, extended to apply to the non-symmetric matrix, used in this paper is as follows:

Given the matrix

$$a = (a_{rk}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{bmatrix} \tag{1}$$

we compute

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ b_{21} & a_{22.1} & a_{23.1} \cdots a_{2n.1} \\ b_{31} & b_{32.1} & a_{33.12} \cdots a_{3n.12} \\ \cdots & \cdots & \cdots \\ b_{n1} & b_{n2.1} & b_{n3.12} \cdots b_{nn \cdot 12 \cdots n-1} \end{bmatrix} \tag{2}$$

where

$$\begin{aligned} b_{r1} &= a_{r1}/a_{11} \\ a_{2k.1} &= a_{2k} - b_{21}a_{1k} \\ b_{r2.1} &= a_{r2} - b_{r1}a_{12})/a_{22.1} \\ a_{3k.12} &= a_{3k} - b_{32}a_{1k} - b_{32.1}a_{2k \cdot 1} \\ b_{r3.12} &= (a_{r3} - b_{r1}a_{13} - b_{r2.1}a_{23.1})/a_{33.12} \end{aligned} \tag{3}$$

and in general

$$\begin{aligned} a_{rk \cdot 12 \cdots j} &= a_{rk \cdot 12 \cdots j-1} - \frac{a_{jk \cdot 12 \cdots j-1}\, a_{rj \cdot 12 \cdots j-1}}{a_{jj \cdot 12 \cdots j-1}}, \\ b_{rk \cdot 12 \cdots j} &= \frac{a_{rk \cdot 12 \cdots j}}{a_{kk \cdot 12 \cdots j}}. \end{aligned} \tag{4}$$

It should be noted that Crout's presentation [8] is similar to that used here except that Crout divides the elements of each row by the leading element while we divide the elements of columns.

The notation used above, introduced by one of the authors [2], parallels that used extensively in multiple correlation and regression theory. It differs somewhat from the notation used by Gauss. See [12; 69].

Since every b is the ratio of two a's it follows that every b can be written in terms of a's so that the formulas can be written in terms of a's alone. This is what Gauss did although he used [ ]'s instead of a's. Gauss also used letters to indicate the primary subscripts and a single secondary subscript to indicate the number of eliminations. Thus our $a_{22.1}$ was written by Gauss as $[bb, 1]$ and $a_{33.12}$ appeared as $[cc, 2]$.

It is in the interest of less extensive notation and it makes our notation somewhat closer to that introduced by Gauss if we replace

$$a_{rk\cdot 12\cdots j} \quad \text{by} \quad a_{rk\cdot(j)}$$

$$b_{rk\cdot 12\cdots j} \quad \text{by} \quad b_{rk\cdot(j)}\,.$$

This shortened notation can always be used when the secondary subscripts include *all* the integers from 1 to j. In this modified notation the formulas (4) become

$$\begin{aligned} a_{rk\cdot(j)} &= a_{rk\cdot(j-1)} - \frac{a_{jk\cdot(j-1)}\,a_{rj\cdot(j-1)}}{a_{jj\cdot(j-1)}} \\ b_{rk\cdot(j)} &= \frac{a_{rk\cdot(j)}}{a_{kk\cdot(j)}}\,. \end{aligned} \tag{5}$$

3. Solution by matrix factorization. The values of matrix (2) are in general not final answers to proposed problems but they are values from which final answers can be computed. The matrix (2) exhibits essentially both the triangular matrix of the $a_{rk\cdot(j)}$ which we call $\mathfrak{t}$ and the triangular matrix $b_{rk\cdot(j)}$ which we call $\mathfrak{s}$. (The diagonal entries of the $\mathfrak{s}$ matrix are all unity and do not appear.) Hence (2) is really $\mathfrak{s} - \mathfrak{J} + \mathfrak{t}$.

A basic property, useful in most problems involving the use of (2), is that $\mathfrak{s}$ and $\mathfrak{t}$ are factors of $\mathfrak{a}$. Thus

$$\mathfrak{a} = \mathfrak{s}\mathfrak{t} \quad \text{and} \quad \mathfrak{a} - \mathfrak{s}\mathfrak{t} = 0. \tag{6}$$

That this is true in the symmetric case was proved in an earlier paper [7; 85]. That this is also true for the non-symmetric case is now shown in a similar manner.

Let $\mathfrak{t}_1$ be a matrix (n by n) with the first row composed of elements a_{1k} and all other elements 0. Let $\mathfrak{s}_1$ be a similar matrix with first column elements $b_{r1} = \dfrac{a_{r1}}{a_{11}}$ and all other elements 0. Then $\mathfrak{a} - \mathfrak{s}_1\mathfrak{t}_1 = \mathfrak{a}_1 = (a_{rk\cdot 1})$ is a matrix (n by n) with all elements of the first column and first row 0.

Next let $\mathfrak{t}_2$ be a matrix (n by n) with the second row elements $a_{2k\cdot 1}$ and all other elements 0. Let $\mathfrak{s}_2$ be a matrix (n by n) with second column elements $b_{r2\cdot 1}$ and all other elements 0. Then $\mathfrak{a}_1 - \mathfrak{s}_2\mathfrak{t}_2 = \mathfrak{a}_2 = (a_{rk\cdot(2)})$ is a matrix (n by n) with each element of the first two columns and first two rows equal to 0.

This process is continued through n successive steps, an additional row and column being made identically zero at each step. We have then

$$\mathfrak{a} - \mathfrak{s}_1\mathfrak{t}_1 - \mathfrak{s}_2\mathfrak{t}_2 - \cdots - \mathfrak{s}_n\mathfrak{t}_n = \mathfrak{a}_{n+1} = 0. \tag{7}$$

Now consider the triangular matrix

$$\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3 + \cdots + \mathfrak{t}_n$$

with its rows composed of the non-zero rows of $\mathfrak{t}$. Consider also the triangular matrix $\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_2 + \cdots + \mathfrak{s}_n$. Then $\mathfrak{st} = \mathfrak{s}_1\mathfrak{t}_1 + \mathfrak{s}_2\mathfrak{t}_2 + \cdots + \mathfrak{s}_n\mathfrak{t}_n$ since $\mathfrak{s}_i\mathfrak{t}_j = 0$ for $i \neq j$; and (7) becomes

$$\mathfrak{a} - \mathfrak{st} = 0 \quad \text{or} \quad \mathfrak{a} = \mathfrak{st}.$$

4. Gaussian computation of inverse (and adjugate) without formal reduction of auxiliary identity matrix. The inverse of $\mathfrak{a}$, $\mathfrak{a}^{-1} = \mathfrak{C} = (c_{rk})$ can be calculated directly from the matrices $\mathfrak{s}$ and $\mathfrak{t}$ of (2). The adjugate $\mathfrak{D} = (d_{rk})$ can be calculated by multiplication by the determinant of the matrix and this can be calculated by the well known formula

$$\Delta = a_{11}a_{22\cdot1}a_{33\cdot(2)} \cdots a_{nn\cdot(n\cdot1)} \,. \tag{8}$$

The theory is presented in some detail and illustrated for the case $n = 4$ after which a more general matrix presentation is given. The matrix equation $\mathfrak{aC} = \mathfrak{J}$ is equivalent to the following 4^2 simultaneous equations in the 4^2 unknowns (c_{rk}):

$$\text{(9)} \quad \begin{array}{rcccc} & k=1 & k=2 & k=3 & k=4 \\ a_{11}\,c_{1k} + a_{12}\,c_{2k} + a_{13}\,c_{3k} + a_{14}\,c_{4k} = & 1 & 0 & 0 & 0 \\ a_{21}\,c_{1k} + a_{22}\,c_{2k} + a_{23}\,c_{3k} + a_{24}\,c_{4k} = & 0 & 1 & 0 & 0 \\ a_{31}\,c_{1k} + a_{32}\,c_{2k} + a_{33}\,c_{3k} + a_{34}\,c_{4k} = & 0 & 0 & 1 & 0 \\ a_{41}\,c_{1k} + a_{42}\,c_{2k} + a_{43}\,c_{3k} + a_{44}\,c_{4k} = & 0 & 0 & 0 & 1 \end{array}$$

Now since $\mathfrak{Ca} = \mathfrak{J}$ also we have $\mathfrak{a}'\mathfrak{C}' = \mathfrak{J}$ and there results another set of 4^2 equations in the 4^2 unknowns (c_{rk}).

$$\text{(10)} \quad \begin{array}{rcccc} & r=1 & r=2 & r=3 & r=4 \\ a_{11}\,c_{r1} + a_{21}\,c_{r2} + a_{31}\,c_{r3} + a_{41}\,c_{r4} = & 1 & 0 & 0 & 0 \\ a_{12}\,c_{r1} + a_{22}\,c_{r2} + a_{32}\,c_{r3} + a_{42}\,c_{r4} = & 0 & 1 & 0 & 0 \\ a_{13}\,c_{r1} + a_{23}\,c_{r2} + a_{33}\,c_{r3} + a_{43}\,c_{r4} = & 0 & 0 & 1 & 0 \\ a_{14}\,c_{r1} + a_{24}\,c_{r2} + a_{34}\,c_{r3} + a_{44}\,c_{r4} = & 0 & 0 & 0 & 1 \end{array}$$

Fisher [11; 150] has shown that the equations (9) could be solved by reducing the unit matrix on the right. One of the authors has shown how to calculate the inverse of a symmetric matrix by Gaussian methods without reducing the unit matrix [1]. We now show how to reduce the non-symmetric matrix similarly. By the same process used in getting from matrix (1) to matrix (2), we can reduce the 4^2 equations of (9) to the 4^2 auxiliary equations below.

$$\text{(11)} \quad \begin{array}{rcccc} & k=1 & k=2 & k=3 & k=4 \\ a_{11}\,c_{1k} + a_{12}\,c_{2k} + a_{13}\,c_{3k} + a_{14}\,c_{4k} = & 1 & 0 & 0 & 0 \\ a_{22\cdot1}\,c_{2k} + a_{23\cdot1}\,c_{3k} + a_{24\cdot1}\,c_{4k} = & * & 1 & 0 & 0 \\ a_{33\cdot(2)}\,c_{3k} + a_{34\cdot(2)}\,c_{4k} = & * & * & 1 & 0 \\ a_{44\cdot(3)}\,c_{4k} = & * & * & * & 1 \end{array}$$

The terms marked * can be computed by the process. However if we do not compute these terms we have ten equations with the right hand terms either 1 or 0.

In a similar way the 4^2 equations of (10) can be reduced to the 4^2 auxiliary equations below. As above we may neglect the calculation of the diagonal terms, and of all terms below the diagonal, and still have six equations (with terms on the right zero).

$$
\begin{array}{lrcccc}
 & & r=1 & r=2 & r=3 & r=4 \\
 & c_{r1} + b_{21}\, c_{r2} + b_{31}\; c_{r3} + b_{41}\; c_{r4} = & * & 0 & 0 & 0 \\
(12) & c_{r2} + b_{32\cdot 1}\, c_{r3} + b_{42\cdot 1}\; c_{r4} = & * & * & 0 & 0 \\
 & c_{r3} + b_{43\cdot(2)}\, c_{r4} = & * & * & * & 0 \\
 & c_{r4} = & * & * & * & *
\end{array}
$$

The ten equations of (11) with the six equations of (12) are sufficient for determining the inverse matrix. Solve (11) for $k = 4$; then solve (12) for $r = 4$; then solve (11) for $k = 3$; then solve (12) for $r = 3$; etc. Each equation can be solved completely on the machine to give a value of a c_{rk}.

It should be noted that Gaussian methods are approximation methods since they are division methods. For a discussion and treatment of the errors resulting the reader is referred to papers by Hotelling [9] and Satterthwaite [10] to which further reference is made in the next section.

Different forms for presentation of the results may be used. We suggest the following form which presents first the matrix (1), then the terms of the matrix (2). The terms of the matrix $\mathfrak{C}'$ are then computed by (11) and (12) and placed diagonally adjacent to the terms of (2). The transpose of $\mathfrak{C}$ is used so that the check multiplication by $\mathfrak{a}$ may be most easily accomplished. The result of this multiplication which next appears shows that the computed value of $\mathfrak{a}$ is correct to three places. The final matrix of Table I gives the value of the adjugate, $\mathfrak{D}$, as found by multiplying each element of the inverse by $(26)(52.308)(39.356)(43.071) = 2{,}305{,}300$ (to five places).

It is possible to check the accuracy of the entries of each row and column of the matrix (2) separately by using a check sum to the right of each row and at the bottom of each column. We have not taken the space to show check sums and they are not particularly needed after one gets a little practice with the method. In any case $\mathfrak{a}\mathfrak{a}^{-1}$ should be computed as a final check.

A more general matrix presentation results from the use of (6). The matrix equation $\mathfrak{a}\mathfrak{C} = \mathfrak{J}$ becomes $\mathfrak{s}\mathfrak{t}\mathfrak{C} = \mathfrak{J}$ and hence the auxiliary equation becomes

$$(13)\qquad \mathfrak{t}\mathfrak{C} = \mathfrak{s}^{-1}.$$

Now since $\mathfrak{s}$ is triangular with unit diagonal terms and zeros above the diagonal, it follows that $\mathfrak{s}^{-1}$ also has unit diagonal terms with zeros above the diagonal. Hence we can select $\frac{n(n+1)}{2}$ equations from the n^2 equation of (13) which demand no further knowledge of the entries of $\mathfrak{s}^{-1}$. A similar treatment of the matrix equation $\mathfrak{a}'\mathfrak{C}' = \mathfrak{J}$, $\mathfrak{t}'\mathfrak{s}'\mathfrak{C}' = \mathfrak{J}$ and

$$(14)\qquad \mathfrak{s}'\mathfrak{C}' = (\mathfrak{t}')^{-1}$$

yields $\frac{n(n-1)}{2}$ equations involving zero terms of $(\mathfrak{t}')^{-1}$. These two sets of equations taken together in the proper order are sufficient for calculating the n^2 values in the inverse.

It may be of interest to note that this is also a procedure for calculating $\mathfrak{t}^{-1}\mathfrak{s}^{-1}$ when $\mathfrak{t}$ and $\mathfrak{s}$ are known without the calculation of $\mathfrak{t}^{-1}$ and $\mathfrak{s}^{-1}$ separately since

$$\mathfrak{C} = \mathfrak{a}^{-1} = \mathfrak{t}^{-1}\mathfrak{s}^{-1}. \tag{15}$$

5. The method of multiplication and subtraction with division. We now present a different method, based upon the work of Hermite [15] and Chiò [16]

TABLE I

Suggested form for calculation

26		−10		15		32	
19		45		−14		−8	
−12		16		27		13	
32		29		−35		28	
26		−10		15		32	
	.02873		−.00696		.01825		−.00283
.73077		52.308		−24.962		−31.385	
	.02436		.01239		.01440		−.02267
−.46154		.21765		39.356		34.600	
	−.02302		.01572		.00791		.01991
1.23077		.78970		−.85753		43.071	
	−.01519		.00419		−.02041		.02322
	1.000		0.000		0.000		0.000
	0.000		1.000		0.000		0.000
	0.000		0.000		1.000		0.000
	0.000		0.000		0.000		1.000
66231		−16045		42072		−6524	
56157		28563		33196		52261	
−53068		36239		18235		45899	
−35018		9659		−47051		53529	

together with important modifications suggested by the work of Dodgson [17]. Current presentations of the basic method include the "method of condensation" [18; 45–48] and in compact forms, the "method of multiplication and subtraction" of one of the authors [2; 197–202].

In Gaussian methods we *divide* each element of a column by the leading (diagonal) element of that column. In the method of multiplication and subtraction we use the leading element as a "pivot" forming a number of two-rowed determinants. Thus we use the leading elements as *multipliers* rather than as divisors. No divisions are made in this method. This is a very real advantage when the elements of the original matrix contain only two (or three) digits each and when $n < 7$ (or 5). In such cases we can use this method to compute *exactly* the values of any minor of the determinant of the matrix and even the adjugate itself.

It is perhaps well to mention here that error control is difficult with division (Gaussian) methods. Even if many significant places are carried the errors may be significant, cumulative, and difficult to measure. The techniques suggested by the papers of Hotelling [9] and Satterthwaite [10] are most useful in developing error control in matrix calculation. However, where accuracy is important, and when the number of digits is not excessive, there appears to be merit in calculating the exact values.

In the method of multiplication and subtraction, we compute from the matrix (1) the following matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & A_{22\cdot 1} & A_{23\cdot 1} & \cdots & A_{2n\cdot 1} \\ a_{31} & A_{32\cdot 1} & A_{33\cdot(2)} & \cdots & A_{3n\cdot(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & A_{n2\cdot 1} & A_{n3\cdot(2)} & \cdots & A_{nn\cdot(n-1)} \end{bmatrix} \tag{16}$$

where

$$\begin{aligned} A_{rk\cdot 1} &= a_{11}a_{rk} - a_{1k}a_{r1} \\ A_{rk\cdot(2)} &= A_{22\cdot 1}A_{rk\cdot 1} - A_{2k\cdot 1}A_{r2\cdot 1} \end{aligned} \tag{17}$$

and in general

$$A_{rk\cdot(j)} = A_{jj\cdot(j-1)}A_{rk\cdot(j-1)} - A_{jk\cdot(j-1)}A_{rj\cdot(j-1)}.$$

This notation is similar to that used in connection with Gaussian methods above.

In the method of multiplication and subtraction with division, we compute from the matrix (1) the following matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & B_{22\cdot 1} & B_{23\cdot 1} & \cdots & B_{2n\cdot 1} \\ a_{31} & B_{32\cdot 1} & B_{33\cdot(2)} & \cdots & B_{3n\cdot(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & B_{n2\cdot 1} & B_{n3\cdot(2)} & \cdots & B_{nn\cdot(n-1)} \end{bmatrix} \tag{18}$$

where

$$\begin{aligned} B_{rk\cdot 1} &= a_{11}a_{rk} - a_{1k}a_{r1} \\ B_{rk\cdot(2)} &= \frac{B_{22\cdot 1}B_{rk\cdot 1} - B_{2k\cdot 1}B_{r2\cdot 1}}{a_{11}} \\ B_{rk\cdot(3)} &= \frac{B_{33\cdot(2)}B_{rk\cdot(2)} - B_{3k\cdot(2)}B_{r3\cdot(2)}}{B_{22\cdot 1}} \end{aligned} \tag{19}$$

and in general

$$B_{rk\cdot(j)} = \frac{B_{jj\cdot(j-1)}B_{rk\cdot(j-1)} - B_{jk\cdot(j-1)}B_{rj\cdot(j-1)}}{B_{j-1,j-1\cdot(j-2)}} \tag{20}$$

with $B_{rk\cdot 1}$ and $B_{rk\cdot(2)}$ as defined in (19).

In general the method calls for the calculation of entries according to the method of multiplication and subtraction but in addition calls for the division by the leading element of the second preceding row or column. Since this division must be exact, as is shown in the next section, we have at each stage a good numerical check on the work as well as an *exact* value of the entry. Furthermore it is shown in the next section that the value of $B_{rk\cdot(j)}$ is the exact value of the determinant

$$(21)\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & a_{2k} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & a_{3k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & a_{jj} & a_{jk} \\ a_{r1} & a_{r2} & a_{r3} & \cdots & a_{rj} & a_{rk} \end{vmatrix}$$

All the recorded entries (themselves values of determinants) are calculated on the machine. The only limitation is the number of places the machine provides. For the trivial problems (composed of small integers) found in most texts of College Algebra, one can calculate the values readily without machines. For example the determinant

$$\begin{vmatrix} 2 & 1 & -3 & 4 \\ 3 & 2 & 2 & 1 \\ -2 & -1 & 1 & 3 \\ 4 & -3 & 2 & 1 \end{vmatrix} \quad \text{yields at once} \quad \begin{vmatrix} 2 & 1 & -3 & 4 \\ 3 & 1 & 13 & -10 \\ -2 & 0 & -2 & 7 \\ 4 & -10 & 73 & -397 \end{vmatrix}$$

and the value of Δ is -397. All the other entries are also minors of Δ.

Dodson introduced a method of multiplication and subtraction with division as early as 1866 [17]. He however used a moving pivot. For our purposes it seems preferable to use a fixed pivot as we suggest in this paper.

6. Proofs of theorems involving the $B_{rk\cdot(j)}$.

(*a*) *First theorem.* We first prove that the numerator $B_{jj\cdot(j-1)}B_{rk\cdot(j-1)} - B_{jk\cdot(j-1)}B_{rj\cdot(j-1)}$ in the definition of $B_{rk\cdot(j)}$ is *exactly* divisible by the denominator $B_{j-1,j-1\cdot(j-2)}$. To do this we expand the terms of this numerator of (20) with the continued use of

$$(22)\qquad B_{rk\cdot(j-1)} = \frac{B_{j-1,j-1\cdot(j-2)}\,B_{rk\cdot(j-2)} - B_{j-1,k\cdot(j-2)}\,B_{r,j-1\cdot(j-2)}}{B_{j-2,j-2\cdot(j-3)}}$$

(which is (20) with j replaced by $j - 1$) and then we multiply and cancel. It is found that $B_{j-1,j-1\cdot(j-2)}$ is a factor of all non-cancellable terms so the exact divisibility is proved.

(*b*) *Second theorem.* We next prove that $B_{rk\cdot(j)}$ is the value of the determinant

(21). We illustrate first for $j = 3$ and then give a more general proof. When $j = 3$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{1k} \\ a_{21} & a_{22} & a_{23} & a_{2k} \\ a_{31} & a_{32} & a_{33} & a_{3k} \\ a_{r1} & a_{r2} & a_{r3} & a_{rk} \end{vmatrix} = \frac{1}{a_{11}^3} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{1k} \\ 0 & B_{22\cdot1} & B_{23\cdot1} & B_{2k\cdot1} \\ 0 & B_{32\cdot1} & B_{33\cdot1} & B_{3k\cdot1} \\ 0 & B_{r2\cdot1} & B_{r3\cdot1} & B_{rk\cdot1} \end{vmatrix}$$

$$= \frac{1}{a_{11}^2} \begin{vmatrix} B_{22\cdot1} & B_{23\cdot1} & B_{2k\cdot1} \\ B_{32\cdot1} & B_{33\cdot1} & B_{3k\cdot1} \\ B_{r2\cdot1} & B_{r3\cdot1} & B_{rk\cdot1} \end{vmatrix} = \frac{1}{B_{22\cdot1}} \begin{vmatrix} B_{33\cdot(2)} & B_{3k\cdot(2)} \\ B_{r3\cdot(2)} & B_{rk\cdot(2)} \end{vmatrix} = B_{rk\cdot(3)} .$$

In the more general case we designate the determinant (21) by $| a_{rk} |$ and reduce the order by the "condensation" method just illustrated. It is understood that the values of $B_{rk\cdots}$ used in the following proof have primary subscripts larger than secondary subscripts since the rank of the resulting determinant decreases with each condensation

$$(23)\qquad \begin{aligned} | a_{rk} | &= \frac{1}{a_{11}^{j-1}} | B_{rk\cdot1} | = \frac{1}{B_{22\cdot1}^{j-2}} | B_{rk\cdot(2)} | \\ &= \frac{1}{B_{33\cdot(2)}^{j-3}} | B_{rk\cdot(3)} | = \cdots = \frac{1}{B_{j-1,j-1\cdot(j-2)}} | B_{rk\cdot(j-1)} | = B_{rk\cdot(j)} . \end{aligned}$$

It is to be noted that the first theorem, since each $B_{rk\cdot(j)}$ can be interpreted as a determinant by the second theorem, is a corollary of a well known theorem [19; 33]. In a conventional determinantal notation it might appear as

$$(24)\qquad \Delta\Delta_{jk:rj} = \Delta_{rk}\Delta_{jj} - \Delta_{rj}\Delta_{jk}$$

where the first subscripts indicate deleted rows and the second subscripts deleted columns.

(c) *Third theorem.* We next relate the values of $B_{rk\cdot(j)}$ and the values $a_{rk\cdot(j)}$ and $b_{rk\cdot(j)}$. With the use of the second theorem (23) and (8) we have

$$(25)\qquad \frac{B_{rk\cdot(j)}}{a_{rk\cdot(j)}} = \frac{a_{11}\, a_{22\cdot1}\, a_{33\cdot(2)} \cdots a_{jj\cdot(j-1)}\, a_{rk\cdot(j)}}{a_{rk\cdot(j)}} = B_{jj\cdot(j-1)}$$

and with the additional use of (4)

$$(26)\qquad \frac{B_{rk\cdot(j)}}{b_{rk\cdot(j)}} = \frac{a_{11}\, a_{22\cdot1}\, a_{33\cdot(2)} \cdots, a_{jj\cdot(j-1)}\, a_{rk\cdot(j)}}{\dfrac{a_{rk\cdot(j)}}{a_{kk\cdot(j)}}} = B_{kk\cdot(j)} .$$

These formulas may be written in the form

$$(27)\qquad \begin{aligned} B_{rk\cdot(j)} &= B_{jj\cdot(j-1)}a_{rk\cdot(j)} \\ B_{rk\cdot(j)} &= B_{kk\cdot(j)}b_{rk\cdot(j)} \end{aligned}$$

and since $B_{jj\cdot(j-1)}$ and $B_{j+1,j+1\cdot(j)}$ are diagonal terms, it follows that the matrix (18) can be obtained from the matrix (2) by multiplication by diagonal matrices.

(d) *Fourth Theorem.* A fourth theorem gives explicit matrix formulation to these results and shows how the values of the matrix (18) can be used in factoring the matrix (1). Now (27) and (28) can be written in the form

$$\mathfrak{T} = \mathfrak{M}_T \mathfrak{t} \tag{29}$$

$$\mathfrak{S} = \mathfrak{M}_S \mathfrak{s} \tag{30}$$

where $\mathfrak{M}_T$ is the diagonal matrix which multiplies $\mathfrak{t}$ to get $\mathfrak{T}$ and $\mathfrak{M}_S$ is the diagonal matrix which multiplies $\mathfrak{s}$ to get $\mathfrak{S}$. The values of the $\mathfrak{T}$ matrix are the values of (18) with $r \leqq k$ while the values of the $\mathfrak{S}$ matrix are the values of (18) with $r \geqq k$. The diagonal matrix $\mathfrak{M}_T$ is composed of diagonal elements $[1, a_{11}, B_{22\cdot 1} \cdots B_{n-1,n-1\cdot(n-2)}]$ while the matrix $\mathfrak{M}_s$ is composed of diagonal elements $[a_{11}, B_{22\cdot 1}, B_{33\cdot(2)} \cdots B_{nn\cdot(n-1)}]$. The basic matrix factorization equation (6) then appears as

$$\mathfrak{a} = \mathfrak{M}_s^{-1} \mathfrak{M}_T^{-1} \mathfrak{S} \mathfrak{T}. \tag{31}$$

It is to be noted that exact values of elements of all these matrices are available if the inverse diagonal matrices are written in fractional form, subject of course to practical limitations such as number of places of computing machine, etc.

7. Computation of the adjugate matrix. We now present matrix formulas which enable one to compute the adjugate of $\mathfrak{a}$ compactly with the method of multiplication and subtraction with division. If (9) is the determinant of $\mathfrak{a}$ and $\mathfrak{D}$ is the adjugate of $\mathfrak{a}$, we have

$$\begin{aligned}
\mathfrak{a}\mathfrak{D} &= \mid a \mid \mathfrak{J} \\
\mathfrak{s}\mathfrak{t}\mathfrak{D} &= \mid a \mid \mathfrak{J} \\
\mathfrak{t}\mathfrak{D} &= \mid a \mid \mathfrak{s}^{-1} \\
\mathfrak{M}_t \mathfrak{t}\mathfrak{D} &= \mathfrak{M}_t \mid a \mid \mathfrak{s}^{-1} \\
\mathfrak{T}\mathfrak{D} &= \mathfrak{M}_t \mid a \mid \mathfrak{s}^{-1}
\end{aligned} \tag{32}$$

and similarly

$$\begin{aligned}
\mathfrak{a}'\mathfrak{D}' &= \mid a \mid \mathfrak{J} \\
\mathfrak{t}'\mathfrak{s}'\mathfrak{D}' &= \mid a \mid \mathfrak{J} \\
\mathfrak{s}'\mathfrak{D}' &= \mid a \mid (\mathfrak{t}')^{-1} \\
\mathfrak{M}_s' \mathfrak{s}'\mathfrak{D}' &= \mathfrak{M}_s \mid a \mid (\mathfrak{t}')^{-1} \\
\mathfrak{S}'\mathfrak{D}' &= \mathfrak{M}_s \mid a \mid (\mathfrak{t}')^{-1}.
\end{aligned} \tag{33}$$

The computational procedure in getting the adjugate is very similar to that used in getting the inverse in section 4. $\mathfrak{T}$ and $\mathfrak{S}$ are triangular matrices while

$\mathfrak{s}^{-1}$ and $\mathfrak{t}^{-1}$ are the matrices used before. The values of $\mathfrak{M}_t[1, a_{11}, B_{22\cdot 1}, \cdots B_{n-1,n-1\cdot(n-2)}]$ $\mathfrak{M}_s[a_{11}, B_{22-1}, B_{33\cdot(2)}, \cdots B_{nn\cdot(n-1)}]$ and $|a|$ are first computed by (18) so that $\mathfrak{M}_t|a|$ and $\mathfrak{M}_s|a|$ can be calculated. Without further calculation we are able to select $\dfrac{n(n+1)}{2}$ equations from the matrix equation (32) having known coefficients on the right $\left(\dfrac{n(n-1)}{2}\text{ of which are zero}\right)$ and $\dfrac{n(n-1)}{2}$ equations from the matrix equation (33) having zero coefficients on the right. These constitute the n^2 equations necessary to determine the n^2 values of d_{rk}. These values of d_{rk} can all be calculated directly on the machine and, what is more useful in discovering calculational errors, the divisions yielding the d_{rk} must be exact.

For $n = 4$ these n^2 equations are

$$
(34)\quad
\begin{array}{rcccc}
 & k=1 & k=2 & k=3 & k=4 \\
a_{11}d_{1k} + a_{12}d_{2k} + a_{13}d_{3k} + a_{14}d_{4k} = & |a| & 0 & 0 & 0 \\
B_{22\cdot1}d_{2k} + B_{23\cdot1}d_{3k} + B_{24\cdot1}d_{4k} = & * & a_{11}|a| & 0 & 0 \\
B_{33\cdot(2)}d_{3k} + B_{34\cdot(2)}d_{4k} = & * & * & B_{22\cdot1}|a| & 0 \\
B_{44\cdot2}d_{4k} = & * & * & * & B_{33\cdot(2)}|a|
\end{array}
$$

$$
(35)\quad
\begin{array}{rcccc}
 & r=1 & r=2 & r=3 & r=4 \\
a_{11}d_{r1} + a_{21}d_{r2} + a_{31}d_{r3} + a_{41}d_{r4} = & * & 0 & 0 & 0 \\
B_{22\cdot1}d_{r2} + B_{32\cdot1}d_{r3} + B_{42\cdot1}d_{r4} = & * & * & 0 & 0 \\
B_{33\cdot(2)}d_{r3} + B_{43\cdot(2)}d_{r4} = & * & * & * & 0
\end{array}
$$

The process is similar to that of section 4. An illustration for the case $n = 4$ is given in Table II. The matrix of the B's is directly below the matrix $\mathfrak{a}$ and the calculated values of the elements of $\mathfrak{D}'$ (obtained by solving (34) and (35)) are placed diagonally in the cells with the B's. The values of the transpose of $\mathfrak{D}$ are used so that the check, premultiplication by $\mathfrak{a}$, is easily carried out. The next matrix in Table II exhibits $\mathfrak{a}\mathfrak{D} = |a|\mathfrak{I}$. The last matrix of Table II is a five decimal place approximation to $\mathfrak{C}'$ which is obtained by dividing the entries of $\mathfrak{D}'$ by $|a|$. Since we know these are the correct five decimal place values of $\mathfrak{C}'$, we may compare the corresponding values of Table I to see how much those are in error. It should be noticed that the approximation to $\mathfrak{C}'$ may be readily carried to more than five decimal places if desired.

As with the Gaussian methods, it is possible here, also, to check each row and column individually by using check sums.

The work necessary for the computation of the adjugate from the matrix of the B's can be shortened somewhat by the use of the fact that the adjugate is composed of the cofactors of the a_{rk}. Now the cofactors of the four terms in the lower right hand corner are $d_{n-1,n-1} = B_{n-1,n-1\cdot(n-2)}$; $d_{n-1,n} = -B_{n-1,n\cdot(n-2)}$; $d_{n,n-1} = -B_{n,n-1\cdot(n-2)}$; and $d_{nn} = B_{nn\cdot(n-2)}$ and these are available from the calculation of the B's though $B_{nn\cdot(n-2)}$ is not recorded. (See the lower right four entries of the B's and a's in Table II above). With these four values immediately available, the use of but $n^2 - 4$ additional equations is demanded, or this additional information can be used in checking.

TABLE II

Suggested form for computation of adjugate (with check) and then inverse

26		−10		15		32	
19		45		−14		−8	
−12		16		27		13	
32		29		−35		28	
26		−10		15		32	
	66233		−16033		42069		−6503
19		1360		−649		−816	
	56151		28558		33194		−52258
−12		296		53524		47056	
	−53068		36236		18224		45899
32		1074		−45899		2305327	
	−35013		9659		−47056		53524
	2305327		0		0		0
	0		2305327		0		0
	0		0		2305327		0
	0		0		0		2305327
	.02873		−.00695		.01825		−.00282
	.02436		.01239		.01440		−.02267
	−.02302		.01572		.00791		.01991
	−.01519		.00419		−.02041		.02322

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